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Corrections

Page 472

also see Answer Book

A

SCHOOL ALGEBRA.

BY

G. A. WENTWORTH,

AUTHOR OF A SERIES OF TEXT-BOOKS IN MATHEMATICS.

TEACHERS' EDITION.

BOSTON, U.S.A.:
PUBLISHED BY GINN & COMPANY.
1896.

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PREFACE.

THIS edition is intended for teachers, and *for them only*. The publishers will make every effort to keep the book from pupils; and teachers are urged to exercise the utmost care not to lose their copies, or to leave them where pupils can have access to them. Teachers must also feel bound in honor not to sell their copies to any persons except to the publishers.

It is hoped that young teachers will derive great advantage from studying the systematic arrangement of the algebraic work, for such attention has been paid to this as the limitation of the page would allow.

It is also expected that many teachers who are pressed for time will find great relief by not being obliged to work out every problem in the Algebra.

G. A. WENTWORTH.

EXETER, N. H.
September, 1890.

SCHOOL ALGEBRA.

Exercise 1. Page 8.

1. $7 + (3 - 2) = 7 + 3 - 2 = 8.$
2. $7 - (3 - 2) = 7 - 3 + 2 = 6.$
3. $7 - (3 + 2) = 7 - 3 - 2 = 2.$
4. $5 \times (2 + 3) = 5 \times 5 = 25.$
5. $(5 + 3) \div 2 = 8 \div 2 = 4.$
6. $5 \times (3 - 2) = 5 \times 1 = 5.$
7. $(7 - 3) \times (5 - 2) = 4 \times 3 = 12.$
8. $(8 - 2) \div (5 - 2) = 6 \div 3 = 2.$
9. $3 \times (12 - 6 - 2) = 3 \times 4 = 12.$

Exercise 2. Page 26.

1. $7a - bc = 7 - 2 \times 3 = 7 - 6 = 1.$
2. $ac + b = 3 + 2 = 5.$
3. $4ab - c = 4 \times 2 - 3 = 8 - 3 = 5.$
4. $6ab - b - c = 6 \times 2 - 2 - 3 = 12 - 2 - 3 = 7.$
5. $2a - b + c = 2 - 2 + 3 = 3.$
6. $ab + bc - ac = 2 + 2 \times 3 - 3 = 2 + 6 - 3 = 5.$
7. $b^2 + a^2 + c^2 = 2^2 + 1^2 + 3^2 = 4 + 1 + 9 = 14.$
8. $2ab - 5bc^2 = 2 \times 2 - 5 \times 2 \times 3^2 = 4 - 90 = -86.$
9. $\sqrt{4}\,abc = 2 \times 1 \times 2 \times 3 = 12.$
10. $\sqrt{6abc} = \sqrt{6 \times 1 \times 2 \times 3} = \sqrt{36} = 6.$
11. $c^3 - b^3 = 3^3 - 2^3 = 27 - 8 = 19.$
12. $\sqrt[3]{c^2 - a^3} = \sqrt[3]{3^2 - 1^3} = \sqrt[3]{8} = 2.$
13. $a - 2(b + c) = 1 - 2(2 + 3) = 1 - 2 \times 5 = -9.$
14. $(a + b)^2 + 2(c - a)^3 = (1 + 2)^2 + 2(3 - 1)^3 = 3^2 + 2 \times 2^3 = 9 + 16 = 25.$

15. $\sqrt{6bc} - (b - c) = \sqrt{6 \times 2 \times 3} - (2 - 3) = \sqrt{36} - (-1) = 6 + 1 = 7.$

16. $$\begin{aligned} 6b - 10bc \div 12a + 2c &= 6 \times 2 - 10 \times 2 \times 3 \div 12 + 2 \times 3 \\ &= 12 - 60 \div 12 + 6 \\ &= 12 - \tfrac{60}{12} + 6 \\ &= 12 - 5 + 6 \\ &= 13. \end{aligned}$$

17. $$\begin{aligned} 5c \div (b - a) - (b + a) &= 15 \div (2 - 1) - (2 + 1) \\ &= 15 \div 1 - 3 \\ &= 15 - 3 \\ &= 12. \end{aligned}$$

18. $\sqrt[3]{6b^2c^2} = \sqrt[3]{6 \times 2^2 \times 3^2} = \sqrt[3]{6 \times 4 \times 9} = \sqrt[3]{216} = 6.$

19. $7a - bc + 6d = 7 - 2 \times 3 + 6 \times 0 = 7 - 6 = 1.$

20. $ac + b - d = 3 + 2 - 0 = 5.$

21. $4ab - cd - d = 4 \times 2 - 3 \times 0 - 0 = 8.$

22. $2a - b + c = 2 - 2 + 3 = 3.$

23. $ab + bc - ad = 2 + 2 \times 3 - 0 = 8.$

24. $2ab - 5bc = 2 \times 2 - 5 \times 2 \times 3 = 4 - 30 = -26.$

25. $\sqrt{4}\,abcd = \sqrt{4} \times 1 \times 2 \times 3 \times 0 = 0.$

26. $\sqrt{4abcd} = \sqrt{4 \times 1 \times 2 \times 3 \times 0} = 0.$

27. $b - c + d = 2 - 3 + 0 = -1.$

28. $a - 2(b + c) = 1 - 2(2 + 3) = 1 - 10 = -9.$

29. $$\begin{aligned} 2b(3 - 5c) \div (a - 2c) &= 2 \times 2(3 - 5 \times 3) \div (1 - 2 \times 3) \\ &= 2 \times 2(-12) \div (-5) = -48 \div (-5) = \tfrac{48}{5}. \end{aligned}$$

30. $2(a + b)^2 = 2(1 + 2)^2 = 2 \times 3^2 = 18.$

Exercise 3. Page 27.

1. $(5 - 2) + (3 + 1) = 5 - 2 + 3 + 1 = 7.$

2. $(6 - 2) - (2 - 3) = 6 - 2 - 2 + 3 = 5.$

3. $(-3 + 4) - (2 + 5) = -3 + 4 - 2 - 5 = -6.$

4. $-6 - (2 - 3 - 1) = -6 - 2 + 3 + 1 = -4.$

5. $2 - (5 - 7 + 8) = 2 - 5 + 7 - 8 = -4.$

6. $8-(7-5+4)=8-7+5-4=2.$

7. $10-(5-6-7)=10-5+6+7=18.$

8. $2-(3-3+4)=2-3+3-4=-2.$

9. $(5-10)+(3-2)=5-10+3-2=-4.$

10. $-7+(3+2-4)=-7+3+2-4=-6.$

11. $23+(2-7-6)=23+2-7-6=12.$

12. $24-(2-7-5)=24-2+7+5=34.$

13. $11+(1-4-3)=11+1-4-3=5.$

14. $(15-3-5)-7=15-3-5-7=0.$

15. $10-(5-2-1)=10-5+2+1=8.$

16. $-17-(5-10-13)=-17-5+10+13=1.$

17. $-5+(3+2+7)=-5+3+2+7=7.$

18. $-(2-3+4)-1=-2+3-4-1=-4.$

19. $-(10-8-12+10)=-10+8+12-10=0.$

20. $+(8-2-3-1)=8-2-3-1=2.$

21. $a+b+c=1+2-3=0.$

22. $a-b+c=1-2-3=-4.$

23. $a-b-c=1-2+3=2.$

24. $a-(-b)+c=a+b+c=1+2-3=0.$

25. $a-(-b)-c=a+b-c=1+2+3=6.$

26. $(-a)+(-b)+(-c)=-a-b-c=-1-2+3=0.$

Exercise 4. Page 31.

1. $a.$ 2. $0.$ 3. $-4a^2b.$ 4. $7xy+4ax.$

5.
$$\begin{array}{r}a+b\\a-b\\\hline 2a\end{array}$$

6.
$$\begin{array}{rrr} & x^2 & -x\\ x^3 & -x^2 & \\\hline x^3 & & -x\end{array}$$

7.
$$\begin{array}{rrr}5x^2 & +6x & -2\\3x^2 & -7x & +2\\\hline 8x^2 & -x & \end{array}$$

8.
$$\begin{array}{rrr}3x^2 & -2xy & +y^2\\x^2 & -2xy & +3y^2\\\hline 4x^2 & -4xy & +4y^2\end{array}$$

9.

$$\begin{array}{rrr} ax^2 + & bx - & 4 \\ 3ax^2 - & 2bx + & 4 \\ -4ax^2 - & 2bx + & 5 \\ \hline & -3bx + & 5 \end{array}$$

11.

$$\begin{array}{rrr} 3ab - & 2ax^2 + & 3a^2x \\ 4ab + & 5ax^2 - & 6a^2x \\ ab - & ax^2 + & a^2x \\ -8ab + & ax^2 - & 5a^2x \\ \hline & 3ax^2 - & 7a^2x \end{array}$$

10.

$$\begin{array}{rrr} 5x + & 3y + & z \\ 3x + & 2y + & 3z \\ x - & 3y - & 5z \\ \hline 9x + & 2y - & z \end{array}$$

12.

$$\begin{array}{rrrrr} a^4 - & 2a^3 + & 3a^2 - & a + & 7 \\ 2a^4 - & 3a^3 + & 2a^2 - & a + & 6 \\ -a^4 - & 2a^3 + & 2a^2 & & -5 \\ \hline 2a^4 - & 7a^3 + & 7a^2 - & 2a + & 8 \end{array}$$

13.

$$\begin{array}{rrrrrr} 3a^2 - & ab + & ac - & 3b^2 + & 4bc - & c^2 \\ -5a^2 - & ab - & ac & & +5bc & \\ & +2ab & & & -4bc + & 5c^2 \\ -4a^2 & & & + b^2 - & 5bc + & 2c^2 \\ \hline -6a^2 & & & -2b^2 & & +6c^2 \end{array}$$

14.

$$\begin{array}{rrrrr} x^4 - & 3x^3 + & 2x^2 - & 4x + & 7 \\ 3x^4 + & 2x^3 + & x^2 - & 5x - & 6 \\ -4x^4 + & 3x^3 - & 3x^2 + & 9x - & 2 \\ 2x^4 - & x^3 + & x^2 - & x + & 1 \\ \hline 2x^4 + & x^3 + & x^2 - & x & \end{array}$$

15.

$$\begin{array}{rrrrrrr} 3xy^2 - & 4x^2y + & x^3 & & & & \\ -11xy^2 & & +5x^3 - & 12xz^2 & & & \\ & + x^2y & - & xz^2 - & 7y^3 & & \\ & & - & 4xz^2 & & + y^2 - & z^3 \\ \hline -8xy^2 - & 3x^2y + & 6x^3 - & 17xz^2 - & 7y^3 + & y^2 - & z^3 \end{array}$$

16.

$$\begin{array}{rrrrr} a^4 - & 2a^3 + & 3a^2 & & \\ & a^3 + & a^2 + & a & \\ 4a^4 + & 5a^3 & & & \\ & & 2a^2 + & 3a - & 2 \\ & & - a^2 - & 2a - & 3 \\ \hline 5a^4 + & 4a^3 + & 5a^2 + & 2a - & 5 \end{array}$$

17.

$$\begin{array}{rrrr} x^3 - & x^2y + & 2xy^2 - & y^3 \\ 2x^3 - & 3x^2y - & 4xy^2 - & 7y^3 \\ x^3 & - & 8xy^2 - & 7y^3 \\ \hline 4x^3 - & 4x^2y - & 10xy^2 - & 15y^3 \end{array}$$

Exercise 5. Page 33.

1. $$\begin{array}{r} 8a - 4b - 2c \\ 2a - 3b - 3c \\ \hline 6a - b + c \end{array}$$

2. $$\begin{array}{r} 3a - 4b + 3c \\ 2a - 8b - c - d \\ \hline a + 4b + 4c + d \end{array}$$

3. $$\begin{array}{r} 7a^2 - 9x - 1 \\ 5a^2 - 6x - 3 \\ \hline 2a^2 - 3x + 2 \end{array}$$

4. $$\begin{array}{r} 2x^2 - 2ax + a^2 \\ x^2 - ax - a^2 \\ \hline x^2 - ax + 2a^2 \end{array}$$

5. $$\begin{array}{r} 4a - 3b - 3c \\ 2a - 3b + 4c \\ \hline 2a \qquad - 7c \end{array}$$

6. $$\begin{array}{r} 5x^2 + 7x + 4 \\ 3x^2 - 7x + 2 \\ \hline 2x^2 + 14x + 2 \end{array}$$

7. $$\begin{array}{r} 2ax + 3by + 5 \\ 3ax - 3by - 5 \\ \hline - ax + 6by + 10 \end{array}$$

8. $$\begin{array}{r} 4a^2 - 6ab + 2b^2 \\ 3a^2 + ab + b^2 \\ \hline a^2 - 7ab + b^2 \end{array}$$

9. $$\begin{array}{r} 4a^2b + 7ab^2 + 9 \\ - 3ab^2 + 8 \\ \hline 4a^2b + 10ab^2 + 1 \end{array}$$

10. $$\begin{array}{r} 5a^2c + 6a^2b - 8a^3 \qquad \\ - 5a^2c + 6a^2b \qquad + b^3 \\ \hline 10a^2c \qquad - 8a^3 - b^3 \end{array}$$

11. $$\begin{array}{r} a^2 - b^2 \\ b^2 \\ \hline a^2 - 2b^2 \end{array}$$

12. $$\begin{array}{r} a^2 - b^2 \\ a^2 \quad\; \\ \hline - b^2 \end{array}$$

13. $$\begin{array}{r} b^2 \\ a^2 - b^2 \\ \hline - a^2 + 2b^2 \end{array}$$

14. $$\begin{array}{r} a^2 \quad\; \\ a^2 - b^2 \\ \hline b^2 \end{array}$$

15. $$\begin{array}{r} x^4 + 3ax^3 - 2bx^2 + 3cx - 4d \qquad \\ 3x^4 + ax^3 \qquad + 6cx + d - 4b^2 \\ \hline - 2x^4 + 2ax^3 - 2bx^2 - 3cx - 5d + 4b^2 \end{array}$$

16. Adding A, C, B, and D together, we have:

$$\begin{array}{r}
3a^2 - 2ab + 5b^2 \\
7a^2 - 8ab + 5b^2 \\
9a^2 - 5ab + 3b^2 \\
11a^2 - 3ab - 4b^2 \\
\hline
30a^2 - 18ab + 9b^2
\end{array}$$

17. Adding A, $-C$, $-B$, and D together, we have:

$$\begin{array}{r}
3a^2 - 2ab + 5b^2 \\
-7a^2 + 8ab - 5b^2 \\
-9a^2 + 5ab - 3b^2 \\
11a^2 - 3ab - 4b^2 \\
\hline
-2a^2 + 8ab - 7b^2
\end{array}$$

18. Adding C, $-A$, $-B$, and D together, we have:

$$\begin{array}{r}
7a^2 - 8ab + 5b^2 \\
-3a^2 + 2ab - 5b^2 \\
-9a^2 + 5ab - 3b^2 \\
11a^2 - 3ab - 4b^2 \\
\hline
6a^2 - 4ab - 7b^2
\end{array}$$

19. Adding A, C, $-B$, and $-D$ together, we have:

$$\begin{array}{r}
3a^2 - 2ab + 5b^2 \\
7a^2 - 8ab + 5b^2 \\
-9a^2 + 5ab - 3b^2 \\
-11a^2 + 3ab + 4b^2 \\
\hline
-10a^2 - 2ab + 11b^2
\end{array}$$

20. Adding A, $-C$, B, and D together, we have:

$$\begin{array}{r}
3a^2 - 2ab + 5b^2 \\
-7a^2 + 8ab - 5b^2 \\
9a^2 - 5ab + 3b^2 \\
11a^2 - 3ab - 4b^2 \\
\hline
16a^2 - 2ab - b^2
\end{array}$$

21. Adding A, C, $-B$, and D together, we have:

$$\begin{array}{r}
3a^2 - 2ab + 5b^2 \\
7a^2 - 8ab + 5b^2 \\
-9a^2 + 5ab - 3b^2 \\
11a^2 - 3ab - 4b^2 \\
\hline
12a^2 - 8ab + 3b^2
\end{array}$$

Exercise 6. Page 35.

1.
$$\begin{aligned}
a - b - [a - (b - c) - c] &= a - b - [a - b + c - c] \\
&= a - b - [a - b] \\
&= a - b - a + b \\
&= 0.
\end{aligned}$$

2.
$$\begin{aligned}
m - [n - (p - m)] &= m - [n - p + m] \\
&= m - n + p - m \\
&= p - n.
\end{aligned}$$

3.
$$\begin{aligned}
2x - \{y + [4z - (y + 2x)]\} &= 2x - \{y + [4z - y - 2x]\} \\
&= 2x - \{y + 4z - y - 2x\} \\
&= 2x - \{4z - 2x\} \\
&= 2x - 4z + 2x \\
&= 4x - 4z.
\end{aligned}$$

4.
$$\begin{aligned}3a-\{2b-[5c-(3a+b)]\}&=3a-\{2b-[5c-3a-b]\}\\&=3a-\{2b-5c+3a+b\}\\&=3a-\{3b-5c+3a\}\\&=3a-3b+5c-3a\\&=5c-3b.\end{aligned}$$

5.
$$\begin{aligned}a-\{b+[c-(d-b)+a]-2b\}&=a-\{b+[c-d+b+a]-2b\}\\&=a-\{b+c-d+b+a-2b\}\\&=a-\{a+c-d\}\\&=a-a-c+d\\&=d-c.\end{aligned}$$

6.
$$\begin{aligned}3x-[9-(2x+7)+3x]&=3x-[9-2x-7+3x]\\&=3x-[2+x]\\&=3x-2-x\\&=2x-2.\end{aligned}$$

7.
$$\begin{aligned}2x-[y-(x-2y)]&=2x-[y-x+2y]\\&=2x-[3y-x]\\&=2x-3y+x\\&=3x-3y.\end{aligned}$$

8.
$$\begin{aligned}a-[2b+(3c-2b)+a]&=a-[2b+3c-2b+a]\\&=a-[3c+a]\\&=a-3c-a\\&=-3c.\end{aligned}$$

9.
$$\begin{aligned}&(a-x+y)-(b-x-y)+(a+b-2y)\\&\quad=a-x+y-b+x+y+a+b-2y\\&\quad=2a.\end{aligned}$$

10.
$$\begin{aligned}&3a-[-4b+(4a-b)-(2a-5b)]\\&\quad=3a-[-4b+4a-b-2a+5b]\\&\quad=3a-[2a]\\&\quad=3a-2a\\&\quad=a.\end{aligned}$$

11.
$$\begin{aligned}&4c-[a-(2b-3c)+c]+[a-(2b-5c-a)]\\&\quad=4c-[a-2b+3c+c]+[a-2b+5c+a]\\&\quad=4c-[a-2b+4c]+[2a-2b+5c]\\&\quad=4c-a+2b-4c+2a-2b+5c\\&\quad=a+5c.\end{aligned}$$

12. $x+(y-z)-[(3x-2y)+z]+[x-(y-2z)]$
$=x+y-z-[3x-2y+z]+[x-y+2z]$
$=x+y-z-3x+2y-z+x-y+2z$
$=-x+2y.$

13. $a-[2a+(a-2a)+2a]-5a-\{6a-[(a+2a)-a]\}$
$=a-[2a+(-a)+2a]-5a-\{6a-[3a-a]\}$
$=a-[2a-a+2a]-5a-\{6a-2a\}$
$=a-[3a]-5a-\{4a\}$
$=a-3a-5a-4a$
$=-11a.$

14. $2x-(3y+z)-\{b-(c-b)+c-[a-(c-b)]\}$
$=2x-3y-z-\{b-c+b+c-[a-c+b]\}$
$=2x-3y-z-\{b-c+b+c-a+c-b\}$
$=2x-3y-z-\{b-a+c\}$
$=2x-3y-z-b+a-c.$

15. $a-[b+c-a-(a+b)-c]+(2a-\overline{b+c})$
$=a-[b+c-a-a-b-c]+(2a-b-c)$
$=a-[-2a]+2a-b-c$
$=a+2a+2a-b-c$
$=5a-b-c.$

16. $10-x-\{-x-[x-(x-\overline{5-x})]\}$
$=10-x-\{-x-[x-(x-5+x)]\}$
$=10-x-\{-x-[x-(2x-5)]\}$
$=10-x-\{-x-[x-2x+5]\}$
$=10-x-\{-x-[-x+5]\}$
$=10-x-\{-x+x-5\}$
$=10-x-\{-5\}$
$=10-x+5$
$=15-x.$

17. $2x-\{2x+(y-z)-3z+[2x-(y-\overline{z-2y})-3z]+4y\}$
$=2x-\{2x+y-z-3z+[2x-(y-z+2y)-3z]+4y\}$
$=2x-\{2x+y-4z+[2x-y+z-2y-3z]+4y\}$
$=2x-\{2x+y-4z+[2x-3y-2z]+4y\}$
$=2x-\{2x+y-4z+2x-3y-2z+4y\}$
$=2x-\{4x+2y-6z\}$
$=2x-4x-2y+6z$
$=-2x-2y+6z.$

18. $$\begin{aligned}&a-[b-\{-c+a-(a-b)-c\}]+[2a-(b-a)]\\&=a-[b-\{-c+a-a+b-c\}]+[2a-b+a]\\&=a-[b-\{b-2c\}]+[3a-b]\\&=a-[b-b+2c]+3a-b\\&=a-[2c]+3a-b\\&=a-2c+3a-b\\&=4a-2c-b.\end{aligned}$$

19. $$\begin{aligned}&a-\{b-[a-(c-b)+\overline{c-a}-(a-b-c)-a]+a\}\\&=a-\{b-[a-c+b+c-a-a+b+c-a]+a\}\\&=a-\{b-[-2a+2b+c]+a\}\\&=a-\{b+2a-2b-c+a\}\\&=a-\{3a-b-c\}\\&=a-3a+b+c\\&=-2a+b+c.\end{aligned}$$

20. $$\begin{aligned}&5a-\{-3a-[3a-(2a-\overline{a-b})-a]+a\}\\&=5a-\{-3a-[3a-(2a-a+b)-a]+a\}\\&=5a-\{-3a-[3a-(a+b)-a]+a\}\\&=5a-\{-3a-[3a-a-b-a]+a\}\\&=5a-\{-3a-[a-b]+a\}\\&=5a-\{-3a-a+b+a\}\\&=5a-\{-3a+b\}\\&=5a+3a-b\\&=8a-b.\end{aligned}$$

Exercise 7. Page 36.

1. $2a-b-3c-(d-3e+5f)$.
2. $x-a-y-(b+z+c)$.
3. $a+b-c-(-4a+b-1)$.
4. $ax+by+cz-(-bx+cy-cz)$.
5. $3a+2b+2c-(5d+3e+4f)$.
6. $x-y+z-(5xy+4xz-3yz)$.
7. $(a+b)x+(b-c)y+(c+a)z$.
8. $(a-b)x+(2a+3)y+(4a-3b-2)z$.
9. $(a-4b)x+(3c-2b)y-(5c+7a)z$.
10. $(a-11c)x+(3a+2b+2c)y-(b+c)z$.
11. $$\begin{aligned}&(-3a+2b-7c-c)x+(4b-5c-c)y-(6c+c)z\\&=(2b-3a-8c)x+(4b-6c)y-7cz.\end{aligned}$$
12. $$\begin{aligned}&-ax+(-5b-3a+b)y+(6a+3c-2b+b)z\\&=-ax-(3a+4b)y+(6a+3c-b)z.\end{aligned}$$
13. $(2a-2m)x-(b+3c)y+(1+3a-5b)z$.
14. $(1-ac)x+(a-mn-1)y-(a-bc+1)z$.

Exercise 8. Page 37.

1.
$$\begin{array}{rrrrrr}
4x^3 & -\ 5ax^2 & +\ 6a^2x & -5a^2 & & \\
3x^3 & +\ 4ax^2 & +\ 2a^2x & & +\ 6a^3 & \\
 & 19ax^2 & -15a^2x & & & -11x^2 \\
10x^3 & -18ax^2 & +\ 7a^2x & & +\ 5a^3 & \\ \hline
17x^3 & & & -5a^2 & +11a^3 & -11x^2
\end{array}$$

2.
$$\begin{array}{rrr}
3ab & +3a & +\ 6b \\
-\ ab & +2a & +\ 4b \\
7ab & -4a & -\ 8b \\
-2ab & +6a & +12b \\ \hline
7ab & +7a & +14b
\end{array}$$

3. $4(5-x)+6(5-x)+3(5-x)-2(5-x)=11(5-x)$.

4.
$$\begin{array}{l}
(a+b)x^2+(b+c)y^2+(a+c)z^2 \\
(b+c)x^2+(a+c)y^2+(a+b)z^2 \\
(a+c)x^2+(a+b)y^2+(b+c)z^2 \\ \hline
(a+b+b+c+c+a)x^2+(b+c+a+c+a+b)y^2+(a+c+a+b+b+c)z^2
\end{array}$$

This is equal to

$$(2a+2b+2c)x^2+(2a+2b+2c)y^2+(2a+2b+2c)z^2$$

or
$$(2a+2b+2c)(x^2+y^2+z^2)$$

or
$$2(a+b+c)(x^2+y^2+z^2)$$

5.
$$\begin{array}{l}
\quad(a+b)x+(b+c)y+(c+a)z \\
\quad(c+a)x-(a+b)y+(b+c)z \\
-(b+c)x+(a+c)y+(a+b)z \\ \hline
(a+b+c+a-b-c)x+(b+c-a-b+a+c)y+(c+a+b+c+a+b)z
\end{array}$$

This is equal to $2ax+2cy+(2a+2b+2c)z$.

6.
$$\begin{array}{rrr}
a^3 & -\ x^2 & \\
a^3 & +\ x^2 & +2ax \\ \hline
 & -2x^2 & -2ax
\end{array}$$

7.
$$\begin{array}{rrr}
3a^2 & +2ax & +\ x^2 \\
a^2 & -\ ax & -\ x^2 \\ \hline
2a^2 & +3ax & +2x^2
\end{array}$$

8.
$$\begin{array}{rrr}
8x^2 & -3ax & +5 \\
5x^2 & +2ax & +5 \\ \hline
3x^2 & -5ax &
\end{array}$$

9.
$$\begin{array}{rrrrr}
a^3 & +3b^2c & +ab^2 & -abc & \\
 & & ab^2 & -abc & +b^3 \\ \hline
a^3 & +3b^2c & & & -b^3
\end{array}$$

10.
$$\begin{array}{l} (a+b)\,x+(a+c)\,y \\ (a-b)\,x-(a-c)\,y \\ \hline (a+b-a+b)\,x+(a+c+a-c)\,y \end{array}$$

This is equal to $2\,bx + 2\,ay$.

11. $7\,a - \{3\,a - [4\,a - (5\,a - 2\,a)]\}$
$= 7\,a - \{3\,a - [4\,a - 3\,a]\}$
$= 7\,a - \{3\,a - a\}$
$= 7\,a - 2\,a$
$= 5\,a.$

12. $3\,a - \{a + b - [a + b + c - (a + b + c + d)]\}$
$= 3\,a - \{a + b - [a + b + c - a - b - c - d]\}$
$= 3\,a - \{a + b - [-d]\}$
$= 3\,a - \{a + b + d\}$
$= 3\,a - a - b - d$
$= 2\,a - b - d.$

13. $(b + a^2)\,x^3 + (1 - c^2 - c - 1)x^2 - (a + b + c)\,x$
or $(b + a^2)\,x^3 - (c^2 + c)\,x^2 - (a + b + c)\,x.$

14. $a^2 - (b^2 - c^2) - [b^2 - (c^2 - a^2)] + [c^2 - (b^2 - a^2)]$
$= a^2 - b^2 + c^2 - [b^2 - c^2 + a^2] + [c^2 - b^2 + a^2]$
$= a^2 - b^2 + c^2 - b^2 + c^2 - a^2 + c^2 - b^2 + a^2$
$= a^2 - 3\,b^2 + 3\,c^2.$

15. $1 - 6 - \{15 - 7 - [3 - (15 - 5 - 56)] - 6\}$
$= -5 - \{8 - [3 - (-46)] - 6\}$
$= -5 - \{8 - [3 + 46] - 6\}$
$= -5 - \{8 - 3 - 46 - 6\}$
$= -5 - \{-47\}$
$= -5 + 47$
$= 42.$

16.
$$\begin{array}{r} 2\,d + 11\,a + 10\,b - 5\,c \\ 5\,a - 3\,b + 2\,c \\ \hline 2\,d + 6\,a + 13\,b - 7\,c \end{array}$$

$$\begin{aligned} 2d+11a+10b-5c &= 14+11+30-25 \\ &= 30, \\ 5a-3b+2c &= 5-9+10 \\ &= 6, \\ 2d+6a+13b-7c &= 14+6+39-35 \\ &= 24 \\ &= 30-6. \end{aligned}$$

17. $$\begin{aligned} a^2+2b^2+3c^2+4d^2 &= 1+18+75+0 \\ &= 94. \end{aligned}$$

In examples 18–20, $a=3,\ b=4,\ c=9,\ s=8.$

18. $$\begin{aligned} s(s-a)(s-b)(s-c) &= 8(8-3)(8-4)(8-9) \\ &= 8\cdot 5\cdot 4\cdot -1 \\ &= -160. \end{aligned}$$

19. $$\begin{aligned} s^2+(s-a)^2+(s-b)^2+(s-c)^2 &= 8^2+(8-3)^2+(8-4)^2+(8-9)^2 \\ &= 64+25+16+1 \\ &= 106. \end{aligned}$$

20. $$\begin{aligned} & s^2-(s-a)(s-b)-(s-b)(s-c)-(s-c)(s-a) \\ &= 8^2-(8-3)(8-4)-(8-4)(8-9)-(8-9)(8-3) \\ &= 64-20+4+5 \\ &= 53. \end{aligned}$$

21. $$\begin{array}{rrrr} x= & a & +2b & -3c \\ y= & -3a & +\ b & +2c \\ z= & 2a & -3b & +\ c \\ \hline \therefore x+y+z=0 & & & \end{array}$$

22. $$\begin{array}{rrrr} x= & a & -2b & +3c \\ y= & 3a & +\ b & -2c \\ z= & -2a & +3b & +\ c \\ \hline \therefore x+y+z= & 2a & +2b & +2c \end{array}$$

23. $$\begin{array}{rrr} & 2y^2 & -\ z^2 \\ x^2 & +5y^2 & +3z^2 \\ \hline -x^2 & -3y^2 & -4z^2 \end{array}$$

24. $$\begin{array}{rrrr} a^3 & -2a^2b & -2ab^2 & +b^3 \\ 5a^3 & -7a^2b & +3ab^2 & \\ \hline -4a^3 & +5a^2b & -5ab^2 & +b^3 \end{array}$$

25. $E-[F-(G-H)]=E-F+G-H.$

$$\begin{array}{rrrr} E= & 5a^3 & +\ 3a^2b & -2b^3 \\ -F= & -3a^3 & +\ 7a^2b & +\ b^3 \\ G= & -\ a^3 & +\ 2a^2b & -\ b^3 \\ -H= & +2a^3 & -\ a^2b & +3b^3 \\ \hline \therefore E-F+G-H= & 3a^3 & +11a^2b & +\ b^3 \end{array}$$

Exercise 9. Page 41.

1. $35bc$.
2. $24xy$.
3. $18a^5$.
4. $6a^6$.
5. $6m^3n^2$.
6. $21a^2bc$.
7. $-14a^4x^2y$.
8. $9a^3b^3$.
9. $20m^3n^3p^4$.
10. $-48a^3b^3$.
11. $6x^9y^5$.
12. $-6a^5b^{10}x^7y^8$.
13. $10a^3 + 6a^2b$.
14. $5a^4b^2c - 5a^3b^2c^2$.
15. $a^2b^2c - a^2bc^2 - ab^2c^2$.
16. $6a^7b^3c - 7a^4b^4c^2$.
17. $a^5bc^2 + a^3b^3c^2 - a^3bc^4$.
18. $20a^3b^3c^2 - 12ab^5c^2 + 8ab^3c^4$.
19. $-2a^2b^3c^2 + 6a^4b^3c^3$.
20. $x^3y^2z^3 - x^4y^4z^2$.
21. $2m^4n^2p^4 + m^3n^2p^3$.
22. $-3x^6y^7z^9 + 3x^5y^8z^9 + 3x^5y^7z^{10}$.
23. $-3x^4 - 6x^2y^2 + 3x^2z$.
24. $15x^3 - 10x^2y - 20x^2$.

Exercise 10. Page 45.

1.
$$\begin{array}{l} x + 10 \\ \underline{x + 6} \\ x^2 + 10x \\ \underline{ + 6x + 60} \\ x^2 + 16x + 60 \end{array}$$

2.
$$\begin{array}{l} x - 2 \\ \underline{x - 3} \\ x^2 - 2x \\ \underline{ - 3x + 6} \\ x^2 - 5x + 6 \end{array}$$

3.
$$\begin{array}{l} x - 3 \\ \underline{x + 5} \\ x^2 - 3x \\ \underline{ + 5x - 15} \\ x^2 + 2x - 15 \end{array}$$

4.
$$\begin{array}{l} x + 3 \\ \underline{x - 3} \\ x^2 + 3x \\ \underline{ - 3x - 9} \\ x^2 \qquad\quad - 9 \end{array}$$

5.
$$\begin{array}{l} x - 11 \\ \underline{x - 1} \\ x^2 - 11x \\ \underline{ - x + 11} \\ x^2 - 12x + 11 \end{array}$$

6.
$$\begin{array}{l} -x + 2 \\ \underline{-x - 3} \\ x^2 - 2x \\ \underline{ + 3x - 6} \\ x^2 + x - 6 \end{array}$$

7.
$$\begin{array}{rrr} -x & -2 & \\ x & -2 & \\ \hline -x^2 & -2x & \\ & +2x & +4 \\ \hline -x^2 & & +4 \end{array}$$

8.
$$\begin{array}{rrr} -x & +4 & \\ x & -4 & \\ \hline -x^2 & +4x & \\ & +4x & -16 \\ \hline -x^2 & +8x & -16 \end{array}$$

9.
$$\begin{array}{rrr} -x & +7 & \\ x & +7 & \\ \hline -x^2 & +7x & \\ & -7x & +49 \\ \hline -x^2 & & +49 \end{array}$$

10.
$$\begin{array}{rrr} x & -7 & \\ x & +7 & \\ \hline x^2 & -7x & \\ & +7x & -49 \\ \hline x^2 & & -49 \end{array}$$

11.
$$\begin{array}{rrr} x & -3 & \\ 2x & +3 & \\ \hline 2x^2 & -6x & \\ & +3x & -9 \\ \hline 2x^2 & -3x & -9 \end{array}$$

12.
$$\begin{array}{rrr} 2x & -3 & \\ x & +3 & \\ \hline 2x^2 & -3x & \\ & +6x & -9 \\ \hline 2x^2 & +3x & -9 \end{array}$$

13.
$$\begin{array}{rrr} x & -7 & \\ 2x & -1 & \\ \hline 2x^2 & -14x & \\ & -x & +7 \\ \hline 2x^2 & -15x & +7 \end{array}$$

14.
$$\begin{array}{rrrr} m & -n & & \\ 2m & +1 & & \\ \hline 2m^2 & -2mn & & \\ & & +m & -n \\ \hline 2m^2 & -2mn & +m & -n \end{array}$$

15.
$$\begin{array}{rrr} m & -a & \\ m & +a & \\ \hline m^2 & -am & \\ & +am & -a^2 \\ \hline m^2 & & -a^2 \end{array}$$

16.
$$\begin{array}{rrr} 3x & +7 & \\ 2x & -3 & \\ \hline 6x^2 & +14x & \\ & -9x & -21 \\ \hline 6x^2 & +5x & -21 \end{array}$$

17.
$$\begin{array}{rrr} 5x & -2y & \\ 5x & +2y & \\ \hline 25x^2 & -10xy & \\ & +10xy & -4y^2 \\ \hline 25x^2 & & -4y^2 \end{array}$$

18.
$$\begin{array}{rrr} 3x & -4y & \\ 2x & +3y & \\ \hline 6x^2 & -8xy & \\ & +9xy & -12y^2 \\ \hline 6x^2 & +xy & -12y^2 \end{array}$$

19.

$$\begin{array}{llll}
x^2 & +y^2 & & \\
x^3 & -y^3 & & \\
\hline
x^5 & +x^3y^2 & & \\
 & & -x^2y^3 & -y^5 \\
\hline
x^5 & +x^3y^2 & -x^2y^3 & -y^5
\end{array}$$

20.

$$\begin{array}{lll}
2x^2 & +3y^2 & \\
x^2 & +y^2 & \\
\hline
2x^4 & +3x^2y^2 & \\
 & +2x^2y^2 & +3y^4 \\
\hline
2x^4 & +5x^2y^2 & +3y^4
\end{array}$$

21.

$$\begin{array}{llllll}
x & +y & +z & & & \\
x & -y & +z & & & \\
\hline
x^2 & +xy & +xz & & & \\
 & -xy & & -y^2 & -yz & \\
 & & +xz & & +yz & +z^2 \\
\hline
x^2 & & +2xz & -y^2 & & +z^2
\end{array}$$

22.

$$\begin{array}{llllll}
x & +2y & -z & & & \\
x & -y & +2z & & & \\
\hline
x^2 & +2xy & -xz & & & \\
 & -xy & & -2y^2 & +yz & \\
 & & +2xz & & +4yz & -2z^2 \\
\hline
x^2 & +xy & +xz & -2y^2 & +5yz & -2z^2
\end{array}$$

23.

$$\begin{array}{lllll}
x^2 & -xy & +y^2 & & \\
x^2 & +xy & +y^2 & & \\
\hline
x^4 & -x^3y & +x^2y^2 & & \\
 & +x^3y & -x^2y^2 & +xy^3 & \\
 & & +x^2y^2 & -xy^3 & +y^4 \\
\hline
x^4 & & +x^2y^2 & & +y^4
\end{array}$$

24.

$$\begin{array}{llll}
m^2 & -mn & +n^2 & \\
m & +n & & \\
\hline
m^3 & -m^2n & +mn^2 & \\
 & +m^2n & -mn^2 & +n^2 \\
\hline
m^3 & & & +n^3
\end{array}$$

25.

$$\begin{array}{llll}
m^2 & +mn & +n^2 & \\
m & -n & & \\
\hline
m^3 & +m^2n & +mn^2 & \\
 & -m^2n & -mn^2 & -n^3 \\
\hline
m^3 & & & -n^3
\end{array}$$

26.

$$\begin{array}{lllll}
a^2 & -3ab & +b^2 & & \\
a^2 & -3ab & -b^2 & & \\
\hline
a^4 & -3a^3b & +a^2b^2 & & \\
 & -3a^3b & +9a^2b^2 & -3ab^3 & \\
 & & -a^2b^2 & +3ab^3 & -b^4 \\
\hline
a^4 & -6a^3b & +9a^2b^2 & & -b^4
\end{array}$$

27.

$$\begin{array}{lllll}
a^2 & -7a & +2 & & \\
a^2 & -2a & +3 & & \\
\hline
a^4 & -7a^3 & +2a^2 & & \\
 & -2a^3 & +14a^2 & -4a & \\
 & & +3a^2 & -21a & +6 \\
\hline
a^4 & -9a^3 & +19a^2 & -25a & +6
\end{array}$$

28.
$$\begin{array}{l}
2x^2-3xy+4y^2\\
3x^2+4xy-5y^2\\ \hline
6x^4-9x^3y+12x^2y^2\\
\qquad +8x^3y-12x^2y^2+16xy^3\\
\qquad\qquad -10x^2y^2+15xy^3-20y^4\\ \hline
6x^4-x^3y-10x^2y^2+31xy^3-20y^4
\end{array}$$

29.
$$\begin{array}{l}
x^2+xy+y^2\\
x^2-xz-z^2\\ \hline
x^4+x^3y+x^2y^2\\
\qquad -x^3z-x^2yz-xy^2z\\
\qquad\qquad -x^2z^2-xyz^2-y^2z^2\\ \hline
x^4+x^3y+x^2y^2-x^3z-x^2yz-xy^2z-x^2z^2-xyz^2-y^2z^2
\end{array}$$

30.
$$\begin{array}{l}
x^2+y^2+z^2-xy-xz-yz\\
x+y+z\\ \hline
x^3+xy^2+xz^2-x^2y-x^2z-xyz\\
\quad -xy^2+x^2y+y^3+yz^2-xyz-y^2z\\
\qquad -xz^2+x^2z-yz^2-xyz+y^2z+z^3\\ \hline
x^3-3xyz+y^3+z^3
\end{array}$$

31.
$$\begin{array}{l}
4a^2-10ab+25a^2b^2\\
5b+2a\\ \hline
20a^2b-50ab^2+125a^2b^3\\
-20a^2b+8a^3+50a^3b^2\\ \hline
-50ab^2+125a^2b^3+8a^3+50a^3b^2
\end{array}$$

32.
$$\begin{array}{l}
x^2+4y\\
y^2+4x\\ \hline
x^2y^2+4y^3\\
\qquad +4x^3+16xy\\ \hline
x^2y^2+4y^3+4x^3+16xy
\end{array}$$

33.
$$\begin{array}{l}
x^2+2xy+8\\
y^2+2xy-8\\ \hline
x^2y^2+2xy^3+8y^2\\
4x^2y^2+2x^3y+16xy\\
\qquad -16xy-8x^2-64\\ \hline
5x^2y^2+2xy^3+2x^3y+8y^2-8x^2-64
\end{array}$$

34.
$$\begin{array}{l}
a^2 + b^2 + 1 - ab - a - b \\
a + 1 + b \\
\hline
a^3 + ab^2 + a - a^2b - a^2 - ab \\
\quad - a \qquad + a^2 - ab + b^2 + 1 - b \\
\quad - ab^2 \qquad + a^2b \qquad - ab - b^2 \qquad + b + b^3 \\
\hline
a^3 \qquad\qquad - 3ab \qquad + 1 \qquad + b^3
\end{array}$$

35.
$$\begin{array}{l}
3x^2 - 2y^2 + 5z^2 \\
8x^2 + 2y^2 - 3z^2 \\
\hline
24x^4 - 16x^2y^2 + 40x^2z^2 \\
\quad + 6x^2y^2 \qquad - 4y^4 + 10y^2z^2 \\
\qquad - 9x^2z^2 \qquad + 6y^2z^2 - 15z^4 \\
\hline
24x^4 - 10x^2y^2 + 31x^2z^2 - 4y^4 + 16y^2z^2 - 15z^4
\end{array}$$

36.
$$\begin{array}{l}
x^2 + y^2 + 2xy - 2x - 2y - 1 \\
x + y - 1 \\
\hline
x^3 + xy^2 + 2x^2y - 2x^2 - 2xy - x \\
\quad + 2xy^2 + x^2y \qquad - 2xy \qquad + y^3 - 2y^2 - y \\
\qquad - x^2 - 2xy + 2x \qquad - y^2 + 2y + 1 \\
\hline
x^3 + 3xy^2 + 3x^2y + y^3 - 3x^2 - 6xy - 3y^2 + x + y + 1
\end{array}$$

37.
$$\begin{array}{l}
a^m + 2a^{m-1} - 3a^{m-2} - 1 \\
a + 1 \\
\hline
a^{m+1} + 2a^m - 3a^{m-1} - a \\
\quad + a^m + 2a^{m-1} - 3a^{m-2} - 1 \\
\hline
a^{m+1} + 3a^m - a^{m-1} - 3a^{m-2} - a - 1
\end{array}$$

38.
$$\begin{array}{l}
a^n - 4a^{n-1} + 5a^{n-2} + a^{n-3} \\
a - 1 \\
\hline
a^{n+1} - 4a^n + 5a^{n-1} + a^{n-2} \\
\quad - a^n + 4a^{n-1} - 5a^{n-2} - a^{n-3} \\
\hline
a^{n+1} - 5a^n + 9a^{n-1} - 4a^{n-2} - a^{n-3}
\end{array}$$

39.
$$\begin{array}{l}
a^{4n+1} - 4a^{3n} + 2a^{2n-1} - a^{n-2} \\
2a^3 - a^2 + a \\
\hline
2a^{4n+4} - 8a^{3n+3} + 4a^{2n+2} - 2a^{n+1} \\
\qquad\qquad - a^{4n+3} + 4a^{3n+2} - 2a^{2n+1} + a^n \\
\qquad\qquad\qquad + a^{4n+2} - 4a^{3n+1} + 2a^{2n} - a^{n-1} \\
\hline
2a^{4n+4} - 8a^{3n+3} + 4a^{2n+2} - 2a^{n+1} - a^{4n+3} + 4a^{3n+2} - 2a^{2n+1} + a^n + a^{4n+2} \\
\qquad\qquad [- 4a^{3n+1} + 2a^{2n} - a^{n-1}
\end{array}$$

40.
$$\begin{array}{llll}
x^n & - y^{n+1} & & \\
x^n & + y^{n+1} & & \\
\hline
x^{2n} & - x^n y^{n+1} & & \\
 & + x^n y^{n+1} & - y^{2n+2} & \\
\hline
x^{2n} & & - y^{2n+2} &
\end{array}$$

Exercise 11. Page 46.

1.
$$\begin{array}{llllll}
a & + b & + c & & & \\
a & + b & - c & & & \\
\hline
a^2 & + ab & + ac & & & \\
 & + ab & & + b^2 & + bc & \\
 & & - ac & & - bc & - c^2 \\
\hline
a^2 & + 2ab & & + b^2 & & - c^2
\end{array}$$

$$\begin{aligned}
\therefore\ & (a + b + c)(a + b - c) - (2ab - c^2) \\
& = a^2 + 2ab + b^2 - c^2 - 2ab + c^2 \\
& = a^2 + b^2.
\end{aligned}$$

2.
$$\begin{array}{lll}
m & - n & \\
m & - n & \\
\hline
m^2 & - mn & \\
 & - mn & + n^2 \\
\hline
m^2 & - 2mn & + n^2
\end{array}$$

$(m + n)m = m^2 + mn;$

$n(n - m) = n^2 - mn.$

$$\begin{aligned}
\therefore\ & (m + n)m - [(m - n)^2 - n(n - m)] \\
& = m^2 + mn - [m^2 - 2mn + n^2 - n^2 + mn] \\
& = m^2 + mn - [m^2 - mn] \\
& = m^2 + mn - m^2 + mn \\
& = 2mn.
\end{aligned}$$

3.
$$\begin{array}{llll}
a - b & & & \\
b + c & & & \\
\hline
ab & - b^2 & & \\
 & & + ac & - bc \\
\hline
ab & - b^2 & + ac & - bc
\end{array}$$

$$\begin{aligned}
\therefore\ & [ac - (a - b)(b + c)] - b[b - (a - c)] \\
& = [ac - (ab - b^2 + ac - bc)] - b[b - (a - c)] \\
& = ac - ab + b^2 - ac + bc - b[b - a + c] \\
& = ac - ab + b^2 - ac + bc - b^2 + ab - bc \\
& = 0.
\end{aligned}$$

4.
$$\begin{array}{l}x-1\\ x-2\\ \hline x^2-x\\ \quad-2x+2\\ \hline x^2-3x+2\end{array}\qquad\qquad\begin{array}{l}x+2\\ x+1\\ \hline x^2+2x\\ \quad+x+2\\ \hline x^2+3x+2\end{array}$$

$$\begin{aligned}\therefore\ &(x-1)(x-2)-3x(x+3)+2[(x+2)(x+1)-3]\\ &=x^2-3x+2-3x^2-9x+2[x^2+3x+2-3]\\ &=x^2-3x+2-3x^2-9x+2x^2+6x-2\\ &=-6x.\end{aligned}$$

5.
$$\begin{array}{l}a-3b\\ a+3b\\ \hline a^2-3ab\\ \quad+3ab-9b^2\\ \hline a^2\qquad\quad-9b^2\end{array}\qquad\qquad\begin{array}{l}a-6b\\ a-6b\\ \hline a^2-6ab\\ \quad-6ab+36b^2\\ \hline a^2-12ab+36b^2\end{array}$$

$$\begin{aligned}\therefore\ &4(a-3b)(a+3b)-2(a-6b)^2-2(a^2+6b^2)\\ &=4(a^2-9b^2)-2(a^2-12ab+36b^2)-2(a^2+6b^2)\\ &=4a^2-36b^2-2a^2+24ab-72b^2-2a^2-12b^2\\ &=24ab-120b^2\end{aligned}$$

6.
$$\begin{array}{l}x+y+z\\ x+y+z\\ \hline x^2+xy+xz\\ \quad+xy\qquad\qquad+y^2+yz\\ \qquad\qquad+xz\qquad\quad+yz+z^2\\ \hline x^2+2xy+2xz+y^2+2yz+z^2\end{array}\qquad\begin{aligned}x(y+z-x)&=xy+xz-x^2\\ y(x+z-y)&=xy+yz-y^2\\ z(x+y-z)&=xz+yz-z^2\end{aligned}$$

$$\begin{aligned}\therefore\ &(x+y+z)^2-x(y+z-x)-y(x+z-y)-z(x+y-z)\\ &=x^2+2xy+2xz+y^2+2yz+z^2-xy-xz+x^2-xy-yz+y^2\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad-xz-yz+z^2\\ &=2x^2+2y^2+2z^2.\end{aligned}$$

7.
$$\begin{aligned}&5[(a-b)x-cy]-2[a(x-y)-bx]-[3ax-(5c-2a)y]\\ &=5[ax-bx-cy]-2[ax-ay-bx]-[3ax-5cy+2ay]\\ &=5ax-5bx-5cy-2ax+2ay+2bx-3ax+5cy-2ay\\ &=-3bx.\end{aligned}$$

Exercise 12. Page 47.

1.
$$\begin{array}{rrrrr}
x^2 & -x & -19 & & \\
x^2 & +2x & -3 & & \\ \hline
x^4 & -x^3 & -19x^2 & & \\
 & +2x^3 & -2x^2 & -38x & \\
 & & -3x^2 & +3x & +57 \\ \hline
x^4 & +x^3 & -24x^2 & -35x & +57
\end{array}$$

2.
$$\begin{array}{rrrrrr}
x^2 & +2x & +1 & & & \\
-x^3 & +2x^2 & -3x & +1 & & \\ \hline
-x^5 & -2x^4 & -x^3 & & & \\
 & +2x^4 & +4x^3 & +2x^2 & & \\
 & & -3x^3 & -6x^2 & -3x & \\
 & & & +x^2 & +2x & +1 \\ \hline
-x^5 & & & -3x^2 & -x & +1
\end{array}$$

3.
$$\begin{array}{rrrrr}
2x^2 & +3x & +2 & & \\
2x^3 & -3x^2 & +2x & & \\ \hline
4x^5 & +6x^4 & +4x^3 & & \\
 & -6x^4 & -9x^3 & -6x^2 & \\
 & & +4x^3 & +6x^2 & +4x \\ \hline
4x^5 & & -x^3 & & +4x
\end{array}$$

4.
$$\begin{array}{rrrrr}
3x^2 & -4x & +5 & & \\
6x^2 & -7x & +8 & & \\ \hline
18x^4 & -24x^3 & +30x^2 & & \\
 & -21x^3 & +28x^2 & -35x & \\
 & & +24x^2 & -32x & +40 \\ \hline
18x^4 & -45x^3 & +82x^2 & -67x & +40
\end{array}$$

5.
$$\begin{array}{rrrrrrr}
x^3 & +x & -y & & & & \\
x^2 & +xy & -y^2 & & & & \\ \hline
x^5 & +x^3 & -x^2y & & & & \\
 & & +x^2y & +x^4y & -xy^2 & & \\
 & & & & -xy^2 & -x^3y^2 & +y^3 \\ \hline
x^5 & +x^3 & & +x^4y & -2xy^2 & -x^3y^2 & +y^3
\end{array}$$

6.
$$\begin{array}{rrrrrr}
 & 7x^2 & -5x & +3 & & \\
-10x^3 & +8x^2 & -6x & +4 & & \\ \hline
-70x^5 & +50x^4 & -30x^3 & & & \\
 & +56x^4 & -40x^3 & +24x^2 & & \\
 & & -42x^3 & +30x^2 & -18x & \\
 & & & +28x^2 & -20x & +12 \\ \hline
-70x^5 & +106x^4 & -112x^3 & +82x^2 & -38x & +12
\end{array}$$

7.
$$\begin{array}{rrrrr}
-4a^2b & +6ab^2 & +b^3 & & \\
-8a^3 & +2a^2b & -ab^2 & & \\ \hline
32a^5b & -48a^4b^2 & -8a^3b^3 & & \\
 & -8a^4b^2 & +12a^3b^3 & +2a^2b^4 & \\
 & & +4a^3b^3 & -6a^2b^4 & -ab^5 \\ \hline
32a^5b & -56a^4b^2 & +8a^3b^3 & -4a^2b^4 & -ab^5
\end{array}$$

8. $$\begin{array}{l} x^2 + ax \quad - b \\ x^2 + 5x \quad - 4 \\ \hline x^4 + ax^3 - bx^2 \\ \qquad + 5x^3 + 5ax^2 - 5bx \\ \qquad\qquad - 4x^2 - 4ax + 4b \\ \hline x^4 + (a+5)x^3 + (5a - b - 4)x^2 - (4a + 5b)x + 4b \end{array}$$

9. $$\begin{array}{l} x^3 - mx^2 + nx + r \\ x^2 + cx \quad + d \\ \hline x^5 - mx^4 + nx^3 \quad + rx^2 \\ \quad + cx^4 - cmx^3 + cnx^2 \quad + crx \\ \qquad\qquad + dx^3 - dmx^2 + dnx + dr \\ \hline x^5 + (c - m)x^4 + (d - cm + n)x^3 + (cn - dm + r)x^2 + (cr + dn)x + dr \end{array}$$

10. $$\begin{array}{l} x^2 - (a+b)x \quad + ab \\ x \quad - c \\ \hline x^3 - (a+b)x^2 + abx \\ \quad - \qquad cx^2 + (a+b)cx - abc \\ \hline x^3 - (a + b + c)x^2 + (ab + ac + bc)x - abc \end{array}$$

11. $$\begin{array}{l} \quad x^3 + x^2y + xy^2 + y^3 \\ -x + y \\ \hline -x^4 - x^3y - x^2y^2 - xy^3 \\ \qquad + x^3y + x^2y^2 + xy^3 + y^4 \\ \hline -x^4 \qquad\qquad\qquad\qquad + y^4 \end{array}$$

12. $$\begin{array}{l} 4x^2 - 6xy + 9y^2 \\ 4x^2 + 6xy + 9y^2 \\ \hline 16x^4 - 24x^3y + 36x^2y^2 \\ \qquad + 24x^3y - 36x^2y^2 + 54xy^3 \\ \qquad\qquad + 36x^2y^2 - 54xy^3 + 81y^4 \\ \hline 16x^4 \qquad\qquad + 36x^2y^2 \qquad\qquad + 81y^4 \end{array}$$

13. $$\begin{array}{l} x^4 - 3x^2 + 5 \\ x^2 + 4 \\ \hline x^6 - 3x^4 + 5x^2 \\ \quad + 4x^4 - 12x^2 + 20 \\ \hline x^6 + x^4 - 7x^2 + 20 \end{array}$$

14. $$\begin{array}{l} x^4 - x^2y^2 + y^4 \\ x^4 + x^2y^2 + y^4 \\ \hline x^8 - x^6y^2 + x^4y^4 \\ \quad + x^6y^2 - x^4y^4 + x^2y^6 \\ \qquad\qquad + x^4y^4 - x^2y^6 + y^8 \\ \hline x^8 \qquad\qquad + x^4y^4 \qquad\qquad + y^8 \end{array}$$

15.
$$\begin{array}{l}
2x^5-3x^3+4x^2-5\\
x^2-8\\
\hline
2x^7-3x^5+4x^4\qquad\qquad-5x^2\\
\qquad-16x^5\qquad\quad+24x^3-32x^2+40\\
\hline
2x^7-19x^5+4x^4+24x^3-37x^2+40
\end{array}$$

16.
$$\begin{array}{l}
a^{2m}-a^my^m+y^{2m}\\
a^m+y^m\\
\hline
a^{3m}-a^{2m}y^m+a^my^{2m}\\
\qquad+a^{2m}y^m-a^my^{2m}+y^{3m}\\
\hline
a^{3m}\qquad\qquad\qquad\qquad+y^{3m}
\end{array}$$

17.
$$\begin{array}{l}
a^{3x}-a^{2x}+a^x-1\\
a^x+1\\
\hline
a^{4x}-a^{3x}+a^{2x}-a^x\\
\qquad+a^{3x}-a^{2x}+a^x-1\\
\hline
a^{4x}\qquad\qquad\qquad\qquad-1
\end{array}$$

18.
$$\begin{array}{l}
a^2+2ab-ac+b^2-bc+c^2\\
a+b+c\\
\hline
a^3+2a^2b-a^2c+ab^2-abc+ac^2\\
\quad+a^2b\qquad+2ab^2-abc\qquad+b^3-b^2c+bc^2\\
\qquad\qquad+a^2c\qquad+2abc-ac^2\qquad+b^2c-bc^2+c^3\\
\hline
a^3+3a^2b\qquad+3ab^2\qquad\qquad+b^3\qquad\qquad+c^3
\end{array}$$

19.
$$\begin{array}{l}
\quad a-2b\\
-2a+b\\
\hline
-2a^2+4ab\\
\qquad+ab-2b^2\\
\hline
-2a^2+5ab-2b^2
\end{array}
\qquad\qquad
\begin{array}{l}
\quad a-3b\\
-a+4b\\
\hline
-a^2+3ab\\
\qquad+4ab-12b^2\\
\hline
-a^2+7ab-12b^2
\end{array}$$

$$\begin{aligned}
\therefore\ (a-2b)(b-2a)-(a-3b)(4b-a)+2ab\\
&=-2a^2+5ab-2b^2+a^2-7ab+12b^2+2ab\\
&=-a^2+10b^2.
\end{aligned}$$

20.
$$\begin{aligned}
(a-b)(a-c)+c(3a-b-c)+2ac-(a-c)+2b\\
&=(-1)(+1)-1(-1+1)+0-(+1)+2\\
&=-1-1+2=0.
\end{aligned}$$

21.
$$\begin{array}{l}
2x+y\\
2x+y\\
\hline
4x^2+2xy\\
\qquad+2xy+y^2\\
\hline
4x^2+4xy+y^2
\end{array}
\qquad\qquad
\begin{array}{l}
x-2y\\
x-2y\\
\hline
x^2-2xy\\
\qquad-2xy+4y^2\\
\hline
x^2-4xy+4y^2
\end{array}$$

$$\begin{array}{l}3x-2y\\3x-2y\\\hline 9x^2-6xy\\\qquad -6xy+4y^2\\\hline 9x^2-12xy+4y^2\end{array}\qquad\qquad\begin{array}{l}2x-3y\\2x-3y\\\hline 4x^2-6xy\\\qquad -6xy+9y^2\\\hline 4x^2-12xy+9y^2\end{array}$$

$$\begin{aligned}(2x+y)^2+(x-2y)^2&=4x^2+4xy+y^2+x^2-4xy+4y^2\\&=5x^2+5y^2\\(3x-2y)^2-(2x-3y)^2&=9x^2-12xy+4y^2-4x^2+12xy-9y^2\\&=5x^2-5y^2\end{aligned}$$

$$\begin{array}{l}5x^2+5y^2\\5x^2-5y^2\\\hline 25x^4+25x^2y^2\\\qquad -25x^2y^2-25y^4\\\hline 25x^4\qquad\qquad\quad -25y^4\end{array}$$

22.

$$\begin{array}{l}a-b\\a-c\\\hline a^2-ab\\\qquad -ac+bc\\\hline a^2-ab-ac+bc\end{array}\qquad\qquad\begin{array}{l}a^2-ab-ac+bc\\b-c\\\hline a^2b\qquad -ab^2-abc+b^2c\\\quad -a^2c\qquad +abc\qquad +ac^2-bc^2\\\hline a^2b-a^2c-ab^2\qquad +b^2c+ac^2-bc^2\end{array}$$

$$\begin{aligned}\therefore\ &a^2(b-c)-b^2(a-c)+c^2(a-b)-(a-b)(a-c)(b-c)\\&=a^2b-a^2c-ab^2+b^2c+ac^2-bc^2-a^2b+a^2c+ab^2-b^2c-ac^2+bc^2\\&=0.\end{aligned}$$

23.

$$\begin{array}{l}2a-b\\2a-b\\\hline 4a^2-2ab\\\qquad -2ab+b^2\\\hline 4a^2-4ab+b^2\end{array}\qquad\qquad\begin{array}{l}a-b\\a-b\\\hline a^2-ab\\\qquad -ab+b^2\\\hline a^2-2ab+b^2\end{array}\qquad\qquad\begin{array}{l}a+b\\a-b\\\hline a^2+ab\\\qquad -ab-b^2\\\hline a^2\qquad\quad -b^2\end{array}$$

$$\begin{aligned}\therefore\ &(2a-b)^2+2b(a+b)-3a^2-(a-b)^2+(a+b)(a-b)\\&=4a^2-4ab+b^2+2ab+2b^2-3a^2-a^2+2ab-b^2+a^2-b^2\\&=a^2+b^2.\end{aligned}$$

Exercise 13. Page 50.

1. $3a^2$.
2. $-7x^2$.
3. $-7z^5$.
4. $2z$.
5. $-5x^6$.
6. 3.
7. $-4a^3b$.
8. $5bz^2$.
9. $6m^2n$.
10. $-7p$.
11. $4rs^2$.
12. $7m^2n$.
13. x.
14. $-2a^2bx^3$.
15. $-4b$.
16. $-2b^3x^4$.
17. $32x^6y^6z^6$.
18. 5.
19. $6b^2x^6y^2$.
20. $5a^3x$.

21. $\dfrac{x^2+2xy}{x}=\dfrac{x^2}{x}+\dfrac{2xy}{x}=x+2y.$

22. $\dfrac{a^2-2ab}{a}=\dfrac{a^2}{a}-\dfrac{2ab}{a}=a-2b.$

23. $\dfrac{4x^6-8x^3}{2x^3}=\dfrac{4x^6}{2x^3}-\dfrac{8x^3}{2x^3}=2x^3-4.$

24. $\dfrac{-6x^3-2x}{-2x}=\dfrac{-6x^3}{-2x}-\dfrac{2x}{-2x}=3x^2+1.$

25. $\dfrac{-8a^5-16a^{10}}{-8a^3}=\dfrac{-8a^5}{-8a^3}-\dfrac{16a^{10}}{-8a^3}=a^2+2a^7.$

26. $\dfrac{27a^3-36a^2}{9a^2}=\dfrac{27a^3}{9a^2}-\dfrac{36a^2}{9a^2}=3a-4.$

27. $\dfrac{-30a^7+20a^9}{-10a^6}=\dfrac{-30a^7}{-10a^6}+\dfrac{20a^9}{-10a^6}=3a-2a^3.$

28. $\dfrac{-12x^6y^4-4x^2y^2}{-4x^2y^2}=\dfrac{-12x^6y^4}{-4x^2y^2}-\dfrac{4x^2y^2}{-4x^2y^2}=3x^4y^2+1.$

29. $\dfrac{-3x^7z^7-6x^5z^5}{-3x^3z^3}=\dfrac{-3x^7z^7}{-3x^3z^3}-\dfrac{6x^5z^5}{-3x^3z^3}=x^4z^4+2x^2z^2.$

30. $\dfrac{3a^3b^3c^7-9a^6b^7c^5}{3a^3b^3c^5}=\dfrac{3a^3b^3c^7}{3a^3b^3c^5}-\dfrac{9a^6b^7c^5}{3a^3b^3c^5}=c^2-3a^3b^4.$

31. $\dfrac{x^2-xy-xz}{-x}=\dfrac{x^2}{-x}-\dfrac{xy}{-x}-\dfrac{xz}{-x}=-x+y+z.$

32. $\dfrac{3a^3-6a^2b-9ab^2}{-3a}=\dfrac{3a^3}{-3a}-\dfrac{6a^2b}{-3a}-\dfrac{9ab^2}{-3a}=-a^2+2ab+3b^2.$

33. $\dfrac{x^5y^2-x^3y^3-x^2y^2}{x^2y^2}=\dfrac{x^5y^2}{x^2y^2}-\dfrac{x^3y^3}{x^2y^2}-\dfrac{x^2y^2}{x^2y^2}=x^3-xy-1.$

34. $$\frac{a^3b^2c-a^2b^3c-a^2bc^3}{abc}=\frac{a^3b^2c}{abc}-\frac{a^2b^3c}{abc}-\frac{a^2bc^3}{abc}=a^2b-ab^2-ac^2.$$

35. $$\frac{8a^3-4a^2b-6ab^2}{-2a}=\frac{8a^3}{-2a}-\frac{4a^2b}{-2a}-\frac{6ab^2}{-2a}=-4a^2+2ab+3b^2.$$

36. $$\frac{5m^3n-10m^2n^2-15mn^3}{5mn}=\frac{5m^3n}{5mn}-\frac{10m^2n^2}{5mn}-\frac{15mn^3}{5mn}$$
$$=m^2-2mn-3n^2.$$

Exercise 14. Page 54.

1. $$\begin{array}{ll} a^2+7a+12 & \underline{a+4} \\ \underline{a^2+4a} & a+3 \\ 3a+12 & \\ \underline{3a+12} & \end{array}$$

2. $$\begin{array}{l|l} a^2-5a+6 & \underline{a-3} \\ \underline{a^2-3a} & a-2 \\ -2a+6 & \\ \underline{-2a+6} & \end{array}$$

3. $$\begin{array}{l|l} x^2+2xy+y^2 & \underline{x+y} \\ \underline{x^2+xy} & x+y \\ xy+y^2 & \\ \underline{xy+y^2} & \end{array}$$

4. $$\begin{array}{l|l} x^2-2xy+y^2 & \underline{x-y} \\ \underline{x^2-xy} & x-y \\ -xy+y^2 & \\ \underline{-xy+y^2} & \end{array}$$

5. $$\begin{array}{l|l} x^2-y^2 & \underline{x-y} \\ \underline{x^2-xy} & x+y \\ xy-y^2 & \\ \underline{xy-y^2} & \end{array}$$

6. $$\begin{array}{l|l} 4x^2+12x+9 & \underline{2x+3} \\ \underline{4x^2+6x} & 2x+3 \\ 6x+9 & \\ \underline{6x+9} & \end{array}$$

7. $$\begin{array}{l|l} 6x^2-11x+4 & \underline{3x-4} \\ \underline{6x^2-8x} & 2x-1 \\ -3x+4 & \\ \underline{-3x+4} & \end{array}$$

8. $$\begin{array}{l|l} 8x^2-10ax-3a^2 & \underline{4x+a} \\ \underline{8x^2+2ax} & 2x-3a \\ -12ax-3a^2 & \\ \underline{-12ax-3a^2} & \end{array}$$

9. $$\begin{array}{l|l} 3a^2-4a-4 & \underline{-a+2} \\ \underline{3a^2-6a} & -3a-2 \\ 2a-4 & \\ \underline{2a-4} & \end{array}$$

10. $$\begin{array}{l|l} a^3-8a\ -3 & \underline{-a\ +3} \\ \underline{a^3-3a^2} & -a^2-3a-1 \\ 3a^2-8a\ -3 & \\ \underline{3a^2-9a} & \\ a\ -3 & \end{array}$$

11.
$$\begin{array}{l|l} a^4 - 5a^3 + 11a^2 - 12a + 6 & a^2 - 3a + 3 \\ \cline{2-2} \underline{a^4 - 3a^3 + 3a^2 \qquad\qquad} & a^2 - 2a + 2 \\ \multicolumn{2}{l}{\quad -2a^3 + 8a^2 - 12a + 6} \\ \multicolumn{2}{l}{\quad \underline{-2a^3 + 6a^2 - 6a \qquad}} \\ \multicolumn{2}{l}{\qquad\qquad 2a^2 - 6a + 6} \\ \multicolumn{2}{l}{\qquad\qquad \underline{2a^2 - 6a + 6}} \end{array}$$

12.
$$\begin{array}{l|l} y^4 + y^3 - 9y^2 - 16y - 4 & y^2 + 4y + 4 \\ \cline{2-2} \underline{y^4 + 4y^3 + 4y^2 \qquad\qquad} & y^2 - 3y - 1 \\ \multicolumn{2}{l}{\quad -3y^3 - 13y^2 - 16y - 4} \\ \multicolumn{2}{l}{\quad \underline{-3y^3 - 12y^2 - 12y \qquad}} \\ \multicolumn{2}{l}{\qquad\qquad - y^2 - 4y - 4} \\ \multicolumn{2}{l}{\qquad\qquad \underline{- y^2 - 4y - 4}} \end{array}$$

13.
$$\begin{array}{l|l} m^4 - 13m^2 + 36 & m^2 + 5m + 6 \\ \cline{2-2} \underline{m^4 + 5m^3 + 6m^2} & m^2 - 5m + 6 \\ \multicolumn{2}{l}{\quad - 5m^3 - 19m^2 + 36} \\ \multicolumn{2}{l}{\quad \underline{- 5m^3 - 25m^2 - 30m \qquad}} \\ \multicolumn{2}{l}{\qquad\qquad 6m^2 + 30m + 36} \\ \multicolumn{2}{l}{\qquad\qquad \underline{6m^2 + 30m + 36}} \end{array}$$

14.
$$\begin{array}{l|l} -s^5 - 3s^2 - s + 1 & s^2 + 2s + 1 \\ \cline{2-2} \underline{-s^5 - 2s^4 - s^3} & -s^3 + 2s^2 - 3s + 1 \\ \multicolumn{2}{l}{\qquad 2s^4 + s^3 - 3s^2 - s + 1} \\ \multicolumn{2}{l}{\qquad \underline{2s^4 + 4s^3 + 2s^2 \qquad\quad}} \\ \multicolumn{2}{l}{\qquad\quad -3s^3 - 5s^2 - s + 1} \\ \multicolumn{2}{l}{\qquad\quad \underline{-3s^3 - 6s^2 - 3s \qquad}} \\ \multicolumn{2}{l}{\qquad\qquad\qquad s^2 + 2s + 1} \\ \multicolumn{2}{l}{\qquad\qquad\qquad \underline{s^2 + 2s + 1}} \end{array}$$

15.
$$\begin{array}{l|l}
b^6 - 2b^3 + 1 & b^2 - 2b + 1 \\ \cline{2-2}
\underline{b^6 - 2b^5 + b^4} & b^4 + 2b^3 + 3b^2 + 2b + 1 \\
\multicolumn{2}{l}{\qquad 2b^5 - b^4 - 2b^3 + 1} \\
\multicolumn{2}{l}{\qquad \underline{2b^5 - 4b^4 + 2b^3}} \\
\multicolumn{2}{l}{\qquad\qquad 3b^4 - 4b^3 + 1} \\
\multicolumn{2}{l}{\qquad\qquad \underline{3b^4 - 6b^3 + 3b^2}} \\
\multicolumn{2}{l}{\qquad\qquad\qquad 2b^3 - 3b^2 + 1} \\
\multicolumn{2}{l}{\qquad\qquad\qquad \underline{2b^3 - 4b^2 + 2b}} \\
\multicolumn{2}{l}{\qquad\qquad\qquad\qquad b^2 - 2b + 1} \\
\multicolumn{2}{l}{\qquad\qquad\qquad\qquad \underline{b^2 - 2b + 1}}
\end{array}$$

16.
$$\begin{array}{l|l}
x^4 + 2x^2y^2 + 9y^4 & x^2 - 2xy + 3y^2 \\ \cline{2-2}
\underline{x^4 - 2x^3y + 3x^2y^2} & x^2 + 2xy + 3y^2 \\
\multicolumn{2}{l}{\qquad 2x^3y - x^2y^2 + 9y^4} \\
\multicolumn{2}{l}{\qquad \underline{2x^3y - 4x^2y^2 + 6xy^3}} \\
\multicolumn{2}{l}{\qquad\qquad 3x^2y^2 - 6xy^3 + 9y^4} \\
\multicolumn{2}{l}{\qquad\qquad \underline{3x^2y^2 - 6xy^3 + 9y^4}}
\end{array}$$

17.
$$\begin{array}{l|l}
a^5 + b^5 & a^4 - a^3b + a^2b^2 - ab^3 + b^4 \\ \cline{2-2}
\underline{a^5 - a^4b + a^3b^2 - a^2b^3 + ab^4} & a + b \\
\multicolumn{2}{l}{\qquad a^4b - a^3b^2 + a^2b^3 - ab^4 + b^5} \\
\multicolumn{2}{l}{\qquad \underline{a^4b - a^3b^2 + a^2b^3 - ab^4 + b^5}}
\end{array}$$

18.
$$\begin{array}{l|l}
1 + 5x^3 - 6x^4 & 1 - x + 3x^2 \\ \cline{2-2}
\underline{1 - x + 3x^2} & 1 + x - 2x^2 \\
\multicolumn{2}{l}{\qquad x - 3x^2 + 5x^3 - 6x^4} \\
\multicolumn{2}{l}{\qquad \underline{x - x^2 + 3x^3}} \\
\multicolumn{2}{l}{\qquad\quad -2x^2 + 2x^3 - 6x^4} \\
\multicolumn{2}{l}{\qquad\quad \underline{-2x^2 + 2x^3 - 6x^4}}
\end{array}$$

19.
$$\begin{array}{l|l}
16x^4 + 8x^2y^2 + 9y^4 & 4x^2 - 4xy + 3y^2 \\ \cline{2-2}
\underline{16x^4 - 16x^3y + 12x^2y^2} & 4x^2 + 4xy + 3y^2 \\
\multicolumn{2}{l}{\qquad 16x^3y - 4x^2y^2 + 9y^4} \\
\multicolumn{2}{l}{\qquad \underline{16x^3y - 16x^2y^2 + 12xy^3}} \\
\multicolumn{2}{l}{\qquad\qquad 12x^2y^2 - 12xy^3 + 9y^4} \\
\multicolumn{2}{l}{\qquad\qquad \underline{12x^2y^2 - 12xy^3 + 9y^4}}
\end{array}$$

20.

$$\begin{array}{l|l}
x^3 + 3x^2y + 3xy^2 + y^3 + z^3 & x + y + z \\ \cline{2-2}
x^3 + x^2y + x^2z & x^2 + 2xy - xz + y^2 - yz + z^2 \\ \cline{1-1}
\end{array}$$

$$\begin{array}{l}
2x^2y - x^2z + 3xy^2 + y^3 + z^3 \\
2x^2y + 2xy^2 + 2xyz \\ \hline
-x^2z + xy^2 - 2xyz + y^3 + z^3 \\
-x^2z - xyz - xz^2 \\ \hline
xy^2 - xyz + xz^2 + y^3 + z^3 \\
xy^2 + y^3 + y^2z \\ \hline
-xyz + xz^2 - y^2z + z^3 \\
-xyz - y^2z - yz^2 \\ \hline
xz^2 + yz^2 + z^3 \\
xz^2 + yz^2 + z^3 \\ \hline
\end{array}$$

21.

$$\begin{array}{l|l}
a^3 - 3abc + b^3 + c^3 & a + b + c \\ \cline{2-2}
a^3 + a^2b + a^2c & a^2 - ab - ac + b^2 - bc + c^2 \\ \cline{1-1}
\end{array}$$

$$\begin{array}{l}
-a^2b - a^2c - 3abc + b^3 + c^3 \\
-a^2b - ab^2 - abc \\ \hline
-a^2c + ab^2 - 2abc + b^3 + c^3 \\
-a^2c - abc - ac^2 \\ \hline
ab^2 - abc + ac^2 + b^3 + c^3 \\
ab^2 + b^3 + b^2c \\ \hline
-abc + ac^2 - b^2c + c^3 \\
-abc - b^2c - bc^2 \\ \hline
ac^2 + bc^2 + c^3 \\
ac^2 + bc^2 + c^3 \\ \hline
\end{array}$$

22.

$$\begin{array}{l|l}
x^3 - 6xyz + 8y^3 + z^3 & x^2 - 2xy - xz + 4y^2 - 2yz + z^2 \\ \cline{2-2}
x^3 - 2x^2y - x^2z + 4xy^2 - 2xyz + xz^2 & x + 2y + z \\ \cline{1-1}
\end{array}$$

$$\begin{array}{l}
2x^2y + x^2z - 4xy^2 - 4xyz - xz^2 + 8y^3 + z^3 \\
2x^2y - 4xy^2 - 2xyz + 8y^3 - 4y^2z + 2yz^2 \\ \hline
x^2z - 2xyz - xz^2 + 4y^2z - 2yz^2 + z^3 \\
x^2z - 2xyz - xz^2 + 4y^2z - 2yz^2 + z^3 \\ \hline
\end{array}$$

23.
$$\begin{array}{l|l}
2x^2 + xy - xz - 3y^2 - 4yz - z^2 & 2x + 3y + z \\ \cline{2-2}
2x^2 + 3xy + xz & x - y - z \\ \cline{1-1}
\quad -2xy - 2xz - 3y^2 - 4yz - z^2 & \\
\quad -2xy - 3y^2 - yz & \\ \cline{1-1}
\qquad -2xz - 3yz - z^2 & \\
\qquad -2xz - 3yz - z^2 & \\ \cline{1-1}
\end{array}$$

24.
$$\begin{array}{l|l}
x^2 - y^2 - 2yz - z^2 & x + y + z \\ \cline{2-2}
x^2 + xy + xz & x - y - z \\ \cline{1-1}
\quad -xy - xz - y^2 - 2yz - z^2 & \\
\quad -xy - y^2 - yz & \\ \cline{1-1}
\qquad -xz - yz - z^2 & \\
\qquad -xz - yz - z^2 & \\ \cline{1-1}
\end{array}$$

25.
$$\begin{array}{l|l}
x^4 + x^2y^2 + y^4 & x^2 + xy + y^2 \\ \cline{2-2}
x^4 + x^3y + x^2y^2 & x^2 - xy + y^2 \\ \cline{1-1}
\quad -x^3y + y^4 & \\
\quad -x^3y - x^2y^2 - xy^3 & \\ \cline{1-1}
\qquad x^2y^2 + xy^3 + y^4 & \\
\qquad x^2y^2 + xy^3 + y^4 & \\ \cline{1-1}
\end{array}$$

26.
$$\begin{array}{l|l}
x^4 - 9x^2 + 12x - 4 & x^2 + 3x - 2 \\ \cline{2-2}
x^4 + 3x^3 - 2x^2 & x^2 - 3x + 2 \\ \cline{1-1}
\quad -3x^3 - 7x^2 + 12x - 4 & \\
\quad -3x^3 - 9x^2 + 6x & \\ \cline{1-1}
\qquad 2x^2 + 6x - 4 & \\
\qquad 2x^2 + 6x - 4 & \\ \cline{1-1}
\end{array}$$

27.
$$\begin{array}{l|l}
y^5 - 2y^4 - 6y^3 + 4y^2 + 13y + 6 & y^3 + 3y^2 + 3y + 1 \\ \cline{2-2}
y^5 + 3y^4 + 3y^3 + y^2 & y^2 - 5y + 6 \\ \cline{1-1}
\quad -5y^4 - 9y^3 + 3y^2 + 13y + 6 & \\
\quad -5y^4 - 15y^3 - 15y^2 - 5y & \\ \cline{1-1}
\qquad 6y^3 + 18y^2 + 18y + 6 & \\
\qquad 6y^3 + 18y^2 + 18y + 6 & \\ \cline{1-1}
\end{array}$$

28.

$$\begin{array}{l|l}
y^4 - 5y^2z^2 + 4z^4 & \underline{y^2 - yz - 2z^2} \\
\underline{y^4 - y^3z - 2y^2z^2} & y^2 + yz - 2z^2 \\
\qquad y^3z - 3y^2z^2 + 4z^4 & \\
\qquad \underline{y^3z - y^2z^2 - 2yz^3} & \\
\qquad\qquad -2y^2z^2 + 2yz^3 + 4z^4 & \\
\qquad\qquad \underline{-2y^2z^2 + 2yz^3 + 4z^4} &
\end{array}$$

29.

$$\begin{array}{l|l}
x^2 - 4y^2 + 12yz - 9z^2 & \underline{x + 2y - 3z} \\
\underline{x^2 + 2xy - 3xz} & x - 2y + 3z \\
\quad -2xy + 3xz - 4y^2 + 12yz - 9z^2 & \\
\quad \underline{-2xy \qquad\quad - 4y^2 + 6yz} & \\
\qquad\qquad 3xz \qquad\; + 6yz - 9z^2 & \\
\qquad\qquad \underline{3xz \qquad\; + 6yz - 9z^2} &
\end{array}$$

30.

$$\begin{array}{l|l}
x^5 - 41x - 120 & \underline{x^2 + 4x + 5} \\
\underline{x^5 + 4x^4 + 5x^3} & x^3 - 4x^2 + 11x - 24 \\
\quad -4x^4 - 5x^3 - 41x - 120 & \\
\quad \underline{-4x^4 - 16x^3 - 20x^2} & \\
\qquad\qquad 11x^3 + 20x^2 - 41x - 120 & \\
\qquad\qquad \underline{11x^3 + 44x^2 + 55x} & \\
\qquad\qquad\qquad -24x^2 - 96x - 120 & \\
\qquad\qquad\qquad \underline{-24x^2 - 96x - 120} &
\end{array}$$

31.

$$\begin{array}{l|l}
x^4 + x^3 - 4x^2 + 5x - 3 & \underline{-x^2 - 2x + 3} \\
\underline{x^4 + 2x^3 - 3x^2} & -x^2 + x - 1 \\
\quad -x^3 - x^2 + 5x - 3 & \\
\quad \underline{-x^3 - 2x^2 + 3x} & \\
\qquad\qquad x^2 + 2x - 3 & \\
\qquad\qquad \underline{x^2 + 2x - 3} &
\end{array}$$

32.

$$\begin{array}{l|l}
-2x^4 + 10x^3 - 11x^2 + x + 6 & \underline{-2x^2 + 4x - 3} \\
\underline{-2x^4 + 4x^3 - 3x^2} & x^2 - 3x - 2 \\
\qquad 6x^3 - 8x^2 + x + 6 & \\
\qquad \underline{6x^3 - 12x^2 + 9x} & \\
\qquad\qquad 4x^2 - 8x + 6 & \\
\qquad\qquad \underline{4x^2 - 8x + 6} &
\end{array}$$

33.
$$\begin{array}{l|l}
5x^6 - 6x^5 + 1 & x^2 - 2x + 1 \\ \cline{2-2}
5x^6 - 10x^5 + 5x^4 & 5x^4 + 4x^3 + 3x^2 + 2x + 1 \\ \cline{1-1}
\quad 4x^5 - 5x^4 + 1 & \\
\quad 4x^5 - 8x^4 + 4x^3 & \\ \cline{1-1}
\qquad 3x^4 - 4x^3 + 1 & \\
\qquad 3x^4 - 6x^3 + 3x^2 & \\ \cline{1-1}
\qquad\quad 2x^3 - 3x^2 + 1 & \\
\qquad\quad 2x^3 - 4x^2 + 2x & \\ \cline{1-1}
\qquad\qquad x^2 - 2x + 1 & \\
\qquad\qquad x^2 - 2x + 1 & \\ \cline{1-1}
\end{array}$$

34.
$$\begin{array}{l|l}
x^4 + 9x^2 + 81 & -x^2 + 3x - 9 \\ \cline{2-2}
x^4 - 3x^3 + 9x^2 & -x^2 - 3x - 9 \\ \cline{1-1}
\quad 3x^3 + 81 & \\
\quad 3x^3 - 9x^2 + 27x & \\ \cline{1-1}
\qquad 9x^2 - 27x + 81 & \\
\qquad 9x^2 - 27x + 81 & \\ \cline{1-1}
\end{array}$$

35.
$$\begin{array}{l|l}
x^3 \qquad\quad - y^3 & x^2 + xy + y^2 \\ \cline{2-2}
x^3 + x^2y + xy^2 & x - y \\ \cline{1-1}
\quad -x^2y - xy^2 - y^3 & \\
\quad -x^2y - xy^2 - y^3 & \\ \cline{1-1}
\end{array}$$

36.
$$\begin{array}{l|l}
x^6 + y^6 & x^2 + y^2 \\ \cline{2-2}
x^6 + x^4y^2 & x^4 - x^2y^2 + y^4 \\ \cline{1-1}
\quad -x^4y^2 + y^6 & \\
\quad -x^4y^2 - x^2y^4 & \\ \cline{1-1}
\qquad + x^2y^4 + y^6 & \\
\qquad\quad x^2y^4 + y^6 & \\ \cline{1-1}
\end{array}$$

37.
$$\begin{array}{l|l}
x^4 + a^2x^2 + a^4 & x^2 - ax + a^2 \\ \cline{2-2}
x^4 - ax^3 + a^2x^2 & x^2 + ax + a^2 \\ \cline{1-1}
\quad ax^3 + a^4 & \\
\quad ax^3 - a^2x^2 + a^3x & \\ \cline{1-1}
\qquad a^2x^2 - a^3x + a^4 & \\
\qquad a^2x^2 - a^3x + a^4 & \\ \cline{1-1}
\end{array}$$

38.
$$\begin{array}{l|l}
a^2 + ab + 2ac - 2b^2 + 7bc - 3c^2 & a - b + 3c \\ \cline{2-2}
a^2 - ab + 3ac & a + 2b - c \\ \cline{1-1}
\quad 2ab - ac - 2b^2 + 7bc - 3c^2 & \\
\quad 2ab \qquad\quad - 2b^2 + 6bc & \\ \cline{1-1}
\qquad - ac \qquad + bc - 3c^2 & \\
\qquad - ac \qquad + bc - 3c^2 & \\ \cline{1-1}
\end{array}$$

39.
$$\begin{array}{l|l}
2a^2 + ab - ac - 3b^2 - 4bc - c^2 & 2a + 3b + c \\ \cline{2-2}
2a^2 + 3ab + ac & a - b - c \\ \cline{1-1}
\quad -2ab - 2ac - 3b^2 - 4bc - c^2 & \\
\quad \underline{-2ab \qquad\quad - 3b^2 - bc} & \\
\qquad\quad -2ac \qquad - 3bc - c^2 & \\
\qquad\quad \underline{-2ac \qquad - 3bc - c^2} &
\end{array}$$

40.
$$\begin{array}{l|l}
15a^4 + 10a^3x + 4a^2x^2 + 6ax^3 - 3x^4 & 3a^2 + 2ax - x^2 \\ \cline{2-2}
15a^4 + 10a^3x - 5a^2x^2 & 5a^2 + 3x^2 \\ \cline{1-1}
\qquad\qquad 9a^2x^2 + 6ax^3 - 3x^4 & \\
\qquad\qquad \underline{9a^2x^2 + 6ax^3 - 3x^4} &
\end{array}$$

41.
$$\begin{array}{l|l}
a^3 - 6ab - 8b^3 - 1 & a - 2b - 1 \\ \cline{2-2}
a^3 - 2a^2b - a^2 & a^2 + 2ab + a + 4b^2 - 2b + 1 \\ \cline{1-1}
\quad 2a^2b + a^2 - 6ab - 8b^3 - 1 & \\
\quad \underline{2a^2b - 4ab^2 - 2ab} & \\
\qquad a^2 + 4ab^2 - 4ab - 8b^3 - 1 & \\
\qquad \underline{a^2 \qquad\quad - 2ab - a} & \\
\qquad\quad 4ab^2 - 2ab + a - 8b^3 - 1 & \\
\qquad\quad \underline{4ab^2 \qquad\qquad - 8b^3 - 4b^2} & \\
\qquad\qquad -2ab + a + 4b^2 - 1 & \\
\qquad\qquad \underline{-2ab \quad + 4b^2 + 2b} & \\
\qquad\qquad\quad a - 2b - 1 & \\
\qquad\qquad\quad \underline{a - 2b - 1} &
\end{array}$$

42.
$$\begin{array}{l|l}
x^{3n} - 3x^{2n}y^n + 3x^ny^{2n} - y^{3n} & x^n - y^n \\ \cline{2-2}
x^{3n} - x^{2n}y^n & x^{2n} - 2x^ny^n + y^{2n} \\ \cline{1-1}
\quad -2x^{2n}y^n + 3x^ny^{2n} - y^{3n} & \\
\quad \underline{-2x^{2n}y^n + 2x^ny^{2n}} & \\
\qquad\qquad x^ny^{2n} - y^{3n} & \\
\qquad\qquad \underline{x^ny^{2n} - y^{3n}} &
\end{array}$$

43.
$$\begin{array}{l|l}
a^{m+n}b^n - 4a^{m+n-1}b^{2n} - 27a^{m+n-2}b^{3n} + 42a^{m+n-3}b^{4n} & a^m + 3a^{m-1}b^n - 6a^{m-2}b^{2n} \\ \cline{2-2}
a^{m+n}b^n + 3a^{m+n-1}b^{2n} - 6a^{m+n-2}b^{3n} & a^nb^n - 7a^{n-1}b^{2n} \\ \cline{1-1}
\qquad -7a^{m+n-1}b^{2n} - 21a^{m+n-2}b^{3n} + 42a^{m+n-3}b^{4n} & \\
\qquad \underline{-7a^{m+n-1}b^{2n} - 21a^{m+n-2}b^{3n} + 42a^{m+n-3}b^{4n}} &
\end{array}$$

Exercise 15. Page 57.

1.
$$\begin{array}{l} \frac{1}{2}a^2b + \frac{1}{3}b^2c^4 + \frac{2}{15} \\ -\frac{3}{10}a^2b - \frac{1}{5}b^2c^4 - \frac{4}{9} \\ \hline \frac{1}{5}a^2b + \frac{2}{15}b^2c^4 - \frac{14}{45} \end{array}$$

2.
$$\begin{array}{l} \frac{5}{2}x^2 + 3ax - \frac{7}{3}a^2 \\ 2x^2 - \frac{3}{2}ax - \frac{1}{2}a^2 \\ \hline \frac{1}{2}x^2 + \frac{9}{2}ax - \frac{11}{6}a^2 \end{array}$$

3.
$$\begin{array}{l} \frac{1}{2}y - \frac{5}{2}a - \frac{3}{4}x + \frac{1}{3}b \\ \frac{1}{3}y + \frac{11}{4}a - \frac{2}{3}x \\ \hline \frac{1}{6}y - \frac{21}{4}a - \frac{1}{12}x + \frac{1}{3}b \end{array}$$

4.
$$\begin{array}{l} \frac{1}{3}c^2 - \frac{1}{4}c - \frac{1}{2} \\ \frac{1}{3}c^2 - \frac{1}{4}c + \frac{1}{2} \\ \hline \frac{1}{9}c^4 - \frac{1}{12}c^3 - \frac{1}{6}c^2 \\ \qquad - \frac{1}{12}c^3 + \frac{1}{16}c^2 + \frac{1}{8}c \\ \qquad\qquad + \frac{1}{6}c^2 - \frac{1}{8}c - \frac{1}{4} \\ \hline \frac{1}{9}c^4 - \frac{1}{6}c^3 + \frac{1}{16}c^2 \qquad - \frac{1}{4} \end{array}$$

5.
$$\begin{array}{l} \frac{1}{2}x - \frac{1}{3}x^2 + \frac{1}{4}x^3 \\ \frac{1}{2}x + \frac{1}{3}x^2 + \frac{1}{4}x^3 \\ \hline \frac{1}{4}x^2 - \frac{1}{6}x^3 + \frac{1}{8}x^4 \\ \qquad + \frac{1}{6}x^3 - \frac{1}{9}x^4 + \frac{1}{12}x^5 \\ \qquad\qquad + \frac{1}{8}x^4 - \frac{1}{12}x^5 + \frac{1}{16}x^6 \\ \hline \frac{1}{4}x^2 \qquad + \frac{5}{36}x^4 \qquad + \frac{1}{16}x^6 \end{array}$$

6.
$$\begin{array}{l} 0.5m^4 - 0.4m^3n + 1.2m^2n^2 + 0.8mn^3 - 1.4n^4 \\ 0.4m^2 - 0.6mn - 0.8n^2 \\ \hline 0.2m^6 - 0.16m^5n + 0.48m^4n^2 + 0.32m^3n^3 - 0.56m^2n^4 \\ \qquad - 0.30m^5n + 0.24m^4n^2 - 0.72m^3n^3 - 0.48m^2n^4 + 0.84mn^5 \\ \qquad\qquad - 0.40m^4n^2 + 0.32m^3n^3 - 0.96m^2n^4 - 0.64mn^5 + 1.12n^6 \\ \hline 0.2m^6 - 0.46m^5n + 0.32m^4n^2 - 0.08m^3n^3 - 2m^2n^4 + 0.2mn^5 + 1.12n^6 \end{array}$$

7.
$$\begin{array}{l|l} \frac{9}{16}a^4 - \frac{7}{8}a^3b + \frac{19}{36}a^2b^2 + \frac{1}{6}ab^3 & \frac{3}{2}a + \frac{1}{3}b \\ \cline{2-2} \frac{9}{16}a^4 + \frac{1}{8}a^3b & \frac{3}{8}a^3 - \frac{2}{3}a^2b + \frac{1}{2}ab^2 \\ \cline{1-1} \qquad - a^3b + \frac{19}{36}a^2b^2 + \frac{1}{6}ab^3 & \\ \qquad - a^3b - \frac{2}{9}a^2b^2 & \\ \cline{1-1} \qquad\qquad \frac{3}{4}a^2b^2 + \frac{1}{6}ab^3 & \\ \qquad\qquad \frac{3}{4}a^2b^2 + \frac{1}{6}ab^3 & \\ \cline{1-1} \end{array}$$

8.
$$\begin{array}{l|l} -\frac{1}{8}d^5 + \frac{5}{6}d^4 - \frac{41}{24}d^3 + d^2 & -\frac{3}{4}d^2 + 2d \\ \cline{2-2} -\frac{1}{8}d^5 + \frac{1}{3}d^4 & \frac{1}{6}d^3 - \frac{2}{3}d^2 + \frac{1}{2}d \\ \cline{1-1} \qquad \frac{1}{2}d^4 - \frac{41}{24}d^3 + d^2 & \\ \qquad \frac{1}{2}d^4 - \frac{4}{3}d^3 & \\ \cline{1-1} \qquad\qquad - \frac{3}{8}d^3 + d^2 & \\ \qquad\qquad - \frac{3}{8}d^3 + d^2 & \\ \cline{1-1} \end{array}$$

Exercise 16. Page 57.

1. $1^3 + 2^3 - 3^3 + 3 \times 1 \times 2 \times 3 = 1 + 8 - 27 + 18 = 0.$

2. $\sqrt{2bc} - a = \sqrt{2 \times 8 \times 9} - 23 = 12 - 23 = -11;$
$\sqrt{2bc - a} = \sqrt{2 \times 8 \times 9 - 23} = \sqrt{144 - 23} = \sqrt{121} = 11.$

3.
$$\begin{array}{llll}
 & a^2b & -\, ab^2 & +\, b^3 \\
a^3 & -\, \frac{1}{2}a^2b & +\, ab^2 & -\, \frac{3}{4}b^3 \\
\hline
a^3 & +\, \frac{1}{2}a^2b & & +\, \frac{1}{4}b^3
\end{array}$$

4.
$$\begin{array}{llll}
a^{2m} & -\, a^mb^m & +\, b^{2m} & \\
a^m & +\, b^m & & \\
\hline
a^{3m} & -\, a^{2m}b^m & +\, a^mb^{2m} & \\
 & +\, a^{2m}b^m & -\, a^mb^{2m} & +\, b^{3m} \\
\hline
a^{3m} & & & +\, b^{3m}
\end{array}$$

5.
$$\begin{array}{llll}
4a^{2m+4} & +\, 6a^{m+3} & +\, 9a^2 & \\
2a^{m+4} & -\, 3a^3 & & \\
\hline
8a^{3m+8} & +\, 12a^{2m+7} & +\, 18a^{m+6} & \\
 & -\, 12a^{2m+7} & -\, 18a^{m+6} & -\, 27a^5 \\
\hline
8a^{3m+8} & & & -\, 27a^5
\end{array}$$

6.
$$\begin{array}{l|l}
x^3 + 30xyz + 8y^3 - 125z^3 & x + 2y - 5z \\
\cline{2-2}
x^3 + 2x^2y - 5x^2z & x^2 - 2xy + 5xz + 4y^2 + 10yz + 25z^2 \\
\cline{1-1}
\quad -2x^2y + 5x^2z + 30xyz + 8y^3 - 125z^2 & \\
\quad -2x^2y - 4xy^2 + 10xyz & \\
\cline{1-1}
\qquad 5x^2z + 4xy^2 + 20xyz + 8y^3 - 125z^3 & \\
\qquad 5x^2z \qquad\qquad + 10xyz - 25xz^2 & \\
\hline
\qquad\qquad 4xy^2 + 10xyz + 25xz^2 + 8y^3 - 125z^3 & \\
\qquad\qquad 4xy^2 \qquad\qquad\qquad\qquad + 8y^3 - 20y^2z & \\
\hline
\qquad\qquad\qquad 10xyz + 25xz^2 \qquad + 20y^2z - 125z^3 & \\
\qquad\qquad\qquad 10xyz \qquad\qquad\qquad + 20y^2z - 50yz^2 & \\
\hline
\qquad\qquad\qquad\quad 25xz^2 \qquad\qquad\qquad + 50yz^2 - 125z^3 & \\
\qquad\qquad\qquad\quad 25xz^2 \qquad\qquad\qquad + 50yz^2 - 125z^3 & \\
\hline
\end{array}$$

7. $(x-a)^2 - (x-b)^2 - (a-b)(a+b-3x)$
$= x^2 - 2ax + a^2 - x^2 + 2bx - b^2 - a^2 + b^2 + 3ax - 3bx$
$= ax - bx.$

8. $$\begin{aligned} x+a-2[2a-b(c-x)] &= x+a-2[2a-bc+bx] \\ &= x+a-4a+2bc-2bx \\ &= (1-2b)x-3a+2bc. \end{aligned}$$

9. $$\begin{array}{llll} 4x^{m+2n-1} & -\ 7x^{2m-3n+2} & +\ 5x^{2n+3m-2} \\ 5x^{2-m-2n} & & \\ \hline 20x & -\ 35x^{m-5n+4} & +\ 25x^{2m} \end{array}$$

10. $$\begin{aligned} \frac{c^2d^{3-2x}-c^4d^{4-2x}-c^{4+m}d^{3-x}}{c^{3-m}d^{2-2x}} &= \frac{c^2d^{3-2x}}{c^{3-m}d^{2-2x}}-\frac{c^4d^{4-2x}}{c^{3-m}d^{2-2x}}-\frac{c^{4+m}d^{3-x}}{c^{3-m}d^{2-2x}} \\ &= c^{m-1}d-c^{m+1}d^2-c^{2m+1}d^{1+x}. \end{aligned}$$

11. $$\begin{aligned} \frac{m^3y^n-m^{1+x}y^{1+n}+m^{5-x}y^{n-4}}{m^{5-x}y^{n-4}} &= \frac{m^3y^n}{m^{5-x}y^{n-4}}-\frac{m^{1+x}y^{1+n}}{m^{5-x}y^{n-4}}+\frac{m^{5-x}y^{n-4}}{m^{5-x}y^{n-4}} \\ &= m^{x-2}y^4-m^{2x-4}y^5+1. \end{aligned}$$

12. $$\begin{aligned} \frac{a^{1+3y}-a^3+a^{2+5y}}{a^{2-x+2y}} &= \frac{a^{1+3y}}{a^{2-x+2y}}-\frac{a^3}{a^{2-x+2y}}+\frac{a^{2+5y}}{a^{2-x+2y}} \\ &= a^{x+y-1}-a^{x-2y+1}+a^{x+3y}. \end{aligned}$$

13. $$\begin{aligned} \frac{x-x^{m-5n+4}+x^{2m}}{x^{2-m-2n}} &= \frac{x}{x^{2-m-2n}}-\frac{x^{m-5n+4}}{x^{2-m-2n}}+\frac{x^{2m}}{x^{2-m-2n}} \\ &= x^{m+2n-1}-x^{2m-3n+2}+x^{3m+2n-2}. \end{aligned}$$

14. $$\begin{aligned} \frac{y^p-y^{3-4m}+y^{4p+1}}{y^{2p-m+1}} &= \frac{y^p}{y^{2p-m+1}}-\frac{y^{3-4m}}{y^{2p-m+1}}+\frac{y^{4p+1}}{y^{2p-m+1}} \\ &= y^{m-p-1}-y^{-3m-2p+2}+y^{m+2p}. \end{aligned}$$

15. $$\begin{array}{l|l} 2x^{3n}-6x^{2n}y^n+6x^ny^{2n}-2y^{3n} & x^n-y^n \\ \cline{2-2} 2x^{3n}-2x^{2n}y^n & 2x^{2n}-4x^ny^n+2y^{2n} \\ \hline \quad -4x^{2n}y^n+6x^ny^{2n}-2y^{3n} & \\ \quad -4x^{2n}y^n+4x^ny^{2n} & \\ \cline{1-1} \qquad\qquad 2x^ny^{2n}-2y^{3n} & \\ \qquad\qquad 2x^ny^{2n}-2y^{3n} & \\ \cline{1-1} \end{array}$$

16. $$\begin{array}{l|l} x^3-2ax^2+a^2x-abx-b^2x+a^2b+ab^2 & x^2-ax+bx-ab \\ \cline{2-2} x^3-\ ax^2+bx^2-abx & x-a-b \\ \hline \quad -\ ax^2-bx^2+a^2x-b^2x+a^2b+ab^2 & \\ \quad -\ ax^2\qquad\ \ +a^2x-abx+a^2b & \\ \cline{1-1} \qquad\qquad -bx^2+abx-b^2x+ab^2 & \\ \qquad\qquad -bx^2+abx-b^2x+ab^2 & \\ \cline{1-1} \end{array}$$

17. $$\begin{array}{l|l} x^{4m}+x^{2m}+1 & x^{2m}-x^{m}+1 \\ \underline{x^{4m}-x^{3m}+x^{2m}} & x^{2m}+x^{m}+1 \\ \qquad x^{3m}+1 & \\ \qquad \underline{x^{3m}-x^{2m}+x^{m}} & \\ \qquad\qquad x^{2m}-x^{m}+1 & \\ \qquad\qquad \underline{x^{2m}-x^{m}+1} & \end{array}$$

18. $$\begin{array}{l|l} 3x^{m+7}-4x^{m+6}-12x^{m+5}-9x^{m+4} & x^{m+4}-3x^{m+3} \\ \underline{3x^{m+7}-9x^{m+6}} & 3x^{3}+5x^{2}+3x \\ \qquad 5x^{m+6}-12x^{m+5}-9x^{m+4} & \\ \qquad \underline{5x^{m+6}-15x^{m+5}} & \\ \qquad\qquad 3x^{m+5}-9x^{m+4} & \\ \qquad\qquad \underline{3x^{m+5}-9x^{m+4}} & \end{array}$$

19. $$\begin{array}{l|l} 6x^{4m+5}-13x^{3m+5}+13x^{2m+5}-13x^{m+5}-5x^{5} & 2x^{2m}-3x^{m}-1 \\ \underline{6x^{4m+5}-\ 9x^{3m+5}-\ 3x^{2m+5}} & 3x^{2m+5}-2x^{m+5}+5x^{5} \\ \qquad -\ 4x^{3m+5}+16x^{2m+5}-13x^{m+5}-5x^{5} & \\ \qquad \underline{-\ 4x^{3m+5}+\ 6x^{2m+5}+\ 2x^{m+5}} & \\ \qquad\qquad 10x^{2m+5}-15x^{m+5}-5x^{5} & \\ \qquad\qquad \underline{10x^{2m+5}-15x^{m+5}-5x^{5}} & \end{array}$$

20. $$\begin{array}{l|l} 12a^{5n-3}-\ a^{4n-2}-20a^{3n-1}+19a^{2n}-10a^{n+1} & 4a^{2n}-3a^{n+1}+2a^{2} \\ \underline{12a^{5n-3}-9a^{4n-2}+\ 6a^{3n-1}} & 3a^{3n-3}+2a^{2n-2}-5a^{n-1} \\ \qquad 8a^{4n-2}-26a^{3n-1}+19a^{2n}-10a^{n+1} & \\ \qquad \underline{8a^{4n-2}-\ 6a^{3n-1}+\ 4a^{2n}} & \\ \qquad\qquad -20a^{3n-1}+15a^{2n}-10a^{n+1} & \\ \qquad\qquad \underline{-20a^{3n-1}+15a^{2n}-10a^{n+1}} & \end{array}$$

21. $(1-a^{2})x^{3}-(a^{2}+a)x^{2}-(2b+a+c+bc)x.$

22. $$\begin{array}{l|l} 1.2a^{4}x-5.494a^{3}x^{2}+4.8a^{2}x^{3}+0.9ax^{4}-x^{5} & 0.6ax-2x^{2} \\ \underline{1.2a^{4}x-4.000a^{3}x^{2}} & 2a^{3}-2.49a^{2}x-0.3ax^{2} \\ \qquad -1.494a^{3}x^{2}+4.8\ \ a^{2}x^{3} & \qquad [+0.5x^{3} \\ \qquad \underline{-1.494a^{3}x^{2}+4.98a^{2}x^{3}} & \\ \qquad\qquad -0.18a^{2}x^{3}+0.9ax^{4} & \\ \qquad\qquad \underline{-0.18a^{2}x^{3}+0.6ax^{4}} & \\ \qquad\qquad\qquad 0.3ax^{4}-x^{5} & \\ \qquad\qquad\qquad \underline{0.3ax^{4}-x^{5}} & \end{array}$$

23.
$$\begin{array}{lllll}
\frac{1}{2}a^2 & -\frac{1}{5}ab & +\frac{1}{3}b^2 & & \\
\frac{1}{2}a & +\frac{1}{5}b & & & \\
\hline
\frac{1}{4}a^3 & -\frac{1}{10}a^2b & +\frac{1}{6}ab^2 & & \\
 & +\frac{1}{10}a^2b & -\frac{1}{25}ab^2 & +\frac{1}{15}b^3 & \\
\hline
\frac{1}{4}a^3 & & +\frac{19}{150}ab^2 & +\frac{1}{15}b^3 &
\end{array}$$

24.
$$\begin{array}{llll}
\frac{2}{3}a^2 & +ab & +\frac{3}{2}b^2 & \\
\frac{1}{2}a & -\frac{1}{3}b & & \\
\hline
\frac{1}{3}a^3 & +\frac{1}{2}a^2b & +\frac{3}{4}ab^2 & \\
 & -\frac{2}{9}a^2b & -\frac{1}{3}ab^2 & -\frac{1}{2}b^2 \\
\hline
\frac{1}{3}a^3 & +\frac{5}{18}a^2b & +\frac{5}{12}ab^2 & -\frac{1}{2}b^3
\end{array}$$

25.
$$\begin{array}{llll|l}
\frac{1}{4}a^3 & & +\frac{1}{72}ab^2 & +\frac{1}{12}b^3 & \frac{1}{2}a+\frac{1}{3}b \\
\cline{5-5}
\frac{1}{4}a^3 & +\frac{1}{6}a^2b & & & \frac{1}{2}a^2-\frac{1}{3}ab+\frac{1}{4}b^2 \\
\cline{1-4}
 & -\frac{1}{6}a^2b & +\frac{1}{72}ab^2 & +\frac{1}{12}b^3 & \\
 & -\frac{1}{6}a^2b & -\frac{1}{9}ab^2 & & \\
\cline{2-4}
 & & \frac{1}{8}ab^2 & +\frac{1}{12}b^3 & \\
 & & \frac{1}{8}ab^2 & +\frac{1}{12}b^3 & \\
\cline{3-4}
\end{array}$$

26.
$$\begin{array}{lll}
\frac{1}{3}x^2 & -\frac{1}{2}xy & +\frac{1}{8}y^2 \\
\frac{1}{4}x^2 & +\frac{1}{5}xy & +\frac{1}{6}y^2 \\
\hline
\frac{1}{12}x^2 & -\frac{7}{10}xy & -\frac{1}{24}y^2
\end{array}$$

27.
$$\begin{array}{lll}
2x^2 & -\frac{1}{2}xy & +y^2 \\
x^2 & +\frac{1}{3}xy & -\frac{1}{2}y^2 \\
\hline
x^2 & -\frac{5}{6}xy & +\frac{3}{2}y^2
\end{array}$$

Exercise 17. Page 64.

1. $x - 5 = 7$

Transpose -5 to right side, $x = 7 + 5$

Combine, $x = 12$

2. $x + 8 = 12$

Transpose 8 to right side, $x = 12 - 8$

Combine, $x = 4$

3. $6x - 12 = 18$

Transpose -12 to right side, $6x = 18 + 12$

Combine, $6x = 30$

Divide by 6, $x = 5$

4. $7x - 3 = 25$
Transpose -3 to right side, $7x = 25 + 3$
Combine, $7x = 28$
Divide by 7, $x = 4$

5. $5x + 8 = 43$
Transpose 8 to right side, $5x = 43 - 8$
Combine, $5x = 35$
Divide by 5, $x = 7$

6. $2x - 5 = 7 + x$
Transpose -5 to right side, and x to left side,
$2x - x = 7 + 5$
Combine, $x = 12$

7. $2x - 4 = 5 - x$
Transpose -4 to right side, and $-x$ to left side,
$2x + x = 5 + 4$
Combine, $3x = 9$
Divide by 3, $x = 3$

8. $5x - 2 = 3x + 4$
Transpose -2 to right side, and $3x$ to left side,
$5x - 3x = 4 + 2$
Combine, $2x = 6$
Divide by 2, $x = 3$

9. $7x - 5 = 6x - 1$
Transpose -5 to right side, and $6x$ to left side,
$7x - 6x = -1 + 5$
Combine, $x = 4$

10. $5x - 3 = 25 - 2x$
Transpose -3 to right side, and $-2x$ to left side,
$5x + 2x = 25 + 3$
Combine, $7x = 28$
Divide by 7, $x = 4$

11. $3x = x + 8$
Transpose x to left side, $2x = 8$
Divide by 2, $x = 4$

12. $2x - 5 = 3x$

Transpose 5 to right side, and $3x$ to left side,

$$2x - 3x = 5$$

Combine, $-x = 5$

Multiply by -1 $x = -5$

13. $2x - 3 = 8 + x$

Transpose -3 to right side, and x to left side,

$$2x - x = 8 + 3$$

Combine, $x = 11$

14. $5x + 4 = 20 + x$

Transpose 4 to right side, and x to left side,

$$5x - x = 20 - 4$$

Combine, $4x = 16$

Divide by 4, $x = 4$

15. $2x - 3 = 7 - x$

Transpose -3 to right side, and $-x$ to left side,

$$2x + x = 7 + 3$$

Combine, $3x = 10$

Divide by 3, $x = \frac{10}{3}$

16. $3x - 4 = 12 - x$

Transpose -4 to right side, and $-x$ to left side,

$$3x + x = 12 + 4$$

Combine, $4x = 16$

Divide by 4, $x = 4$

17. $2x - 5 = 7 - 2x$

Transpose -5 to right side, and $-2x$ to left side,

$$2x + 2x = 7 + 5$$

Combine, $4x = 12$

Divide by 4, $x = 3$

18. $3x + 14 = 2 - x$

Transpose 14 to right side, and $-x$ to left side,

$$3x + x = 2 - 14$$

Combine, $4x = -12$

Divide by 4, $x = -3$

19. $2x-5=4x-11$
Transpose -5 to right side, and $4x$ to left side,
$2x-4x=-11+5$
Combine, $-2x=-6$
Divide by -2, $x=3$

20. $x(x-7)=x^2-70$
Remove parenthesis, $x^2-7x=x^2-70$
Cancel the x^2's, $-7x=-70$
Divide by -7, $x=10$

21. $x(3x-2)=3x(x-1)+2$
Remove parentheses, $3x^2-2x=3x^2-3x+2$
Cancel the x^2's, $-2x=-3x+2$
Transpose $-3x$ to left side, $x=2$

22. $3x-5=4x-10$
Transpose -5 to right side, and $4x$ to left side,
$3x-4x=-10+5$
Combine, $-x=-5$
Multiply by -1, $x=5$

23. $2x-4=14-x$
Transpose -4 to right side, and $-x$ to left side,
$2x+x=14+4$
Combine, $3x=18$
Divide by 3, $x=6$

24. $3x-8=4x-11$
Transpose -8 to right side, and $4x$ to left side,
$3x-4x=-11+8$
Combine, $-x=-3$
Multiply by -1 $x=3$

25. $2x-5=7-x$
Transpose -5 to right side, and $-x$ to left side,
$2x+x=7+5$
Combine, $3x=12$
Divide by 3, $x=4$

26. $2x^2 - 23 = (2x+1)(x-3)$

Remove parentheses, $2x^2 - 23 = 2x^2 - 5x - 3$

Cancel the x^2's, $-23 = -5x - 3$

Transpose $-5x$ to the right side, and -23 to the left side,

$5x = -3 + 23$

Combine, $5x = 20$

Divide by 5, $x = 4$

27. $(x+3)(x-7) - (x-4)(x+1) = 25$

Remove parentheses,

$x^2 - 4x - 21 - x^2 + 3x + 4 = 25$

Combine, $-x - 17 = 25$

Transpose -17 to right side, $-x = 25 + 17$

Combine, $-x = 42$

Multiply by -1, $x = -42$

28. $(2x-1)(x+3) - (x-3)(2x-3) = 72$

Remove parentheses,

$2x^2 + 5x - 3 - 2x^2 + 9x - 9 = 72$

Combine, $14x - 12 = 72$

Transpose -12 to right side, $14x = 72 + 12$

Combine, $14x = 84$

Divide by 14, $x = 6$

29. $x(x-5) = x^2 - 30$

Remove parentheses, $x^2 - 5x = x^2 - 30$

Cancel the x^2's, $-5x = -30$

Divide by -5, $x = 6$

30. $x(x+3) = x^2 + 18$

Remove parentheses, $x^2 + 3x = x^2 + 18$

Cancel the x^2's, $3x = 18$

Divide by 3, $x = 6$

31. $(x-3)(x+1) = x^2 - 3x + 1$

Remove parentheses, $x^2 - 2x - 3 = x^2 - 3x + 1$

Cancel the x^2's, $-2x - 3 = -3x + 1$

Transpose -3 to right side, and $-3x$ to left side,

$-2x + 3x = 1 + 3$

Combine, $x = 4$

32. $(x-1)(x+2)+(x+3)(x-1)=2x(x+4)-(x+1)$

Remove parentheses,

$x^2+x-2+x^2+2x-3=2x^2+8x-x-1$

Combine, $2x^2+3x-5=2x^2+7x-1$

Cancel the x^2's, $3x-5=7x-1$

Transpose -5 to right side, and $7x$ to left side,

$3x-7x=-1+5$

Combine, $-4x=4$

Divide by -4, $x=-1$

33. $x(x+8)-(x+1)(x-2)-5(x+3)+3=0$

Remove parentheses, $x^2+8x-x^2+x+2-5x-15+3=0$

Combine, $4x-10=0$

Transpose -10 to right side, $4x=10$

Divide by 4, $x=\frac{5}{2}$

34. $(x-3)(x+3)-(x-4)(x+4)-x=0$

Remove parentheses, $x^2-9-x^2+16-x=0$

Combine, $7-x=0$

Transpose 7 to right side, $-x=-7$

Divide by -1, $x=7$

Exercise 18. Page 69.

1. If a number is multiplied by 7, the product is 301. Find the number.

Let $x=$ the number.

Then, $7x=$ the number multiplied by 7.

But, $301=$ the number multiplied by 7.

$\therefore 7x=301$

$\therefore x=43,$

Therefore the number is 43.

2. The sum of two numbers is 48, and the greater is five times the less. Find the numbers.

Let $x=$ the less number.

Then, $5x=5$ times the less number.

Hence, $5x=$ the greater number,

and $x+5x=$ the sum of the two numbers.

But, $48=$ the sum of the two numbers.

$$\therefore\ x + 5x = 48$$
$$6x = 48$$
$$\therefore\ x = 8$$
$$5x = 40$$

Therefore the numbers are 8 and 40.

3. The sum of two numbers is 25, and seven times the less exceeds three times the greater by 35. Find the numbers.

Let $x =$ the less number.
Then, $25 - x =$ the greater number.
Now, $7x = 7$ times the less number,
and $3(25 - x) = 3$ times the greater number.
Then, $7x - 3(25 - x) =$ the excess of 7 times the less number over 3 times the greater.
But, $35 =$ this excess.

$$\therefore\ 7x - 3(25 - x) = 35$$
$$7x - 75 + 3x = 35$$
$$10x = 110$$
$$\therefore\ x = 11$$
$$25 - x = 14$$

Therefore the numbers are 11 and 14.

4. Divide 20 in two parts such that four times the greater exceeds three times the less by 17.

Let $x =$ the less part.
Then, $20 - x =$ the greater part,
and $3x = 3$ times the less part.
$4(20 - x) = 4$ times the greater part.
Then, $4(20 - x) - 3x =$ the excess of 4 times the greater part over 3 times the less part.
But $17 =$ this excess.

$$4(20 - x) - 3x = 17$$
$$80 - 4x - 3x = 17$$
$$80 - 7x = 17$$
$$-7x = -63$$
$$\therefore\ x = 9$$
$$20 - x = 11$$

Therefore the parts are 9 and 11.

5. Divide 23 into two parts such that the sum of twice the greater part and three times the less part is 57.

Let $x =$ the less part.
Then, $23 - x =$ the greater part,
and $3x = 3$ times the less part.
$2(23 - x) = 2$ times the greater part.
Then, $2(23 - x) + 3x =$ the sum of twice the greater part and 3 times the smaller part.
But, $57 =$ this sum.

$$\therefore 2(23 - x) + 3x = 57$$
$$46 - 2x + 3x = 57$$
$$46 + x = 57$$
$$x = 11$$
$$23 - x = 12$$

Therefore the two parts are 11 and 12.

6. Divide 19 into two parts such that the greater part exceeds twice the less part by 1 less than twice the less part.

Let $x =$ the less part.
Then, $19 - x =$ the greater part,
and $2x =$ twice the less part.
Then, $19 - x - 2x =$ the excess of the greater part over twice the less part.
But, $2x - 1 =$ this excess.

$$\therefore 19 - x - 2x = 2x - 1$$
$$19 - 3x = 2x - 1$$
$$-5x = -20$$
$$x = 4$$
$$19 - x = 15$$

Therefore the two parts are 4 and 15.

7. A tree 84 feet high was broken so that the part broken off was five times the length of the part left standing. Required the length of each part.

Let $x =$ the number of feet in length of the part left standing.
Then, $84 - x =$ the number of feet in length of the part broken off,
and $5x = 5$ times the number of feet in length of the part left standing.

But the last two numbers are equal.

$$\therefore\ 84 - x = 5x$$
$$-6x = -84$$
$$x = 14$$
$$5x = 70$$

Therefore the length of the part left standing was 14 feet; that of the part broken off was 70 feet.

8. Four times the smaller of two numbers is three times the greater, and their sum is 63. Find the numbers.

Let $x =$ the smaller number.
Then, $63 - x =$ the greater number,
and $4x = 4$ times the smaller number.
$3(63 - x) = 3$ times the greater number.

But these last two numbers are equal.

$$\therefore\ 4x = 3(63 - x)$$
$$4x = 189 - 3x$$
$$7x = 189$$
$$\therefore\ x = 27$$
$$63 - x = 36$$

Therefore the numbers are 27 and 36.

9. A farmer sold a sheep, a cow, and a horse for $216. He sold the cow for seven times as much as the sheep, and the horse for four times as much as the cow. How much did he get for each?

Let $x =$ the number of dollars for which he sold the sheep.
Then, $7x =$ the number of dollars for which he sold the cow,
and $28x =$ the number of dollars for which he sold the horse.
Hence, $x + 7x + 28x =$ the number of dollars for which he sold the three.
But, $216 =$ this number of dollars.

$$\therefore\ x + 7x + 28x = 216$$
$$36x = 216$$
$$\therefore\ x = 6$$
$$7x = 42$$
$$28x = 168$$

Therefore he sold the sheep for $6, the cow for $42, and the horse for $168.

10. Distribute $15 among Thomas, Richard, and Henry so that Thomas and Richard shall each have twice as much as Henry.

Let $x =$ the number of dollars to be given to Henry.

Then $2x =$ the number of dollars to be given to Richard,

and $2x =$ the number of dollars to be given to Thomas.

Hence, $x + 2x + 2x =$ the entire number of dollars to be distributed.

But, $15 =$ this number of dollars.

$$\therefore\ x + 2x + 2x = 15$$
$$5x = 15$$
$$\therefore\ x = 3$$
$$2x = 6$$

Therefore Henry must receive $3, and Thomas and Richard $6 each.

11. Three men, A, B, and C, pay $1000 taxes. B pays four times as much as A, and C pays as much as A and B together. How much does each pay?

Let $x =$ the number of dollars that A pays.

Then, $4x =$ the number of dollars that B pays,

and $x + 4x =$ the number of dollars that C pays.

Hence, $x + 4x + x + 4x =$ the number of dollars that all pay.

But, $1000 =$ this number.

$$\therefore\ x + 4x + x + 4x = 1000$$
$$10x = 1000$$
$$\therefore\ x = 100$$
$$4x = 400$$
$$x + 4x = 500$$

Therefore A pays $100; B, $400; and C, $500.

12. John's age is three times the age of James, and their ages together are 16 years. What is the age of each?

Let $x =$ the number of years in James' age.

Then, $3x =$ the number of years in John's age,

and $x + 3x =$ the number of years in their ages together.

But $16 =$ this number.

$$\therefore\ x + 3x = 16$$
$$4x = 16$$
$$\therefore\ x = 4$$
$$3x = 12$$

Therefore James is 4 years old, and John 12 years old.

13. Twice a certain number increased by 8 is 40. Find the number.

Let $x =$ the number.
Then, $2x =$ twice the number,
and $2x + 8 =$ twice the number increased by 8.
But, $40 =$ twice the number increased by 8.

$$\therefore\ 2x + 8 = 40$$
$$2x = 32$$
$$\therefore\ x = 16$$

Therefore the number is 16.

14. Three times a certain number is 46 more than the number itself. Find the number.

Let $x =$ the number.
Then, $3x = 3$ times the number,
and $x + 46 =$ the number increased by 46.
But the last two numbers are equal.

$$\therefore\ 3x = x + 46$$
$$2x = 46$$
$$\therefore\ x = 23$$

Therefore the number is 23.

15. One number is four times as large as another. If I take the smaller from 12 and the greater from 21, the remainders are equal. What are the numbers?

Let $x =$ the smaller number.
Then, $4x =$ the greater number,
and $12 - x = 12$ decreased by the smaller number.
$21 - 4x = 21$ decreased by the greater number.
But the last two numbers are equal.

$$\therefore\ 12 - x = 21 - 4x$$
$$3x = 9$$
$$\therefore\ x = 3$$
$$4x = 12$$

Therefore the numbers are 3 and 12.

16. The joint ages of a father and son are 70 years. If the age of the son were doubled, he would be 4 years younger than his father. What is the age of each?

Let $x =$ the number of years in the son's age.
Then, $70 - x =$ the number of years in the father's age,
and $2x =$ double the number of years in the son's age.
$2x + 4 =$ double the number of years in the son's age increased by 4.

But this is the number of years in the father's age.

$$\therefore\ 2x + 4 = 70 - x$$
$$3x = 66$$
$$\therefore\ x = 22$$
$$70 - x = 48$$

Therefore the son is 22 years old, and the father 48 years old.

17. A man has 6 sons, each 4 years older than the next younger. The eldest is three times as old as the youngest. What is the age of each?

Let $x =$ the number of years in the age of the youngest son.
Then, $x + 20 =$ the number of years in the age of the oldest son.

But the latter number is three times the former.

$$\therefore\ x + 20 = 3x$$
$$-2x = -20$$
$$\therefore\ x = 10$$
$$x + 20 = 30$$

Therefore the youngest son is 10 years old, and the oldest son is 30 years old.

18. Add \$24 to a certain amount, and the sum will be as much above \$80 as the amount is below \$80. What is the amount?

Let $x =$ the number of dollars in the amount.
Then, $80 - x =$ the number of dollars in the amount below \$80,
and $x + 24 =$ the number of dollars increased by 24.
$x + 24 - 80 =$ the excess of the increased number of dollars above \$80.

$$\therefore\ 80 - x = x + 24 - 80$$
$$-2x = -136$$
$$\therefore\ x = 68$$

Therefore the amount is \$68.

19. Thirty yards of cloth and 40 yards of silk together cost \$330; and the silk costs twice as much per yard as the cloth. How much does each cost per yard?

Let $x =$ the number of dollars the cloth costs per yard.
Then, $2x =$ the number of dollars the silk costs per yard,
and $30x =$ the number of dollars 30 yards of cloth cost.
$80x =$ the number of dollars 40 yards of silk cost.
$30x + 80x =$ the number of dollars both cost.
But, $330 =$ this number of dollars.

$$\therefore\ 30x + 80x = 330$$
$$110x = 330$$
$$\therefore\ x = 3$$
$$2x = 6$$

Therefore the cloth costs \$3 per yard, and the silk \$6 per yard.

20. Find the number whose double diminished by 24 exceeds 80 by as much as the number itself is less than 100.

Let $x =$ the number.
Then, $2x =$ double the number.
$2x - 24 =$ double the number decreased by 24.
$2x - 24 - 80 =$ excess of last number over 80.
$100 - x =$ excess of 100 over the number.

But the last two numbers are equal.

$$\therefore\ 2x - 24 - 80 = 100 - x$$
$$3x = 204$$
$$x = 68$$

Therefore the number is 68.

21. In a company of 180 persons, composed of men, women, and children, there are twice as many men as women, and three times as many women as children. How many are there of each?

Let $x =$ the number of children.
Then, $3x =$ the number of women,
and $6x =$ the number of men.

Hence, $x + 3x + 6x =$ the entire number.
But, $180 =$ the entire number.
$\therefore\ x + 3x + 6x = 180$
$10x = 180$
$\therefore\ x = 18$
$3x = 54$
$6x = 108$

Therefore there are 18 children, 54 women, and 108 men.

22. A banker was asked to pay \$56 in five-dollar and two-dollar bills in such a manner as to pay the same number of each kind of bills. How many bills of each kind must he pay?

Let $x =$ the number of bills of each kind used.
Then $5x =$ the number of dollars paid in \$5 bills,
and $2x =$ the number paid in \$2 bills.
Hence, $5x + 2x =$ the entire number of dollars paid.
But, $56 =$ this number.
$\therefore\ 5x + 2x = 56$
$7x = 56$
$x = 8$

Therefore he must pay 8 bills of each kind.

23. How can \$3.60 be paid in quarters and ten-cent pieces so as to pay twice as many ten-cent pieces as quarters?

Let $x =$ the number of quarters.
Then, $2x =$ the number of ten-cent pieces,
and $25x =$ the number of cents paid in quarters.
$20x =$ the number of cents paid in ten-cent pieces.
$25x + 20x =$ the number of cents paid in all.
But, $360 =$ this number.
$\therefore\ 25x + 20x = 360$
$45x = 360$
$x = 8$
$2x = 16$

Therefore 8 quarters and 16 ten-cent pieces must be paid.

24. I have \$1.98 in ten-cent pieces and three-cent pieces, and have four times as many three-cent pieces as ten-cent pieces. How many have I of each?

Let $x =$ the number of ten-cent pieces.
Then, $4x =$ the number of three-cent pieces,
and $10x =$ the number of cents in ten-cent pieces.
$12x =$ the number of cents in three-cent pieces.
$10x + 12x =$ the entire number of cents.
But, $198 =$ the entire number of cents.

$$\therefore\ 10x + 12x = 198$$
$$22x = 198$$
$$x = 9$$
$$4x = 36$$

Therefore I have 9 ten-cent pieces and 36 three-cent pieces.

25. I have \$17 dollars in two-dollar bills and twenty-five-cent pieces, and have twice as many bills as coins. How many have I of each?

Let $x =$ the number of coins.
Then, $2x =$ the number of bills,
and $25x =$ the number of cents in coins.
$400x =$ the number of cents in bills.
$25x + 400x =$ the entire number of cents.
But, $1700 =$ the entire number of cents.

$$\therefore\ 25x + 400x = 1700$$
$$425x = 1700$$
$$\therefore\ x = 4$$
$$2x = 8$$

Therefore I have 4 coins and 8 bills.

26. I have \$6.50 in silver dollars and ten-cent pieces, and I have 20 coins in all. How many have I of each?

Let $x =$ the number of ten-cent pieces.
Then, $20 - x =$ the number of silver dollars,
and $10x =$ the number of cents in ten-cent pieces.
$100(20 - x) =$ the number of cents in silver dollars.
$10x + 100(20 - x) =$ the entire number of cents.
But, $650 =$ the entire number of cents.

$$\therefore\ 10x + 100(20 - x) = 650$$
$$10x + 2000 - 100x = 650$$
$$-90x = -1350$$
$$\therefore\ x = 15$$
$$20 - x = 5$$

Therefore I have 15 ten-cent pieces and 5 silver dollars.

27. A bought 9 dozen oranges for $2.00. For a part he paid 20 cents per dozen; for the remainder he paid 25 cents a dozen. How many dozen of each kind did he buy?

Let $x =$ the number of dozen bought at 20 cents.
Then, $9 - x =$ the number of dozen bought at 25 cents,
and $20x =$ the number of cents paid for the first lot.
$(9 - x)25 =$ the number of cents paid for the second lot.
$20x + (9 - x)25 =$ the number of cents paid for all.
But, $200 =$ the number of cents paid for all.

$$\therefore\ 20x + (9 - x)25 = 200$$
$$20x - 25x + 225 = 200$$
$$-5x = -25$$
$$\therefore\ x = 5$$
$$9 - x = 4$$

Therefore he bought 5 dozen of the first kind and 4 of the second.

28. A gentleman gave some children 10 cents apiece, and found that he had just 50 cents left. If he had had another half-dollar, he might have given each of them at first 20 cents instead of 10 cents. How many children were there?

Let $x =$ the number of children.
Then, $10x =$ the number of cents given them.
Also, $20x =$ the number of cents he would have given them, had he given each 20 cents.

But the second amount is 100 cents more than the first.

$$\therefore\ 20x = 10x + 100$$
$$10x = 100$$
$$x = 10$$

Therefore there were 10 children.

29. A is twice as old as B and 6 years younger than C. The sum of the ages of A, B, and C is 96 years. What is the age of B?

Let $x =$ the number of years in B's age.
Then, $2x =$ the number of years in A's age,
and $2x + 6 =$ the number of years in C's age.
Hence, $x + 2x + 2x + 6 =$ the number of years in the sum of their ages.
But, $96 =$ the number of years in the sum of their ages.

$$\begin{aligned} \therefore\ x+2x+2x+6&=96\\ 5x&=90\\ \therefore\ x&=18\\ 2x&=36\\ 2x+6&=22 \end{aligned}$$

Therefore B is 18 years old.

30. Divide a line 24 inches long into two parts such that the one part shall be 6 inches longer than the other.

Let $x =$ the number of inches in the shorter part.
Then, $24 - x =$ the number of inches in the larger part.
But the latter part is 6 inches longer than the former.

$$\begin{aligned} \therefore\ 24-x&=x+6\\ -2x&=-18\\ \therefore\ x&=9\\ 24-x&=15 \end{aligned}$$

Therefore the parts are 9 and 15 inches long.

31. Two trains travelling, one at 25 and the other at 30 miles an hour, start at the same time from two places 220 miles apart, and move toward each other. In how many hours will the trains meet?

Let $x =$ the number of hours required.
Then, $25x =$ the number of miles the first train will have travelled,
and $30x =$ the number of miles the second train will have travelled.
Hence, $25x + 30x =$ the number of miles in the whole distance.
But, $220 =$ the number of miles in the whole distance.

$$\begin{aligned} \therefore\ 25x+30x&=220\\ 55x&=220\\ \therefore\ x&=4 \end{aligned}$$

Therefore the trains will meet in 4 hours.

32. A man bought 12 yards of velvet, and if he had bought 1 yard less for the same money, each yard would have cost $1 more. What did the velvet cost a yard?

Let $x =$ the number of dollars which the velvet cost per yard.

Then, $12x =$ the number of dollars which 12 yards cost,

and $x + 1 =$ the number of dollars which the velvet would have cost per yard at \$1 per yard more.

$11(x + 1) =$ the number of dollars 11 yards would have cost at the latter price.

$$\therefore\ 12x = 11(x + 1)$$
$$12x = 11x + 11$$
$$\therefore\ x = 11$$

Therefore the velvet cost \$11 per yard.

33. A and B have together \$8; A and C, \$10; B and C, \$12. How much has each?

Let $x =$ the number of dollars A has.

Then, $8 - x =$ the number of dollars B has,

and $10 - x =$ the number of dollars C has.

Hence, $8 - x + 10 - x =$ the number of dollars B and C have.

But, $12 =$ the number of dollars B and C have.

$$\therefore\ 8 - x + 10 - x = 12$$
$$-2x = -6$$
$$x = 3$$
$$8 - x = 5$$
$$10 - x = 7$$

Therefore A has \$3; B, \$5; and C, \$7.

34. Twelve persons subscribed for a new boat, but two being unable to pay, each of the others had to pay \$4 more than his share. Find the cost of the boat.

Let $x =$ the number of dollars in the share of each person.

Then, $12x =$ the number of dollars subscribed by all.

Also, $x + 4 =$ the number of dollars in the increased share of each person,

and $10(x + 4) =$ the number of dollars paid by all.

But the number of dollars subscribed was equal to the number paid.

$$\therefore\ 12x = 10(x + 4)$$
$$12x = 10x + 40$$
$$2x = 40$$
$$x = 20$$
$$12x = 240$$

Therefore the price of the boat was \$240.

35. A man was hired for 26 days on condition that for every day he worked he was to receive $3, and for every day he was idle he was to pay $1 for his board. At the end of the time he received $58. How many days did he work?

Let $x =$ the number of days he worked.
Then, $26 - x =$ the number of days he was idle,
and $3x =$ the number of dollars due for his work.
$26 - x =$ the number of dollars he paid for his board.
$3x - (26 - x) =$ the number of dollars he actually received.
But $58 =$ the number of dollars he actually received.

$$\therefore\ 3x - (26 - x) = 58$$
$$3x - 26 + x = 58$$
$$4x = 84$$
$$x = 21$$

Therefore he worked 21 days.

36. A man walking 4 miles an hour starts 2 hours after another person, who walks 3 miles an hour. How many miles must the first man walk to overtake the second?

Let $x =$ the number of hours the second man has walked when he is overtaken.
Then, $x - 2 =$ the number of hours the first man has walked,
and $3x =$ the number of miles the second man has walked.
$4(x - 2) =$ the number of miles the first man has walked.
But both have walked the same distance.

$$\therefore\ 3x = 4(x - 2)$$
$$3x = 4x - 8$$
$$-x = -8$$
$$\therefore\ x = 8$$
$$3x = 24$$

Therefore the second man must walk 24 miles to overtake the first.

37. A man swimming in a river which runs 1 mile an hour finds that it takes him three times as long to swim a mile up the river as it does to swim the same distance down. Find his rate of swimming in still water.

Let $x =$ the number of miles he can swim in one hour in still water.

Then, $x - 1 =$ the number of miles he can swim in one hour against the current,

and $x + 1 =$ the number of miles he can swim in one hour with the current.

But he swims 3 times as fast with the current as against it.

$$\therefore\ x + 1 = 3(x - 1)$$
$$x + 1 = 3x - 3$$
$$-2x = -4$$
$$x = 2$$

Therefore he can swim 2 miles an hour in still water.

38. At an election there were two candidates, and 2644 votes were cast. The successful candidate had a majority of 140. How many votes were cast for each?

Let $x =$ the number of votes the successful candidate received.

Then, $2644 - x =$ the number of votes the defeated candidate received,

and $x - (2644 - x) =$ the majority of the successful candidate.

But, $140 =$ his majority.

$$\therefore\ x - (2644 - x) = 140$$
$$x - 2644 + x = 140$$
$$2x = 2784$$
$$x = 1392$$
$$2644 - x = 1252$$

Therefore the candidates received 1392 and 1252 votes respectively.

39. Two persons start from towns 55 miles apart and walk toward each other. One walks at the rate of 4 miles an hour, but stops 2 hours on the way; the other walks at the rate of 3 miles an hour. How many miles will each have travelled when they meet?

Let $x =$ the number of hours the two persons will have travelled,

and $x - 2 =$ the number of hours the first person actually walked.

$4(x - 2) =$ the number of miles the first person will have walked.

$3x =$ the number of miles the second person has walked.

$4(x-2)+3x =$ the number of miles both have walked.

But, $55 =$ the number of miles both have walked.

$$\therefore 4(x-2)+3x=55$$
$$4x-8+3x=55$$
$$7x=63$$
$$x=9$$
$$4(x-2)=28$$
$$3x=27$$

Therefore the first person will have walked 28, the second 27, miles.

40. A had twice as much money as B; but if A gives B $10, B will have three times as much as A. How much has each?

Let $x =$ the number of dollars B has.

Then, $2x =$ the number of dollars A has.

$x+10 =$ the number of dollars B will have if A gives him $10.

$2x-10 =$ the number of dollars A will have if he gives B $10.

But the former number is 3 times the latter.

$$\therefore x+10=3(2x-10)$$
$$x+10=6x-30$$
$$-5x=-40$$
$$\therefore x=8$$
$$2x=16$$

Therefore A has $16, and B $8.

41. If $2x-8$ stands for 20, for what number will $4-x$ stand?

$$2x-8=20$$
$$2x=28$$
$$\therefore x=14$$
$$4-x=4-14$$
$$=-10$$

Therefore $4-x$ stands for -10.

42. A vessel containing 100 gallons was emptied in 10 minutes by two pipes running one at a time. The first pipe discharged 14 gallons a minute, and the second 9 gallons a minute. How many minutes did each pipe run?

Let $x =$ the number of minutes the first pipe runs.
Then, $10 - x =$ the number of minutes the second pipe runs,
and $14x =$ the number of gallons discharged by the first pipe.
$9(10 - x) =$ the number of gallons discharged by the second pipe.
Hence, $14x + 9(10 - x) =$ the entire number of gallons discharged.
But, $100 =$ the entire number of gallons discharged.

$$\therefore\ 14x + 9(10 - x) = 100$$
$$14x + 90 - 9x = 100$$
$$5x = 10$$
$$\therefore\ x = 2$$
$$10 - x = 8$$

Therefore the first pipe runs 2 minutes, and the second 8 minutes.

43. A man has 8 hours for an excursion. How far can he ride out in a carriage which goes at the rate of 9 miles an hour so as to return in time, walking at the rate of 3 miles an hour?

Let $x =$ the number of hours he may ride out.
Then, $8 - x =$ the number of hours for the return journey,
and $9x =$ the number of miles he may ride out.
$3(8 - x) =$ the number of miles he walks back.

But the last two numbers are equal.

$$\therefore\ 9x = 3(8 - x)$$
$$9x = 24 - 3x$$
$$12x = 24$$
$$\therefore\ x = 2$$
$$9x = 18$$

Therefore he may ride out 18 miles.

44. If $3x - 4a = 2x - a$, find the number for which $4x - 7a$ stands.

$$3x - 4a = 2x - a$$
$$x = 3a$$
$$\therefore\ 4x - 7a = 12a - 7a$$
$$= 5a$$

Therefore $4x - 7a$ stands for $5a$.

45. If $7x - a = 9(x - a)$, find the number $\frac{2x - 3a}{x + a}$.

$$7x - a = 9(x - a)$$
$$7x - a = 9x - 9a$$
$$-2x = -8a$$
$$x = 4a$$

$$\therefore \frac{2x-3a}{x+a}=\frac{8a-3a}{4+a}$$
$$=\frac{5a}{5a}$$
$$=1$$

Therefore $\frac{2x-3a}{x+a}$ stands for 1.

46. A man is now twice as old as his son; 15 years ago he was three times as old as his son. Find the age of each.

Let $x =$ the number of years in the son's age.
Then, $2x =$ the number of years in the father's age,
and $x - 15 =$ the number of years in the son's age 15 years ago.
$2x - 15 =$ the number of years in the father's age 15 years ago.

But the father's age 15 years ago was 3 times the son's age.

$$\therefore 2x-15=3(x-15)$$
$$2x-15=3x-45$$
$$-x=-30$$
$$x=30$$
$$2x=60$$

Therefore the son is 30 years old, and the father 60 years old.

47. A man was four times as old as his son 7 years ago, and will be only twice as old as his son 7 years hence. Find the age of each.

Let $x =$ the number of years in the son's age at present.
Then, $x - 7 =$ the number of years in the son's age 7 years ago.
$x + 7 =$ the number of years in the son's age 7 years hence.
Also, $4(x-7) =$ the number of years in the father's age 7 years ago.
$2(x+7) =$ the number of years in the father's age 7 years hence.

But the last number is 14 greater than the preceding one.

$$\therefore 2(x+7)=14+4(x-7)$$
$$2x+14=14+4x-28$$
$$-2x=-28$$
$$x=14$$
$$4(x-7)+7=4\times 7+7$$
$$=35$$

Therefore the son is 14 years old, and the father 35 years old.

48. A, who is 25 years older than B, is 5 years more than twice as old as B. Find the age of each.

Let $x =$ the number of years in B's age.
Then, $x + 25 =$ the number of years in A's age.
Also, $2x + 5 =$ the number of years in A's age.

$$\begin{aligned} \therefore\ x + 25 &= 2x + 5 \\ -x &= -20 \\ x &= 20 \\ x + 25 &= 45 \end{aligned}$$

Therefore A is 45 years old, and B 20 years old.

49. A man is 25 years older than his son; 10 years ago he was six times as old as his son. Find the age of each.

Let $x =$ the number of years in the son's age at present.
Then, $x + 25 =$ the number of years in the father's age at present,
and $x - 10 =$ the number of years in the son's age 10 years ago.
Hence, $6(x - 10) =$ the number of years in the father's age 10 years ago.
But, $x + 25 - 10 =$ the number of years in the father's age 10 years ago.

$$\begin{aligned} \therefore\ x + 25 - 10 &= 6(x - 10) \\ x + 15 &= 6x - 60 \\ -5x &= -75 \\ x &= 15 \\ x + 25 &= 40 \end{aligned}$$

Therefore the son is 15 years old, and the father 40 years old.

50. The difference in the squares of two consecutive numbers is 19. Find the numbers.

Let $x =$ the first number.
Then, $x + 1 =$ the second number
and $x^2 =$ the square of the first number.
$(x + 1)^2 =$ the square of the second number.
$(x + 1)^2 - x^2 =$ the difference of the two squares.
But, $19 =$ this difference.

$$\begin{aligned} \therefore\ (x + 1)^2 - x^2 &= 19 \\ x^2 + 2x + 1 - x^2 &= 19 \\ 2x &= 18 \\ x &= 9 \\ x + 1 &= 10 \end{aligned}$$

Therefore the numbers are 9 and 10.

51. The difference in the squares of two successive odd numbers is 40. Find the numbers.

Let $x =$ the first odd number.

Then, $x + 2 =$ the second odd number.

$x^2 =$ the square of the first number.

$(x + 2)^2 =$ the square of the second number.

$(x + 2)^2 - x^2 =$ the difference of the squares.

But, $40 =$ this difference.

$$\begin{aligned} \therefore\ (x + 2)^2 - x^2 &= 40 \\ x^2 + 4x + 4 - x^2 &= 40 \\ 4x + 4 &= 40 \\ 4x &= 36 \\ x &= 9 \\ x + 2 &= 11 \end{aligned}$$

Therefore the numbers are 9 and 11.

Exercise 19. Page 75.

Write the product of

1. $(x + y)^2$.
   $x^2 + 2xy + y^2$.
2. $(x - a)^2$.
   $x^2 - 2ax + a^2$.
3. $(x + 2b)^2$.
   $x^2 + 4bx + 4b^2$.
4. $(3x - 2c)^2$.
   $9x^2 - 12cx + 4c^2$.
5. $(4y - 5)^2$.
   $16y^2 - 40y + 25$.
6. $(3a^2 + 4z^2)^2$.
   $9a^4 + 24a^2z^2 + 16z^4$.
7. $(x + y)(x - y)$.
   $x^2 - y^2$.
8. $(4z - 3)(4z + 3)$.
   $16z^2 - 9$.
9. $(3a^2 + 4b^2)(3a^2 - 4b^2)$.
   $9a^4 - 16b^4$.
10. $(3a - c)(3a - c)$.
    $9a^2 - 6ac + c^2$.
11. $(x + 7b^2)(x + 7b^2)$.
    $x^2 + 14b^2x + 49b^4$.
12. $(ax + 2by)(ax - 2by)$.
    $a^2x^2 - 4b^2y^2$.

Exercise 20. Page 76.

Find the product of

1. $x + y + z$ and $x - y - z$.

$$\begin{aligned} (x + y + z)(x - y - z) &= [x + (y + z)][x - (y + z)] \\ &= x^2 - (y + z)^2 \\ &= x^2 - (y^2 + 2yz + z^2) \\ &= x^2 - y^2 - 2yz - z^2. \end{aligned}$$

2. $x-y+z$ and $x-y-z$.

$$\begin{aligned}(x-y+z)(x-y-z)&=[(x-y)+z][(x-y)-z]\\&=(x-y)^2-z^2\\&=x^2-2xy+y^2-z^2.\end{aligned}$$

3. $ax+by+1$ and $ax+by-1$.

$$\begin{aligned}(ax+by+1)(ax+by-1)&=(ax+by)^2-1\\&=a^2x^2+2abxy+b^2y^2-1.\end{aligned}$$

4. $1+x-y$ and $1-x+y$.

$$\begin{aligned}(1+x-y)(1-x+y)&=[1+(x-y)][1-(x-y)]\\&=1-(x-y)^2\\&=1-x^2+2xy-y^2.\end{aligned}$$

5. $a+2b-3c$ and $a-2b+3c$.

$$\begin{aligned}(a+2b-3c)(a-2b+3c)&=[a+(2b-3c)][a-(2b-3c)]\\&=a^2-(2b-3c)^2\\&=a^2-(4b^2-12bc+9c^2)\\&=a^2-4b^2+12bc-9c^2.\end{aligned}$$

6. a^2-ab+b^2 and a^2+ab+b^2.

$$\begin{aligned}(a^2-ab+b^2)(a^2+ab+b^2)&=[(a^2+b^2)-ab][(a^2+b^2)+ab]\\&=(a^2+b^2)^2-a^2b^2\\&=a^4+2a^2b^2+b^4-a^2b^2\\&=a^4+a^2b^2+b^4.\end{aligned}$$

7. m^2+mn+n^2 and m^2-mn+n^2

$$\begin{aligned}(m^2+mn+n^2)(m^2-mn+n^2)&=[(m^2+n^2)+mn][(m^2+n^2)-mn]\\&=(m^2+n^2)^2-m^2n^2\\&=m^4+2m^2n^2+n^4-m^2n^2\\&=m^4+m^2n^2+n^4.\end{aligned}$$

8. $2+x+x^2$ and $2-x-x^2$.

$$\begin{aligned}(2+x+x^2)(2-x-x^2)&=[2+(x+x^2)][2-(x+x^2)]\\&=4-(x+x^2)^2\\&=4-(x^2+2x^3+x^4)\\&=4-x^2-2x^3-x^4.\end{aligned}$$

9. a^2+a+1 and a^2-a+1.

$$\begin{aligned}(a^2+a+1)(a^2-a+1)&=[(a^2+1)+a][(a^2+1)-a]\\&=(a^2+1)^2-a^2\\&=a^4+2a^2+1-a^2\\&=a^4+a^2+1.\end{aligned}$$

10. $3x+2y-z$ and $3x-2y+z$.

$$\begin{aligned}(3x+2y-z)(3x-2y+z) &= [3x+(2y-z)][3x-(2y-z)]\\ &= 9x^2-(2y-z)^2\\ &= 9x^2-(4y^2-4yz+z^2)\\ &= 9x^2-4y^2+4yz-z^2.\end{aligned}$$

Exercise 21. Page 77.

Write the square of

1. $2x-3y$.
$4x^2-12xy+9y^2$.

2. $a+b+c$.
$a^2+b^2+c^2+2ab+2ac+2bc$.

3. $x+y-z$.
$x^2+y^2+z^2+2xy-2xz-2yz$.

4. $x-y+z$.
$x^2+y^2+z^2-2xy+2xz-2yz$.

5. $x+y+5$.
$x^2+y^2+2xy+10x+10y+25$.

6. $x+2y+3$.
$x^2+4y^2+4xy+6x+12y+9$.

7. $a-b+c$.
$a^2+b^2+c^2-2ab+2ac-2bc$.

8. $3x-2y+4$.
$9x^2+4y^2-12xy+24x-16y+16$.

9. $2x-3y+4z$.
$4x^2+9y^2+16z^2-12xy+16xz-24yz$.

10. $x^2+y^2+z^2$.
$x^4+y^4+z^4+2x^2y^2+2x^2z^2+2y^2z^2$.

11. $2x-y-z$.
$4x^2+y^2+z^2-4xy-4xz+2yz$.

12. $a-2b-3c$.
$a^2+4b^2+9c^2-4ab-6ac+12bc$.

13. $3a-b+2c$.
$9a^2+b^2+4c^2-6ab+12ac-4bc$.

14. $x+2y-3z$.
$x^2+4y^2+9z^2+4xy-6xz-12yz$.

15. $x+y+z+1$.
$x^2+y^2+z^2+2xy+2xz+2x+2yz+2y+2z+1$.

16. $4x + y + z - 2.$

$$16x^2 + y^2 + z^2 + 8xy + 8xz - 16x + 2yz - 4y - 4z + 4.$$

17. $2x - y - z - 3.$

$$4x^2 + y^2 + z^2 - 4xy - 4xz - 12x + 2yz + 6y + 6z + 9.$$

18. $x - 2y - 3z + 4.$

$$x^2 + 4y^2 + 9z^2 - 4xy - 6xz + 8x + 12yz - 16y - 24z + 16.$$

Exercise 22. Page 79.

Find by inspection the product of

1. $(x + 8)(x + 3).$

$$8 + 3 = 11, \quad 8 \times 3 = 24.$$
$$\therefore (x + 8)(x + 3) = x^2 + 11x + 24.$$

2. $(x + 8)(x - 3).$

$$8 - 3 = 5, \quad 8 \times (-3) = -24.$$
$$\therefore (x + 8)(x - 3) = x^2 + 5x - 24.$$

3. $(x - 7)(x + 10).$

$$-7 + 10 = 3, \quad (-7) \times 10 = -70.$$
$$\therefore (x - 7)(x + 10) = x^2 + 3x - 70.$$

4. $(x - 9)(x - 5).$

$$-9 - 5 = -14, \quad (-9) \times (-5) = 45.$$
$$\therefore (x - 9)(x - 5) = x^2 - 14x + 45.$$

5. $(x - 10)(x + 9).$

$$-10 + 9 = -1, \quad (-10) \times 9 = -90.$$
$$\therefore (x - 10)(x + 9) = x^2 - x - 90.$$

6. $(a - 10)(a - 5).$

$$-10 - 5 = -15, \quad (-10) \times (-5) = 50.$$
$$\therefore (a - 10)(a - 5) = a^2 - 15a + 50.$$

7. $(x - 3a)(x + 2a).$

$$-3a + 2a = -a, \quad (-3a) \times (2a) = -6a^2.$$
$$\therefore (x - 3a)(x + 2a) = x^2 - ax - 6a^2.$$

8. $(a + 2b)(a - 4b).$

$$2b - 4b = -2b, \quad (2b) \times (-4b) = -8b^2.$$
$$\therefore (a + 2b)(a - 4b) = a^2 - 2ab - 8b^2.$$

9. $(a-12)(a-3)$.

$$-12-3=-15, \quad (-12)\times(-3)=36.$$

$$\therefore (a-12)(a-3)=a^2-15a+36.$$

10. $(a+2b)(a+4b)$.

$$2b+4b=6b, \quad (2b)\times(4b)=8b^2.$$

$$\therefore (a+2b)(a+4b)=a^2+6ab+8b^2.$$

11. $(a-3b)(a+7b)$.

$$-3b+7b=4b, \quad (-3b)\times(7b)=-21b^2.$$

$$\therefore (a-3b)(a+7b)=a^2+4ab-21b^2.$$

12. $(a+2b)(a-9b)$.

$$2b-9b=-7b, \quad (2b)\times(-9b)=-18b^2.$$

$$\therefore (a+2b)(a-9b)=a^2-7ab-18b^2.$$

13. $(x-3a)(x-4a)$.

$$-3a-4a=-7a, \quad (-3a)\times(-4a)=12a^2.$$

$$\therefore (x-3a)(x-4a)=x^2-7ax+12a^2.$$

14. $(x+4z)(x-2z)$.

$$4z-2z=2z, \quad (4z)\times(-2z)=-8z^2.$$

$$\therefore (x+4z)(x-2z)=x^2+2xz-8z^2.$$

15. $(x+6y)(x-5y)$.

$$6y-5y=y, \quad (6y)\times(-5y)=-30y^2.$$

$$\therefore (x+6y)(x-5y)=x^2+xy-30y^2.$$

16. $(x^2-9)(x^2+8)$.

$$-9+8=-1, \quad (-9)\times 8=-72.$$

$$\therefore (x^2-9)(x^2+8)=x^4-x^2-72.$$

17. $(x^2+2y^2)(x^2-3y^2)$.

$$2y^2-3y^2=-y^2, \quad (2y^2)\times(-3y^2)=-6y^4$$

$$\therefore (x^2+2y^2)(x^2-3y^2)=x^4-x^2y^2-6y^4.$$

18. $(x^2+8y^2)(x^2-4y^2)$.

$$8y^2-4y^2=4y^2, \quad (8y^2)\times(-4y^2)=-32y^4.$$

$$\therefore (x^2+8y^2)(x^2-4y^2)=x^4+4x^2y^2-32y^4.$$

19. $(ab-8)(ab+5)$.

$$-8+5=-3, \quad (-8)\times 5=-40,$$

$$\therefore (ab-8)(ab+5)=a^2b^2-3ab-40.$$

20. $(ab - 7xy)(ab + 3xy)$.

$$-7xy + 3xy = -4xy, \quad (-7xy) \times (3xy) = -21x^2y^2.$$

$$\therefore\ (ab - 7xy)(ab + 3xy) = a^2b^2 - 4abxy - 21x^2y^2.$$

21. $(x - 3y)(x - 3y)$.

$$x^2 - 6xy + 9y^2.$$

22. $(x + 6)(x + 6)$.

$$x^2 + 12x + 36.$$

23. $(a - 3b)(a - 3b)$.

$$a^2 - 6ab + 9b^2.$$

24. $(x - c)(x - d)$.

$$x^2 - (c + d)x + cd.$$

25. $(x + a)(x - b)$.

$$x^2 + (a - b)x - ab.$$

26. $(x - a)(x + b)$.

$$x^2 - (a - b)x - ab.$$

27. $[(a + b) + 2][(a + b) - 4]$.

$$2 - 4 = -2, \quad 2 \times (-4) = -8.$$

$$\therefore\ [(a + b) + 2][(a + b) - 4] = (a + b)^2 - 2(a + b) - 8.$$

28. $[(x + y) - 2][(x + y) + 4]$.

$$-2 + 4 = 2, \quad (-2) \times 4 = -8.$$

$$\therefore\ [(x + y) - 2][(x + y) + 4] = (x + y)^2 + 2(x + y) - 8.$$

29. $(x + y - 7)(x + y + 10)$.

$$-7 + 10 = 3, \quad -7 \times 10 = -70.$$

$$\begin{aligned}\therefore\ (x + y - 7)(x + y + 10) &= [(x + y) - 7][(x + y) + 10] \\ &= (x + y)^2 + 3(x + y) - 70.\end{aligned}$$

30. $(x - y - 7)(x - y - 10)$.

$$-7 - 10 = -17, \quad (-7) \times (-10) = 70.$$

$$\begin{aligned}\therefore\ (x - y - 7)(x - y - 10) &= [(x - y) - 7][(x - y) - 10] \\ &= (x - y)^2 - 17(x - y) + 70.\end{aligned}$$

Exercise 23. Page 80.

Find the product of

1. $3x - y$ and $2x + y$.

$$\begin{aligned}(3x - y)(2x + y) &= 6x^2 + 3xy - 2xy - y^2 \\ &= 6x^2 + xy - y^2.\end{aligned}$$

2. $4x - 3y$ and $3x - 2y$.

$$\begin{aligned}(4x - 3y)(3x - 2y) &= 12x^2 - 8xy - 9xy + 6y^2 \\ &= 12x^2 - 17xy + 6y^2.\end{aligned}$$

3. $5x-4y$ and $3x-4y$.

$$\begin{aligned}(5x-4y)(3x-4y) &= 15x^2-20xy-12xy+16y^2\\ &= 15x^2-32xy+16y^2.\end{aligned}$$

4. $x-7y$ and $2x-5y$.

$$\begin{aligned}(x-7y)(2x-5y) &= 2x^2-5xy-14xy+35y^2\\ &= 2x^2-19xy+35y^2.\end{aligned}$$

5. $11x-2y$ and $7x+y$.

$$\begin{aligned}(11x-2y)(7x+y) &= 77x^2+11xy-14xy-2y^2\\ &= 77x^2-3xy-2y^2.\end{aligned}$$

6. $10x-3y$ and $10x-7y$.

$$\begin{aligned}(10x-3y)(10x-7y) &= 100x^2-70xy-30xy+21y^2\\ &= 100x^2-100xy+21y^2.\end{aligned}$$

7. $3a^2-2b^2$ and $2a^2+3b^2$.

$$\begin{aligned}(3a^2-2b^2)(2a^2+3b^2) &= 6a^4+9a^2b^2-4a^2b^2-6b^4\\ &= 6a^4+5a^2b^2-6b^4.\end{aligned}$$

8. a^2+b^2 and $a-b$.

$$(a^2+b^2)(a-b)=a^3-a^2b+ab^2-b^3.$$

9. $3a^2-2b^2$ and $2a+3b$.

$$(3a^2-2b^2)(2a+3b)=6a^3+9a^2b-4ab^2-6b^3.$$

10. a^2-b^2 and $a+b$.

$$(a^2-b^2)(a+b)=a^3+a^2b-ab^2-b^3.$$

Exercise 24. Page 81.

Write by inspection the quotient of

1. $\dfrac{a^2-4}{a-2}$.

$a+2$.

2. $\dfrac{9-x^2}{3+x}$.

$3-x$.

3. $\dfrac{16-a^2}{4+a}$.

$4-a$.

4. $\dfrac{x^2-25}{x-5}$.

$x+5$.

5. $\dfrac{36-x^2}{6+x}$.

$6-x$.

6. $\dfrac{9a^2-b^2}{3a-b}$.

$3a+b$.

7. $\frac{25x^2-36b^2}{5x+6b}$.

$5x-6b$.

8. $\frac{49c^2-d^2}{7c-d}$.

$7c+d$.

9. $\frac{9a^2-1}{3a+1}$.

$3a-1$.

10. $\frac{16-4a^2}{4-2a}$.

$4+2a$.

11. $\frac{9a^4-25y^4}{3a^2+5y^2}$.

$3a^2-5y^2$.

12. $\frac{4x^{16}-9y^6}{2x^8-3y^8}$.

$2x^8+3y^8$.

13. $\frac{4x^{10}-a^8}{2x^5-a^4}$.

$2x^5+a^4$.

14. $\frac{a^2b^6c^8-x^{12}}{ab^3c^4+x^6}$.

$ab^3c^4-x^6$.

15. $\frac{x^4a^8-b^{10}}{x^2a^4-b^5}$.

$x^2a^4+b^5$.

16. $\frac{a^2-(b+c)^2}{a-(b+c)}$.

$a+b+c$.

17. $\frac{a^2-(3b-4c)^2}{a-(3b-4c)}$.

$a+(3b-4c)=a+3b-4c$.

18. $\frac{1-(x-y)^2}{1+(x-y)}$.

$1-(x-y)=1-x+y$.

19. $\frac{(3x-y)^2-16z^2}{(3x-y)+4z}$.

$(3x-y)-4z=3x-y-4z$.

20. $\frac{(x+3a)^2-9x^2}{(x+3a)-3x}$.

$(x+3a)+3x=4x+3a$.

21. $\frac{1-(7a-5b)^2}{1+(7a-5b)}$.

$1-(7a-5b)=1-7a+5b$.

22. $\frac{(3x+2y)^2-4z^2}{(3x+2y)-2z}$.

$(3x+2y)+2z=3x+2y+2z$.

23. $\frac{(a-b)^2-(c-y)^2}{(a-b)+(c-y)}$.

$(a-b)-(c-y)=a-b-c+y$.

24. $\frac{x^2-(y+z)^2}{x-(y+z)}$.

$x+(y+z)=x+y+z$.

25. $\frac{(x-2y)^2-25}{(x-2y)+5}$.

$(x-2y)-5=x-2y-5$.

26. $\frac{(2x+y)^2-9z^4}{(2x+y)-3z^2}$.

$(2x+y)+3z^2=2x+y+3z^2$.

Exercise 25. Page 82.

Write by inspection the quotient of

1. $\frac{1-8x^3}{1-2x}$.

$1+2x+4x^2$.

2. $\frac{1+8x^3}{1+2x}$.

$1-2x+4x^2$.

3. $\frac{27a^3-b^3}{3a-b}$.

$9a^2+3ab+b^2$.

4. $\frac{27a^3+b^3}{3a+b}$.

$9a^2-3ab+b^2$.

5. $\frac{64x^3+27y^3}{4x+3y}$.

$16x^2-12xy+9y^2$.

6. $\frac{64x^3-27y^3}{4x-3y}$.

$16x^2+12xy+9y^2$.

7. $\frac{1-27z^3}{1-3z}$.

$1+3z+9z^2$.

8. $\frac{1+27z^3}{1+3z}$.

$1-3z+9z^2$.

9. $\frac{a^3b^3-c^3}{ab-c}$.

$a^2b^2+abc+c^2$.

10. $\frac{a^3b^3+c^3}{ab+c}$.

$a^2b^2-abc+c^2$.

11. $\frac{64+y^3}{4+y}$.

$16-4y+y^2$.

12. $\frac{343-8a^3}{7-2a}$.

$49+14a+4a^2$.

13. $\frac{8a^3+b^6}{2a+b^2}$.

$4a^2-2ab^2+b^4$.

14. $\frac{x^6+729y^3}{x^2+9y}$.

$x^4-9x^2y+81y^2$.

15. $\frac{a^6-27b^3}{a^2-3b}$.

$a^4+3a^2b+9b^2$.

16. $\frac{8x^3-64y^6}{2x-4y^2}$.

$4x^2+8xy^2+16y^4$.

Exercise 26. Page 83

Find the quotient of

1. $\frac{x^6-y^6}{x-y}$.

$x^5+x^4y+x^3y^2+x^2y^3+xy^4+y^5$.

2. $\frac{x^6-y^6}{x+y}$.

$x^5-x^4y+x^3y^2-x^2y^3+xy^4-y^5$.

3. $\frac{x^4-1}{x-1}$.

x^3+x^2+x+1.

4. $\frac{x^4-1}{x+1}$.

x^3-x^2+x-1.

5. $\frac{x^4-16}{x-2}$.

x^3+2x^2+4x+8.

6. $\frac{x^5-32}{x-2}$.

$x^4+2x^3+4x^2+8x+16$.

7. $\frac{x^5+32}{x+2}$.

$x^4-2x^3+4x^2-8x+16$.

8. $\frac{1-m^4}{1-m}$.

$1+m+m^2+m^3$.

9. $\frac{1+m^5}{1+m}$.

$1-m+m^2-m^3+m^4$.

Exercise 27. Page 85.

Resolve into two factors:

1. $3x^2-6x^3$.

$3x^2-6x^3=3x^2(1-2x)$.

2. $2a^2-4a$.

$2a^2-4a=2a(a-2)$.

3. $5ab-5a^2b^3$.

$5ab-5a^2b^3=5ab(1-ab^2)$.

4. $3a^3-a^2+a$.

$3a^3-a^2+a=a(3a^2-a+1)$.

5. $x^3+x^2y-xy^2$.

$x^3+x^2y-xy^2=x(x^2+xy-y^2)$.

6. $a^4-a^3b+a^2b^2$.

$a^4-a^3b+a^2b^2=a^2(a^2-ab+b^2)$.

7. $3a^2b-4ab^2$.

$3a^2b-4ab^2=ab(3a-4b)$.

8. $8x^3y^2+4x^2y^3$.

$8x^3y^2+4x^2y^3=4x^2y^2(2x+y)$.

9. $3x^4-9x^2-6x^3$.

$3x^4-9x^2-6x^3=3x^2(x^2-3-2x)$.

10. $8a^2x^2-4a^2b+12a^2y^2$.

$8a^2x^2-4a^2b+12a^2y^2=4a^2(2x^2-b+3y^2)$.

11. $8a^3b^2c^2-4a^2b^3c^3$.

$8a^3b^2c^2-4a^2b^3c^3=4a^2b^2c^2(2a-bc)$.

12. $15a^3x-10a^3y+5a^3z$.

$15a^3x-10a^3y+5a^3z=5a^3(3x-2y+z)$.

Exercise 28. Page 86.

Resolve into factors:

1. $ax - bx + ay - by.$

$$\begin{aligned} ax - bx + ay - by &= (ax - bx) + (ay - by) \\ &= x(a - b) + y(a - b) \\ &= (x + y)(a - b). \end{aligned}$$

2. $ax - bx - ay + by.$

$$\begin{aligned} ax - bx - ay + by &= (ax - bx) - (ay - by) \\ &= x(a - b) - y(a - b) \\ &= (x - y)(a - b). \end{aligned}$$

3. $ax - cy - ay + cx.$

$$\begin{aligned} ax - cy - ay + cx &= (ax + cx) - (ay + cy) \\ &= x(a + c) - y(a + c) \\ &= (x - y)(a + c). \end{aligned}$$

4. $2ab - 3ac - 2by + 3cy.$

$$\begin{aligned} 2ab - 3ac - 2by + 3cy &= (2ab - 3ac) - (2by - 3cy) \\ &= a(2b - 3c) - y(2b - 3c) \\ &= (a - y)(2b - 3c). \end{aligned}$$

5. $x^2 + ax - bx - ab.$

$$\begin{aligned} x^2 + ax - bx - ab &= (x^2 + ax) - (bx + ab) \\ &= x(x + a) - b(x + a) \\ &= (x - b)(x + a). \end{aligned}$$

6. $x^2 + xy - ax - ay.$

$$\begin{aligned} x^2 + xy - ax - ay &= (x^2 + xy) - (ax + ay) \\ &= x(x + y) - a(x + y) \\ &= (x - a)(x + y). \end{aligned}$$

7. $x^2 - xy - 6x + 6y.$

$$\begin{aligned} x^2 - xy - 6x + 6y &= (x^2 - xy) - (6x - 6y) \\ &= x(x - y) - 6(x - y) \\ &= (x - 6)(x - y). \end{aligned}$$

8. $2x^2 - 3xy + 4ax - 6ay.$

$$\begin{aligned} 2x^2 - 3xy + 4ax - 6ay &= (2x^2 - 3xy) + (4ax - 6ay) \\ &= x(2x - 3y) + 2a(2x - 3y) \\ &= (x + 2a)(2x - 3y). \end{aligned}$$

9. $a^2b - abx - ac + cx.$

$$\begin{aligned} a^2b - abx - ac + cx &= (a^2b - abx) - (ac - cx) \\ &= ab(a - x) - c(a - x) \\ &= (ab - c)(a - x). \end{aligned}$$

10. $a^2bx + b^2cx - a^2cy - bc^2y.$

$$\begin{aligned} a^2bx + b^2cx - a^2cy - bc^2y &= (a^2bx + b^2cx) - (a^2cy + bc^2y) \\ &= bx(a^2 + bc) - cy(a^2 + bc) \\ &= (bx - cy)(a^2 + bc). \end{aligned}$$

11. $3x^2 - 5y^2 - 6x^3 + 10xy^2.$

$$\begin{aligned} 3x^2 - 5y^2 - 6x^3 + 10xy^2 &= (3x^2 - 6x^3) - (5y^2 - 10xy^2) \\ &= 3x^2(1 - 2x) - 5y^2(1 - 2x) \\ &= (3x^2 - 5y^2)(1 - 2x). \end{aligned}$$

12. $8ax - 10bx - 12a + 15b.$

$$\begin{aligned} 8ax - 10bx - 12a + 15b &= (8ax - 10bx) - (12a - 15b) \\ &= 2x(4a - 5b) - 3(4a - 5b) \\ &= (2x - 3)(4a - 5b). \end{aligned}$$

13. $2x^3 - 3x^2 - 4x + 6.$

$$\begin{aligned} 2x^3 - 3x^2 - 4x + 6 &= (2x^3 - 3x^2) - (4x - 6) \\ &= x^2(2x - 3) - 2(2x - 3) \\ &= (x^2 - 2)(2x - 3). \end{aligned}$$

14. $6x^4 + 8x^3 - 9x^2 - 12x.$

$$\begin{aligned} 6x^4 + 8x^3 - 9x^2 - 12x &= (6x^4 + 8x^3) - (9x^2 + 12x) \\ &= 2x^3(3x + 4) - 3x(3x + 4) \\ &= (2x^3 - 3x)(3x + 4) \\ &= x(2x^2 - 3)(3x + 4). \end{aligned}$$

15. $ax^4 + bx^3 - ax - b.$

$$\begin{aligned} ax^4 + bx^3 - ax - b &= (ax^4 + bx^3) - (ax + b) \\ &= x^3(ax + b) - 1(ax + b) \\ &= (x^3 - 1)(ax + b) \\ &= (x - 1)(x^2 + x + 1)(ax + b). \end{aligned}$$

16. $3cx^4 - 2dx^3 - 9cx^2 + 6dx.$

$$\begin{aligned} 3cx^4 - 2dx^3 - 9cx^3 + 6dx &= (3cx^4 - 2dx^3) - (9cx^2 - 6dx) \\ &= x^3(3cx - 2d) - 3x^2(3cx - 2d) \\ &= (x^3 - 3x)(3cx - 2d) \\ &= x(x^2 - 3)(3cx - 2d). \end{aligned}$$

17. $1 + 15x^4 - 5x - 3x^3$.

$$\begin{aligned} 1 + 15x^4 - 5x - 3x^3 &= (1 - 5x) - (3x^3 - 15x^4) \\ &= 1(1 - 5x) - 3x^3(1 - 5x) \\ &= (1 - 3x^3)(1 - 5x). \end{aligned}$$

18. $ax^2 + a^2x + a + x$.

$$\begin{aligned} ax^2 + a^2x + a + x &= (ax^2 + a^2x) + (a + x) \\ &= ax(x + a) + (a + x) \\ &= (ax + 1)(a + x). \end{aligned}$$

19. $(a + b)(c + d) - 3c(a + b)$.

$$\begin{aligned} (a + b)(c + d) - 3c(a + b) &= (a + b)(c + d - 3c) \\ &= (a + b)(d - 2c). \end{aligned}$$

20. $(x - y)^2 + 2y(x - y)$.

$$\begin{aligned} (x - y)^2 + 2y(x - y) &= (x - y)(x - y + 2y) \\ &= (x - y)(x + y). \end{aligned}$$

Exercise 29. Page 88.

Resolve into factors:

1. $a^2 - 6ab + 9b^2$.
$(a - 3b)(a - 3b)$.

2. $4a^2 + 4ab + b^2$.
$(2a + b)(2a + b)$.

3. $a^2 - 4ab + 4b^2$.
$(a - 2b)(a - 2b)$.

4. $x^2 + 6xy + 9y^2$.
$(x + 3y)(x + 3y)$.

5. $4x^2 - 12ax + 9a^2$.
$(2x - 3a)(2x - 3a)$.

6. $a^2 - 10ab + 25b^2$.
$(a - 5b)(a - 5b)$.

7. $4a^2 - 4a + 1$.
$(2a - 1)(2a - 1)$.

8. $49y^2 - 14yz + z^2$.
$(7y - z)(7y - z)$.

9. $x^2 - 16x + 64$.
$(x - 8)(x - 8)$.

10. $9x^2 + 24xy + 16y^2$.
$(3x + 4y)(3x + 4y)$.

11. $16a^2 + 8ax + x^2$.
$(4a + x)(4a + x)$.

12. $25 + 80x + 64x^2$.
$(5 + 8x)(5 + 8x)$.

13. $49x^2 - 28xy + 4y^2$.
$(7x - 2y)(7x - 2y)$.

14. $1 - 20b + 100b^2$.
$(1 - 10b)(1 - 10b)$.

15. $81a^2 + 126ab + 49b^2$.
$(9a + 7b)(9a + 7b)$.

16. $m^2n^2 - 16mna^2 + 64a^4$.
$(mn - 8a^2)(mn - 8a^2)$.

17. $4a^2 - 20ax + 25x^2$.
$(2a - 5x)(2a - 5x)$.

18. $121a^2 + 198ay + 81y^2$.
$(11a + 9y)(11a + 9y)$.

19. $a^2b^4c^6 - 2ab^2c^3x^8 + x^{16}$.
$(ab^2c^3 - x^8)(ab^2c^3 - x^8)$.

20. $49 - 140k^2 + 100k^4$.
$(7 - 10k^2)(7 - 10k^2)$.

Exercise 30. Page 89.

Resolve into factors:

1. $a^2 - 4$.
$(a + 2)(a - 2)$.

2. $1 - x^2$.
$(1 + x)(1 - x)$.

3. $x^2 - 9y^2$.
$(x + 3y)(x - 3y)$.

4. $4a^2 - 49b^2$.
$(2a + 7b)(2a - 7b)$.

5. $x^2 - 4y^2$.
$(x + 2y)(x - 2y)$.

6. $25 - 16a^2$.
$(5 + 4a)(5 - 4a)$.

7. $16 - 25y^2$.
$(4 + 5y)(4 - 5y)$.

8. $a^2b^2 - 1$.
$(ab + 1)(ab - 1)$.

9. $x^2 - 100$.
$(x + 10)(x - 10)$.

10. $121a^2 - 36b^2$.
$(11a + 6b)(11a - 6b)$.

11. $81x^2 - 4y^2$.
$(9x + 2y)(9x - 2y)$.

12. $64a^4 - b^4$.
$(8a^2 + b^2)(8a^2 - b^2)$.

13. $m^2n^2 - 36$.
$(mn + 6)(mn - 6)$.

14. $x^4 - 144$.
$(x^2 + 12)(x^2 - 12)$.

15. $x^2 - 25$.
$(x + 5)(x - 5)$.

16. $49 - 100y^2$.
$(7 + 10y)(7 - 10y)$.

17. $1 - 49x^6$.
$(1 + 7x^3)(1 - 7x^3)$.

18. $4 - 121y^8$.
$(2 + 11y^4)(2 - 11y^4)$.

19. $1 - 169a^6$.
$(1 + 13a^3)(1 - 13a^3)$.

20. $a^2b^2 - 4c^6$.
$(ab + 2c^3)(ab - 2c^3)$.

21. $9x^8 - a^6$.
$(3x^4 + a^3)(3x^4 - a^3)$.

22. $4x^{16} - y^{20}$.
$(2x^8 + y^{10})(2x^8 - y^{10})$.

23. $49a^{14} - y^{12}$.
$(7a^7 + y^6)(7a^7 - y^6)$.

24. $64a^2 - 9b^6$.
$(8a + 3b^3)(8a - 3b^3)$.

25. $81a^4b^4 - c^4$.
$(9a^2b^2 + c^2)(9a^2b^2 - c^2)$.

26. $4a^2c - 9c^5$.
$c(2a + 3c^2)(2a - 3c^2)$.

27. $20a^3b^3 - 5ab$.
$5ab(2ab + 1)(2ab - 1)$.

28. $3a^5 - 12a^3c^2$.
$3a^3(a + 2c)(a - 2c)$.

29. $9a^2 - 81b^2$.
$(3a + 9b)(3a - 9b)$.

30. $25 - 64y^2$.
$(5 + 8y)(5 - 8y)$.

31. $16x^{17} - 9xy^6$.
$x(4x^8 + 3y^3)(4x^8 - 3y^3)$.

32. $25x^{10} - 16a^8x^8$.
$x^8(5x + 4a^4)(5x - 4a^4)$.

33. $36a^2x^2 - 49a^4$.
$a^2(6x + 7a)(6x - 7a)$.

34. $x^2 - 16y^2$.
$(x + 4y)(x - 4y)$.

35. $1 - 400x^4$.
$(1 + 20x^2)(1 - 20x^2)$.

36. $4a^2c - 9c^3$.
$c(2a + 3c)(2a - 3c)$.

Exercise 31. Page 91.

Resolve into factors:

1. $(x + y)^2 - z^2$.
$(x + y + z)(x + y - z)$.

2. $(x - y)^2 - z^2$.
$(x - y + z)(x - y - z)$.

3. $(x - 2y)^2 - 4z^2$.
$(x - 2y + 2z)(x - 2y - 2z)$.

4. $(a + 3b)^2 - 16c^2$.
$(a + 3b + 4c)(a + 3b - 4c)$.

5. $x^2 - (y - z)^2$.
$\{x + (y - z)\}\{x - (y - z)\} = (x + y - z)(x - y + z)$.

6. $a^2 - (3b - 2c)^2$.
$\{a + (3b - 2c)\}\{a - (3b - 2c)\} = (a + 3b - 2c)(a - 3b + 2c)$.

7. $b^2 - (2a + 3c)^2$.
$\{b + (2a + 3c)\}\{b - (2a + 3c)\} = (b + 2a + 3c)(b - 2a - 3c)$.

8. $1-(x+5b)^2$.

$\{1+(x+5b)\}\{1-(x+5b)\}=(1+x+5b)(1-x-5b)$.

9. $9a^2-(x-3c)^2$.

$\{3a+(x-3c)\}\{3a-(x-3c)\}=(3a+x-3c)(3a-x+3c)$.

10. $16a^2-(2y-3z)^2$.

$\{4a+(2y-3z)\}\{4a-(2y-3z)\}=(4a+2y-3z)(4a-2y+3z)$.

11. $(a-b)^2-(c-d)^2$.

$\{(a-b)+(c-d)\}\{(a-b)-(c-d)\}$
$=(a-b+c-d)(a-b-c+d)$.

12. $(2a+b)^2-25c^2$.

$(2a+b+5c)(2a+b-5c)$.

13. $(x+2y)^2-(2x-y)^2$.

$\{(x+2y)+(2x-y)\}\{(x+2y)-(2x-y)\}$
$=(x+2y+2x-y)(x+2y-2x+y)$
$=(3x+y)(3y-x)$.

14. $(x+3)^2-(3x-4)^2$.

$\{(x+3)+(3x-4)\}\{(x+3)-(3x-4)\}$
$=(x+3+3x-4)(x+3-3x+4)$
$=(4x-1)(7-2x)$.

15. $(a+b-c)^2-(a-b-c)^2$.

$\{(a+b-c)+(a-b-c)\}\{(a+b-c)-(a-b-c)\}$
$=(a+b-c+a-b-c)(a+b-c-a+b+c)$
$=(2a-2c)(2b)$
$=4b(a-c)$.

16. $(a-3x)^2-(3a-2x)^2$.

$\{(a-3x)+(3a-2x)\}\{(a-3x)-(3a-2x)\}$
$=(a-3x+3a-2x)(a-3x-3a+2x)$
$=(4a-5x)(-2a-x)$
$=-(4a-5x)(2a+x)$
$=(5x-4a)(x+2a)$.

17. $(2a-1)^2-(3a+1)^2$.

$\{(2a-1)+(3a+1)\}\{(2a-1)-(3a+1)\}$
$=(2a-1+3a+1)(2a-1-3a-1)$
$=5a(-a-2)$
$=-5a(a+2)$.

18. $(x-5)^2-(x+y-5)^2.$

$$\begin{aligned}&\{(x-5)+(x+y-5)\}\{(x-5)-(x+y-5)\}\\&\qquad=(x-5+x+y-5)(x-5-x-y+5)\\&\qquad=(2x+y-10)(-y)\\&\qquad=-y(2x+y-10).\end{aligned}$$

19. $(2a+b-c)^2-(a-2b+c)^2.$

$$\begin{aligned}&\{(2a+b-c)+(a-2b+c)\}\{(2a+b-c)-(a-2b+c)\}\\&\qquad=(2a+b-c+a-2b+c)(2a+b-c-a+2b-c)\\&\qquad=(3a-b)(a+3b-2c)\end{aligned}$$

20. $(a+2b-3c)^2-(a+5c)^2.$

$$\begin{aligned}&\{(a+2b-3c)+(a+5c)\}\{(a+2b-3c)-(a+5c)\}\\&\qquad=(a+2b-3c+a+5c)(a+2b-3c-a-5c)\\&\qquad=(2a+2b+2c)(2b-8c)\\&\qquad=4(a+b+c)(b-4c).\end{aligned}$$

Exercise 32. Page 92.

Resolve into factors:

1. $a^2+2ab+b^2-4c^2.$

$$\begin{aligned}a^2+2ab+b^2-4c^2&=(a^2+2ab+b^2)-4c^2\\&=(a+b)^2-4c^2\\&=(a+b+2c)(a+b-2c).\end{aligned}$$

2. $x^2-2xy+y^2-9a^2.$

$$\begin{aligned}x^2-2xy+y^2-9a^2&=(x^2-2xy+y^2)-9a^2\\&=(x-y)^2-9a^2\\&=(x-y+3a)(x-y-3a).\end{aligned}$$

3. $b^2-x^2+4ax-4a^2.$

$$\begin{aligned}b^2-x^2+4ax-4a^2&=b^2-(x^2-4ax+4a^2)\\&=b^2-(x-2a)^2\\&=(b+x-2a)(b-x+2a).\end{aligned}$$

4. $4a^2+4ab+b^2-x^2.$

$$\begin{aligned}4a^2+4ab+b^2-x^2&=(4a^2+4ab+b^2)-x^2\\&=(2a+b)^2-x^2\\&=(2a+b+x)(2a+b-x).\end{aligned}$$

5. $a^2 - x^2 - y^2 - 2xy$.

$$\begin{aligned} a^2 - x^2 - y^2 - 2xy &= a^2 - (x^2 + 2xy + y^2) \\ &= a^2 - (x + y)^2 \\ &= (a + x + y)(a - x - y). \end{aligned}$$

6. $1 - a^2 - 2ab - b^2$.

$$\begin{aligned} 1 - a^2 - 2ab - b^2 &= 1 - (a^2 + 2ab + b^2) \\ &= 1 - (a + b)^2 \\ &= (1 + a + b)(1 - a - b). \end{aligned}$$

7. $a^2 + b^2 + 2ab - 16a^2b^2$.

$$\begin{aligned} a^2 + b^2 + 2ab - 16a^2b^2 &= (a^2 + 2ab + b^2) - 16a^2b^2 \\ &= (a + b)^2 - 16a^2b^2 \\ &= (a + b + 4ab)(a + b - 4ab). \end{aligned}$$

8. $4a^2 - 9a^2 + 6a - 1$.

$$\begin{aligned} 4a^2 - 9a^2 + 6a - 1 &= 4a^2 - (9a^2 - 6a + 1) \\ &= 4a^2 - (3a - 1)^2 \\ &= (2a + 3a - 1)(2a - 3a + 1) \\ &= (5a - 1)(1 - a). \end{aligned}$$

9. $a^2 + b^2 - c^2 - d^2 - 2ab - 2cd$.

$$\begin{aligned} a^2 + b^2 - c^2 - d^2 - 2ab - 2cd &= (a^2 - 2ab + b^2) - (c^2 + 2cd + d^2) \\ &= (a - b)^2 - (c + d)^2 \\ &= (a - b + c + d)(a - b - c - d). \end{aligned}$$

10. $x^2 + y^2 - 2xy - 2ab - a^2 - b^2$.

$$\begin{aligned} x^2 + y^2 - 2xy - 2ab - a^2 - b^2 &= (x^2 - 2xy + y^2) - (a^2 + 2ab + b^2) \\ &= (x - y)^2 - (a + b)^2 \\ &= (x - y + a + b)(x - y - a - b) \end{aligned}$$

11. $9x^2 - 6x + 1 - a^2 - 4ab - 4b^2$.

$$\begin{aligned} 9x^2 - 6x + 1 - a^2 - 4ab - 4b^2 &= (9x^2 - 6x + 1) - (a^2 + 4ab + 4b^2) \\ &= (3x - 1)^2 - (a + 2b)^2 \\ &= (3x - 1 + a + 2b)(3x - 1 - a - 2b). \end{aligned}$$

12. $a^2 + 2ab - x^2 - 6xy - 9y^2 + b^2$.

$$\begin{aligned} a^2 + 2ab - x^2 - 6xy - 9y^2 + b^2 &= (a^2 + 2ab + b^2) - (x^2 + 6xy + 9y^2) \\ &= (a + b)^2 - (x + 3y)^2 \\ &= (a + b + x + 3y)(a + b - x - 3y). \end{aligned}$$

13. $x^2-2x+1-b^2+2by-y^2.$

$$\begin{aligned} x^2-2x+1-b^2+2by-y^2 &= (x^2-2x+1)-(b^2-2by+y^2) \\ &= (x-1)^2-(b-y)^2 \\ &= (x-1+b-y)(x-1-b+y). \end{aligned}$$

14. $9-6x+x^2-a^2-8ab-16b^2.$

$$\begin{aligned} 9-6x+x^2-a^2-8ab-16b^2 &= (9-6x+x^2)-(a^2+8ab+16b^2) \\ &= (3-x)^2-(a+4b)^2 \\ &= (3-x+a+4b)(3-x-a-4b). \end{aligned}$$

15. $4-4x+x^2-4a-1-4a^2.$

$$\begin{aligned} 4-4x+x^2-4a-1-4a^2 &= (4-4x+x^2)-(1+4a+4a^2) \\ &= (2-x)^2-(1+2a)^2 \\ &= (2-x+1+2a)(2-x-1-2a) \\ &= (3-x+2a)(1-x-2a). \end{aligned}$$

16. $a^4-a^2-9+b^4+6a-2a^2b^2.$

$$\begin{aligned} a^4-a^2-9+b^4+6a-2a^2b^2 &= (a^4-2a^2b^2+b^4)-(a^2-6a+9) \\ &= (a^2-b^2)^2-(a-3)^2 \\ &= (a^2-b^2+a-3)(a^2-b^2-a+3). \end{aligned}$$

Exercise 33. Page 93.

Resolve into factors:

1. $x^4+x^2y^2+y^4.$

$$\begin{aligned} x^4+x^2y^2+y^4 &= (x^4+2x^2y^2+y^4)-x^2y^2 \\ &= (x^2+y^2)^2-x^2y^2 \\ &= (x^2+y^2+xy)(x^2+y^2-xy) \\ &= (x^2+xy+y^2)(x^2-xy+y^2). \end{aligned}$$

2. $x^4+x^2+1.$

$$\begin{aligned} x^4+x^2+1 &= (x^4+2x^2+1)-x^2 \\ &= (x^2+1)^2-x^2 \\ &= (x^2+1+x)(x^2+1-x) \\ &= (x^2+x+1)(x^2-x+1). \end{aligned}$$

3. $9a^4-15a^2+1.$

$$\begin{aligned} 9a^4-15a^2+1 &= (9a^4-6a^2+1)-9a^2 \\ &= (3a^2-1)^2-9a^2 \\ &= (3a^2-1+3a)(3a^2-1-3a) \\ &= (3a^2+3a-1)(3a^2-3a-1). \end{aligned}$$

4. $16a^4 - 17a^2 + 1$.

$$\begin{aligned}16a^4 - 17a^2 + 1 &= (16a^4 - 8a^2 + 1) - 9a^2\\ &= (4a^2 - 1)^2 - 9a^2\\ &= (4a^2 - 1 + 3a)(4a^2 - 1 - 3a)\\ &= (4a^2 + 3a - 1)(4a^2 - 3a - 1).\end{aligned}$$

5. $4a^4 - 13a^2 + 1$.

$$\begin{aligned}4a^4 - 13a^2 + 1 &= (4a^4 - 4a^2 + 1) - 9a^2\\ &= (2a^2 - 1)^2 - 9a^2\\ &= (2a^2 - 1 + 3a)(2a^2 - 1 - 3a)\\ &= (2a^2 + 3a - 1)(2a^2 - 3a - 1).\end{aligned}$$

6. $9a^4 + 26a^2b^2 + 25b^4$.

$$\begin{aligned}9a^4 + 26a^2b^2 + 25b^4 &= (9a^4 + 30a^2b^2 + 25b^4) - 4a^2b^2\\ &= (3a^2 + 5b^2)^2 - 4a^2b^2\\ &= (3a^2 + 5b^2 + 2ab)(3a^2 + 5b^2 - 2ab)\\ &= (3a^2 + 2ab + 5b^2)(3a^2 - 2ab + 5b^2).\end{aligned}$$

7. $4x^4 - 21x^2y^2 + 9y^4$.

$$\begin{aligned}4x^4 - 21x^2y^2 + 9y^4 &= 4x^4 - 12x^2y^2 + 9y^4 - 9x^2y^2\\ &= (2x^2 - 3y^2)^2 - 9x^2y^2\\ &= (2x^2 - 3y^2 + 3xy)(2x^2 - 3y^2 - 3xy)\\ &= (2x^2 + 3xy - 3y^2)(2x^2 - 3xy - 3y^2).\end{aligned}$$

8. $4a^4 - 29a^2c^2 + 25c^4$.

$$\begin{aligned}4a^4 - 29a^2c^2 + 25c^4 &= (4a^4 - 20a^2c^2 + 25c^4) - 9a^2c^2\\ &= (2a^2 - 5c^2)^2 - 9a^2c^2\\ &= (2a^2 - 5c^2 + 3ac)(2a^2 - 5c^2 - 3ac)\\ &= (2a^2 + 3ac - 5c^2)(2a^2 - 3ac - 5c^2).\end{aligned}$$

9. $4a^4 + 16a^2c^2 + 25c^4$.

$$\begin{aligned}4a^4 + 16a^2c^2 + 25c^4 &= (4a^4 + 20a^2c^2 + 25c^4) - 4a^2c^2\\ &= (2a^2 + 5c^2)^2 - 4a^2c^2\\ &= (2a^2 + 5c^2 + 2ac)(2a^2 + 5c^2 - 2ac)\\ &= (2a^2 + 2ac + 5c^2)(2a^2 - 2ac + 5c^2).\end{aligned}$$

10. $25x^4 + 31x^2y^2 + 16y^4$.

$$\begin{aligned}25x^4 + 31x^2y^2 + 16y^4 &= (25x^4 + 40x^2y^2 + 16y^4) - 9x^2y^2\\ &= (5x^2 + 4y^2)^2 - 9x^2y^2\\ &= (5x^2 + 4y^2 + 3xy)(5x^2 + 4y^2 - 3xy)\\ &= (5x^2 + 3xy + 4y^2)(5x^2 - 3xy + 4y^2).\end{aligned}$$

Exercise 34. Page 95.

Resolve into factors:

1. $x^2+8x+15.$
$(x+3)(x+5).$

2. $x^2-8x+15.$
$(x-3)(x-5).$

3. $x^2+2x-15.$
$(x-3)(x+5).$

4. $x^2-3x-10.$
$(x-5)(x+2).$

5. $x^2+5ax+6a^2.$
$(x+2a)(x+3a).$

6. $x^2-5ax+6a^2.$
$(x-2a)(x-3a).$

7. $x^2-2x-15.$
$(x+3)(x-5).$

8. $x^2+5x+6.$
$(x+2)(x+3).$

9. $x^2-5x+6.$
$(x-2)(x-3).$

10. $x^2+x-6.$
$(x-2)(x+3).$

11. $x^2-x-6.$
$(x+2)(x-3).$

12. $x^2+6x+5.$
$(x+1)(x+5).$

13. $x^2-6x+5.$
$(x-1)(x-5).$

14. $x^2+4x-5.$
$(x-1)(x+5).$

15. $x^2-4x-5.$
$(x+1)(x-5).$

16. $x^2+9x+18.$
$(x+3)(x+6).$

17. $x^2-9x+18.$
$(x-3)(x-6).$

18. $x^2+3x-18.$
$(x-3)(x+6).$

19. $x^2-3x-18.$
$(x+3)(x-6).$

20. $x^2+9x+8.$
$(x+1)(x+8).$

21. $x^2-9x+8.$
$(x-1)(x-8).$

22. $x^2+7x-8.$
$(x-1)(x+8).$

23. $x^2-7x-8.$
$(x+1)(x-8).$

24. $x^2+7x+10.$
$(x+2)(x+5).$

25. $x^2-7x+10.$
$(x-2)(x-5).$

26. $x^2+3x-10.$
$(x-2)(x+5).$

27. $x^2 - 5ax - 50a^2$.
$(x + 5a)(x - 10a)$.

28. $x^2y^2 - 3xy - 4$.
$(xy + 1)(xy - 4)$.

29. $a^2 - 3ax - 54x^2$.
$(a + 6x)(a - 9x)$.

30. $a^6 - a^3c - 2c^2$.
$(a^3 + c)(a^3 - 2c)$.

31. $y^2 - 28yz + 187z^2$.
$(y - 11z)(y - 17z)$.

32. $x^2 + ax - 6a^2$.
$(x - 2a)(x + 3a)$.

33. $x^2 - ax - 6a^2$.
$(x + 2a)(x - 3a)$.

34. $x^2 + 5xy + 4y^2$.
$(x + y)(x + 4y)$.

35. $x^2 - 3xy - 4y^2$.
$(x + y)(x - 4y)$.

36. $x^2 - 5xy + 4y^2$.
$(x - y)(x - 4y)$.

37. $x^2 + 3xy - 4y^2$.
$(x - y)(x + 4y)$.

38. $x^2 + 3xy + 2y^2$.
$(x + y)(x + 2y)$.

39. $a^2 - 7ab + 10b^2$.
$(a - 2b)(a - 5b)$.

40. $a^2x^2 - 3ax - 54$.
$(ax + 6)(ax - 9)$.

41. $x^2 - 7x - 44$.
$(x + 4)(x - 11)$.

42. $x^2 + x - 132$.
$(x - 11)(x + 12)$.

43. $x^2 - 15x + 50$.
$(x - 5)(x - 10)$.

44. $a^2 - 23a + 120$.
$(a - 8)(a - 15)$.

45. $a^2 + 17a - 390$.
$(a - 13)(a + 30)$.

46. $c^2 + 25c - 150$.
$(c - 5)(c + 30)$.

47. $c^2 - 58c + 57$.
$(c - 1)(c - 57)$.

48. $a^4 - 11a^2b^3 + 30b^6$.
$(a^2 - 5b^3)(a^2 - 6b^3)$.

49. $z^2 + 9zy + 20y^2$.
$(z + 4y)(z + 5y)$.

50. $x^2y^2 + 19xyz + 48z^2$.
$(xy + 3z)(xy + 16z)$.

51. $a^2b^2 - 13abc + 22c^2$.
$(ab - 2c)(ab - 11c)$.

52. $a^2 - 16ab - 36b^2$.
$(a + 2b)(a - 18b)$.

53. $x^2 + 17xy + 30y^2$.
$(x + 2y)(x + 15y)$.

54. $x^2 - 7xy - 18y^2$.
$(x + 2y)(x - 9y)$.

55. $c^2 + c - 20$.
$(c - 4)(c + 5)$.

56. $a^2 + 16ab - 260b^2$.
$(a - 10b)(a + 26b)$.

57. $y^2 - 5yz - 84z^2$.
$(y + 7z)(y - 12z)$.

58. $x^2 - 11x - 152$.
$(x + 8)(x - 19)$.

59. $18 - 3x - x^2 = -(x^2 + 3x - 18)$.
$-(x - 3)(x + 6)$.

60. $33 + 8x - x^2 = -(x^2 - 8x - 33)$.
$-(x + 3)(x - 11)$.

61. $78 - 7x - x^2$.
$-(x^2 + 7x - 78) = -(x - 6)(x + 13)$.

Exercise 35. Page 98.

Resolve into factors:

1. $2x^2 + 5x + 3$.
$(2x + 3)(x + 1)$.

2. $3x^2 - x - 2$.
$(3x + 2)(x - 1)$.

3. $5x^2 - 8x + 3$.
$(5x - 3)(x - 1)$.

4. $24x^2 - 2xy - 15y^2$.
$(4x + 3y)(6x - 5y)$.

5. $36x^2 - 19xy - 6y^2$.
$(4x - 3y)(9x + 2y)$.

6. $15x^2 + 19xy + 6y^2$.
$(3x + 2y)(5x + 3y)$.

7. $6x^2 + 7x + 2$.
$(2x + 1)(3x + 2)$.

8. $6x^2 - x - 2$.
$(2x + 1)(3x - 2)$.

9. $15x^2 + 14x - 8$.
$(3x + 4)(5x - 2)$.

10. $8x^2 - 10x + 3$.
$(2x - 1)(4x - 3)$.

11. $18x^2 + 9x - 2$.
$(3x + 2)(6x - 1)$.

12. $24x^2 - 14xy - 5y^2$.
$(4x + y)(6x - 5y)$.

13. $24x^2 - 38xy + 15y^2$.
$(4x - 3y)(6x - 5y)$.

14. $15x^2 - 26xy + 8y^2$.
$(3x - 4y)(5x - 2y)$.

15. $9x^2 + 6xy - 8y^2$.
$(3x - 2y)(3x + 4y)$.

16. $6x^2 - xy - 35y^2$.
$(2x - 5y)(3x + 7y)$.

17. $10x^2 - 21xy - 10y^2$.
$(2x - 5y)(5x + 2y)$.

18. $14x^2 - 55xy + 21y^2$.
$(2x - 7y)(7x - 3y)$.

19. $6x^2 - 23xy + 20y^2$.
$(2x - 5y)(3x - 4y)$

20. $6x^2 + 35xy - 6y^2$.
$(6x - y)(x + 6y)$.

Exercise 36. Page 100.

Resolve into factors:

1. $a^3 + 8b^3$.
$(a + 2b)(a^2 - 2ab + 4b^2)$.

2. $a^3 - 27a^6$.
$(a - 3a^2)(a^2 + 3a^3 + 9a^4)$.

3. $a^3 + 64$.
$(a + 4)(a^2 - 4a + 16)$.

4. $125a^3 + 1$.
$(5a + 1)(25a^2 - 5a + 1)$.

5. $27x^3y^3 - 1$.
$(3xy - 1)(9x^2y^2 + 3xy + 1)$.

6. $a^3 + 27b^3$.
$(a + 3b)(a^2 - 3ab + 9b^2)$.

7. $x^3y^3 - 64$.
$(xy - 4)(x^2y^2 + 4xy + 16)$.

8. $64a^6 + 125b^3$.
$(4a^2 + 5b)(16a^4 - 20a^2b + 25b^2)$

9. $216a^6 - b^3$.
$(6a^2 - b)(36a^4 + 6a^2b + b^2)$.

10. $64a^3 - 27b^3$.
$(4a - 3b)(16a^2 + 12ab + 9b^2)$.

11. $343 - x^3$.
$(7 - x)(49 + 7x + x^2)$.

12. $a^3b^3 + 343$.
$(ab + 7)(a^2b^2 - 7ab + 49)$.

13. $8a^3 - b^6$.
$(2a - b^2)(4a^2 + 2ab^2 + b^4)$.

14. $216m^3 + n^6$.
$(6m + n^2)(36m^2 - 6mn^2 + n^4)$.

15. $(a + b)^3 - 1$.

$$\{(a + b) - 1\}\{(a + b)^2 + (a + b) + 1\}$$
$$= (a + b - 1)(a^2 + 2ab + b^2 + a + b + 1).$$

16. $(a - b)^3 + 1$.

$$\{(a - b) + 1\}\{(a - b)^2 - (a - b) + 1\}$$
$$= (a - b + 1)(a^2 - 2ab + b^2 - a + b + 1).$$

17. $(2x + y)^3 - (x - y)^3$.

$$\{(2x + y) - (x - y)\}\{(2x + y)^2 + (2x + y)(x - y) + (x - y)^2\}$$
$$= (2x+y-x+y)(4x^2+4xy + y^2 + 2x^2 - xy - y^2 + x^2 - 2xy + y^2)$$
$$= (x + 2y)(7x^2 + xy + y^2).$$

18. $1 - (a - b)^3$.

$$\{1 - (a - b)\}\{1 + (a - b) + (a - b)^2\}$$
$$= (1 - a + b)(1 + a - b + a^2 - 2ab + b^2).$$

19. $8x^3-(x-y)^3$.

$$\begin{aligned}&\{2x-(x-y)\}\{4x^2+2x(x-y)+(x-y)^2\}\\&\quad=(2x-x+y)(4x^2+2x^2-2xy+x^2-2xy+y^2)\\&\quad=(x+y)(7x^2-4xy+y^2).\end{aligned}$$

20. $8(x+y)^3+z^3$.

$$\begin{aligned}&\{2(x+y)+z\}\{4(x+y)^2-2(x+y)z+z^2\}\\&\quad=(2x+2y+z)(4x^2+8xy+4y^2-2xz-2yz+z^2).\end{aligned}$$

21. $729y^3-64z^3$.

$$(9y-4z)(81y^2+36yz+16z^2).$$

22. $(a+b)^3-(a-b)^3$.

$$\begin{aligned}&\{(a+b)-(a-b)\}\{(a+b)^2+(a+b)(a-b)+(a-b)^2\}\\&\quad=(a+b-a+b)(a^2+2ab+b^2+a^2-b^2+a^2-2ab+b^2)\\&\quad=2b(3a^2+b^2).\end{aligned}$$

23. $729a^6+216c^6$.

$$(9a^2+6c^2)(81a^4-54a^2c^2+36c^4).$$

24. $x^3y^3-512z^3$.

$$(xy-8z)(x^2y^2+8xyz+64z^2).$$

Exercise 37. Page 103.

Resolve into factors:

1. a^3-9a.

$$a^3-9a=a(a+3)(a-3).$$

2 $x^2y^2-4xy^4-3x^2y^3$.

$$x^2y^2-4xy^4-3x^2y^3=xy^2(x-4y^2-3xy).$$

3. x^3+x^2+x+1.

$$\begin{aligned}x^3+x^2+x+1&=(x^3+x^2)+(x+1)\\&=x^2(x+1)+(x+1)\\&=(x^2+1)(x+1).\end{aligned}$$

4. $x^2-2y+2x-xy$.

$$\begin{aligned}x^2-2y+2x-xy&=(x^2-xy)+(2x-2y)\\&=x(x-y)+2(x-y)\\&=(x+2)(x-y).\end{aligned}$$

5. $3x^3 + 2x^2 - 9x - 6$.

$$\begin{aligned}3x^3 + 2x^2 - 9x - 6 &= (3x^3 + 2x^2) - (9x + 6)\\ &= x^2(3x + 2) - 3(3x + 2)\\ &= (x^2 - 3)(3x + 2).\end{aligned}$$

6. $x^2 - 14x + 49$.

$$x^2 - 14x + 49 = (x - 7)(x - 7).$$

7. $36x^2 - 49y^2$.

$$36x^2 - 49y^2 = (6x + 7y)(6x - 7y).$$

8. $x^4 - y^4$.

$$x^4 - y^4 = (x^2 + y^2)(x^2 - y^2) = (x^2 + y^2)(x + y)(x - y).$$

9. $(x - y)^2 - b^2$.

$$(x - y)^2 - b^2 = (x - y + b)(x - y - b).$$

10. $x^3 + y^3$.

$$x^3 + y^3 = (x + y)(x^2 - xy + y^2).$$

11. $x^6 - y^6$.

$$\begin{aligned}x^6 - y^6 &= (x^3 + y^3)(x^3 - y^3)\\ &= (x+y)(x^2-xy+y^2)(x-y)(x^2+xy+y^2).\end{aligned}$$

12. $x^6 + y^6$.

$$x^6 + y^6 = (x^2 + y^2)(x^4 - x^2y^2 + y^4).$$

13. $x^2 - (a - b)^2$.

$$[x + (a - b)][x - (a - b)] = (x + a - b)(x - a + b).$$

14. $m^2 + 2mn + n^2 - 1$.

$$\begin{aligned}m^2 + 2mn + n^2 - 1 &= (m + n)^2 - 1\\ &= (m + n + 1)(m + n - 1).\end{aligned}$$

15. $a^2 - (m + n)^2$.

$$a^2 - (m + n)^2 = (a + m + n)(a - m - n).$$

16. $x^2 - 11x + 18$.

$$x^2 - 11x + 18 = (x - 2)(x - 9).$$

17. $x^2 + 4x - 45$.

$$x^2 + 4x - 45 = (x - 5)(x + 9).$$

18. $x^2 + 13x + 36$.

$$x^2 + 13x + 36 = (x + 4)(x + 9).$$

19. $x^2 - 13x - 48$.

$$x^2 - 13x - 48 = (x + 3)(x - 16).$$

20. $x^2+9x-36$.

$$x^2+9x-36=(x-3)(x+12).$$

21. $10x^2+x-21$.

$$10x^2+x-21=(2x+3)(5x-7).$$

22. $6x^2-x-12$.

$$6x^2-x-12=(2x-3)(3x+4).$$

23. $12x^2-x-1$.

$$12x^2-x-1=(3x-1)(4x+1).$$

24. $12x^2-x-20$.

$$12x^2-x-20=(3x-4)(4x+5).$$

25. $9a^2+12a+4$.

$$9a^2+12a+4=(3a+2)(3a+2).$$

26. $a^2-b^2-c^2+2bc$.

$$\begin{aligned}a^2-b^2-c^2+2bc&=a^2-(b^2-2bc+c^2)\\&=a^2-(b-c)^2\\&=(a+b-c)(a-b+c).\end{aligned}$$

27. $x^4+x^2y^2+y^4$.

$$\begin{aligned}x^4+x^2y^2+y^4&=x^4+2x^2y^2+y^4-x^2y^2\\&=(x^2+y^2)^2-x^2y^2\\&=(x^2+y^2+xy)(x^2+y^2-xy)\\&=(x^2+xy+y^2)(x^2-xy+y^2).\end{aligned}$$

28. $z^2-6z-40$.

$$z^2-6z-40=(z+4)(z-10).$$

29. $x^2-7x-60$.

$$x^2-7x-60=(x+5)(x-12).$$

30. $a^2-19a+84$.

$$a^2-19a+84=(a-7)(a-12).$$

31. $x^2+2ax+3bx+6ab$.

$$\begin{aligned}x^2+2ax+3bx+6ab&=(x^2+2ax)+(3bx+6ab)\\&=x(x+2a)+3b(x+2a)\\&=(x+3b)(x+2a).\end{aligned}$$

32. $x^2+m^2-n^2-2mx.$

$$\begin{aligned}x^2+m^2-n^2-2mx &= (x^2-2mx+m^2)-n^2\\ &= (x-m)^2-n^2\\ &= (x-m+n)(x-m-n).\end{aligned}$$

33. $4x^4-x^2.$

$$4x^4-x^2=x^2(4x^2-1)=x^2(2x+1)(2x-1).$$

34. $x^{12}+y^{12}.$

$$x^{12}+y^{12}=(x^4)^3+(y^4)^3=(x^4+y^4)(x^8-x^4y^4+y^8).$$

35. $9x^4+21x^2y^2+25y^4.$

$$\begin{aligned}9x^4+21x^2y^2+25y^4 &= (9x^4+30x^2y^2+25y^4)-9x^2y^2\\ &= (3x^2+5y^2)^2-9x^2y^2\\ &= (3x^2+5y^2+3xy)(3x^2+5y^2-3xy)\\ &= (3x^2+3xy+5y^2)(3x^2-3xy+5y^2).\end{aligned}$$

36. $x^2-4+y^2+2xy.$

$$\begin{aligned}x^2-4+y^2+2xy &= x^2+2xy+y^2-4\\ &= (x+y)^2-4\\ &= (x+y+2)(x+y-2).\end{aligned}$$

37. $2x^2+3xy-2y^2.$

$$2x^2+3xy-2y^2=(2x-y)(x+2y).$$

38. $2a^2-7a+6.$

$$2a^2-7a+6=(2a-3)(a-2).$$

39. $x^4-7x^2+1.$

$$\begin{aligned}x^4-7x^2+1 &= (x^4+2x^2+1)-9x^2\\ &= (x^2+1)^2-9x^2\\ &= (x^2+1+3x)(x^2+1-3x)\\ &= (x^2+3x+1)(x^2-3x+1).\end{aligned}$$

40. $1-a^2-b^2-2ab.$

$$\begin{aligned}1-a^2-b^2-2ab &= 1-(a^2+2ab+b^2)\\ &= 1-(a+b)^2\\ &= (1+a+b)(1-a-b).\end{aligned}$$

41. $3x^4-6x^3+9x^2.$

$$3x^4-6x^3+9x^2=3x^2(x^2-2x+3).$$

42. $x^3-5x^2-2x+10$.

$$\begin{aligned}x^3-5x^2-2x+10&=(x^3-5x^2)-(2x-10)\\&=x^2(x-5)-2(x-5)\\&=(x^2-2)(x-5).\end{aligned}$$

43. $x^2+ax-bx-ab$.

$$\begin{aligned}x^2+ax-bx-ab&=(x^2+ax)-(bx+ab)\\&=x(x+a)-b(x+a)\\&=(x-b)(x+a).\end{aligned}$$

44. $2x^2-3xy+4ax-6ay$.

$$\begin{aligned}2x^2-3xy+4ax-6ay&=(2x^2-3xy)+(4ax-6ay)\\&=x(2x-3y)+2a(2x-3y)\\&=(x+2a)(2x-3y).\end{aligned}$$

45. ax^4+bx^3-ax-b.

$$\begin{aligned}ax^4+bx^3-ax-b&=(ax^4+bx^3)-(ax+b)\\&=x^3(ax+b)-(ax+b)\\&=(x^3-1)(ax+b)\\&=(x-1)(x^2+x+1)(ax+b).\end{aligned}$$

46. a^3+b^3+a+b.

$$\begin{aligned}a^3+b^3+a+b&=(a^3+b^3)+(a+b)\\&=(a+b)(a^2-ab+b^2)+(a+b)\\&=(a+b)(a^2-ab+b^2+1).\end{aligned}$$

47. a^3-b^3+a-b.

$$\begin{aligned}a^3-b^3+a-b&=(a^3-b^3)+(a-b)\\&=(a-b)(a^2+ab+b^2)+(a-b)\\&=(a-b)(a^2+ab+b^2+1).\end{aligned}$$

48. $(x-y)^2-2y(x-y)$.

$$\begin{aligned}(x-y)^2-2y(x-y)&=(x-y)(x-y-2y)\\&=(x-y)(x-3y).\end{aligned}$$

49. $1-10xy+25x^2y^2$.

$$1-10xy+25x^2y^2=(1-5xy)(1-5xy).$$

50. $a^2-b^2+2bc-c^2$.

$$\begin{aligned}a^2-b^2+2bc-c^2&=a^2-(b^2-2bc+c^2)\\&=a^2-(b-c)^2\\&=(a+b-c)(a-b+c).\end{aligned}$$

51. $x^2+4y^2-z^2-4xy$.

$$\begin{aligned} x^2+4y^2-z^2-4xy &= (x^2-4xy+4y^2)-z^2\\ &= (x-2y)^2-z^2\\ &= (x-2y+z)(x-2y-z). \end{aligned}$$

52. $a^2-4b^2-9c^2+12bc$.

$$\begin{aligned} a^2-4b^2-9c^2+12bc &= a^2-(4b^2-12bc+9c^2)\\ &= a^2-(2b-3c)^2\\ &= (a+2b-3c)(a-2b+3c). \end{aligned}$$

53. $4x^2+9y^2-z^2-12xy$.

$$\begin{aligned} 4x^2+9y^2-z^2-12xy &= (4x^2-12xy+9y^2)-z^2\\ &= (2x-3y)^2-z^2\\ &= (2x-3y+z)(2x-3y-z). \end{aligned}$$

54. $(a+b)^2-(c-d)^2$.

$$(a+b)^2-(c-d)^2=(a+b+c-d)(a+b-c+d).$$

55. x^5+y^5.

$$x^5+y^5=(x+y)(x^4-x^3y+x^2y^2-xy^3+y^4).$$

56. $32x^5-c^5$.

$$32x^5-c^5=(2x-c)(16x^4+8cx^3+4c^2x^2+2c^3x+c^4).$$

57. a^6+64y^6.

$$a^6+64y^6=(a^2+4y^2)(a^4-4a^2y^2+16y^4).$$

58. $729-x^6$.

$$\begin{aligned} 729-x^6 &= 3^6-x^6\\ &= (3^3+x^3)(3^3-x^3)\\ &= (3+x)(3^2-3x+x^2)(3-x)(3^2+3x+x^2)\\ &= (3+x)(9-3x+x^2)(3-x)(9+3x+x^2). \end{aligned}$$

59. $x^{12}-y^{12}$.

$$\begin{aligned} x^{12}-y^{12} &= (x^6+y^6)(x^6-y^6)\\ &= (x^2+y^2)(x^4-x^2y^2+y^4)(x^2-y^2)(x^4+x^2y^2+y^4)\\ &= (x^2+y^2)(x^4-x^2y^2+y^4)(x+y)(x-y)(x^2+xy+y^2)(x^2-xy+y^2). \end{aligned}$$

60. $(a+b)^4-1$.

$$\begin{aligned} (a+b)^4-1 &= \{(a+b)^2+1\}\{(a+b)^2-1\}\\ &= (a^2+2ab+b^2+1)(a+b+1)(a+b-1) \end{aligned}$$

61. $a^2 - b^2 + a - b$.

$$\begin{aligned} a^2 - b^2 + a - b &= (a^2 - b^2) + (a - b) \\ &= (a + b)(a - b) + (a - b) \\ &= (a + b + 1)(a - b). \end{aligned}$$

62. $a^2 + a + 3b - 9b^2$.

$$\begin{aligned} a^2 + a + 3b - 9b^2 &= (a^2 - 9b^2) + (a + 3b) \\ &= (a + 3b)(a - 3b) + (a + 3b) \\ &= (a + 3b)(a - 3b + 1). \end{aligned}$$

63. $6a^2 - a - 77$.

$$6a^2 - a - 77 = (3a - 11)(2a + 7).$$

64. $5c^4 - 15c^3 - 90c^2$.

$$\begin{aligned} 5c^4 - 15c^3 - 90c^2 &= 5c^2(c^2 - 3c - 18) \\ &= 5c^2(c + 3)(c - 6). \end{aligned}$$

65. $a^2x - c^2x + a^2y - c^2y$.

$$\begin{aligned} a^2x - c^2x + a^2y - c^2y &= x(a^2 - c^2) + y(a^2 - c^2) \\ &= (x + y)(a^2 - c^2) \\ &= (x + y)(a + c)(a - c). \end{aligned}$$

66. $16x^4 - 81$.

$$\begin{aligned} 16x^4 - 81 &= (4x^2 + 9)(4x^2 - 9) \\ &= (4x^2 + 9)(2x + 3)(2x - 3). \end{aligned}$$

67. $x^4 + x^2 + 1$.

$$\begin{aligned} x^4 + x^2 + 1 &= (x^4 + 2x^2 + 1) - x^2 \\ &= (x^2 + 1)^2 - x^2 \\ &= (x^2 + 1 + x)(x^2 + 1 - x) \\ &= (x^2 + x + 1)(x^2 - x + 1). \end{aligned}$$

68. $27x^3 - 64a^3$.

$$27x^3 - 64a^3 = (3x - 4a)(9x^2 + 12ax + 16a^2)$$

69. $x^9 + y^9$.

$$\begin{aligned} x^9 + y^9 &= (x^3)^3 + (y^3)^3 \\ &= (x^3 + y^3)(x^6 - x^3y^3 + y^6) \\ &= (x + y)(x^2 - xy + y^2)(x^6 - x^3y^3 + y^6). \end{aligned}$$

70. $x^9 - y^9$.

$$\begin{aligned} x^9 - y^9 &= (x^3)^3 - (y^3)^3 \\ &= (x^3 - y^3)(x^6 + x^3y^3 + y^6) \\ &= (x - y)(x^2 + xy + y^2)(x^6 + x^3y^3 + y^6). \end{aligned}$$

71. $a^8 - 256$.

$$\begin{aligned} a^8 - 256 &= a^8 - 2^8 \\ &= (a^4 + 2^4)(a^4 - 2^4) \\ &= (a^4 + 2^4)(a^2 + 2^2)(a^2 - 2^2) \\ &= (a^4 + 2^4)(a^2 + 2^2)(a + 2)(a - 2) \\ &= (a^4 + 16)(a^2 + 4)(a + 2)(a - 2). \end{aligned}$$

72. $x^4 + 16\,a^2x^2 + 256\,a^4$.

$$\begin{aligned} x^4 + 16\,a^2x^2 + 256\,a^4 &= (x^4 + 32\,a^2x^2 + 256\,a^4) - 16\,a^2x^2 \\ &= (x^2 + 16\,a^2)^2 - 16\,a^2x^2 \\ &= (x^2 + 16\,a^2 + 4\,ax)(x^2 + 16\,a^2 - 4\,ax) \\ &= (x^2 + 4\,ax + 16\,a^2)(x^2 - 4\,ax + 16\,a^2). \end{aligned}$$

73. $1 - (x - y)^3$.

$$\begin{aligned} 1 - (x - y)^3 &= \{1 - (x - y)\}\{1 + (x - y) + (x - y)^2\} \\ &= (1 - x + y)(1 + x - y + x^2 - 2\,xy + y^2). \end{aligned}$$

74. $(x + y)^3 + (2\,x - y)^3$.

$$\begin{aligned} &(x + y)^3 + (2\,x - y)^3 \\ &\quad = \{(x + y) + (2\,x - y)\}\{(x + y)^2 - (x + y)(2\,x - y) + (2\,x - y)^2\} \\ &\quad = 3\,x(x^2 + 2\,xy + y^2 - 2\,x^2 - xy + y^2 + 4\,x^2 - 4\,xy + y^2) \\ &\quad = 3\,x(3\,x^2 - 3\,xy + 3\,y^2) \\ &\quad = 9\,x(x^2 - xy + y^2). \end{aligned}$$

75. $x^6 - 216$.

$$\begin{aligned} x^6 - 216 &= x^6 - 6^3 \\ &= (x^2)^3 - 6^3 \\ &= (x^2 - 6)(x^4 + 6\,x^2 + 36). \end{aligned}$$

76. $3\,x^2 + x - 2$.

$$3\,x^2 + x - 2 = (3\,x - 2)(x + 1).$$

77. $2 - 3\,a - 2\,a^2$.

$$2 - 3\,a - 2\,a^2 = (2 + a)(1 - 2\,a).$$

78. $4 - 5\,c - 6\,c^2$.

$$4 - 5\,c - 6\,c^2 = (4 + 3\,c)(1 - 2\,c).$$

79. $2xy - x^2 - y^2 + z^2$.

$$\begin{aligned}2xy - x^2 - y^2 + z^2 &= z^2 - (x^2 - 2xy + y^2)\\ &= z^2 - (x - y)^2\\ &= (z + x - y)(z - x + y).\end{aligned}$$

80. $4a^4 - 9a^2 + 6a - 1$.

$$\begin{aligned}4a^4 - 9a^2 + 6a - 1 &= 4a^4 - (9a^2 - 6a + 1)\\ &= 4a^4 - (3a - 1)^2\\ &= (2a^2 + 3a - 1)(2a^2 - 3a + 1)\\ &= (2a^2 + 3a - 1)(2a - 1)(a - 1).\end{aligned}$$

81. $a^2 - 2ab + b^2 + 12xy - 4x^2 - 9y^2$.

$$\begin{aligned}a^2 - 2ab + b^2 + 12xy - 4x^2 - 9y^2 &= (a^2 - 2ab + b^2) - (4x^2 - 12xy + 9y^2)\\ &= (a - b)^2 - (2x - 3y)^2\\ &= (a - b + 2x - 3y)(a - b - 2x + 3y).\end{aligned}$$

82. $2x^2 - 4xy + 2y^2 + 2ax - 2ay$.

$$\begin{aligned}2x^2 - 4xy + 2y^2 + 2ax - 2ay &= 2x^2 - 4xy + 2y^2 + 2ax + 2ay\\ &= 2(x^2 - 2xy + y^2) + 2a(x - y)\\ &= 2(x - y)(x - y) + 2a(x - y)\\ &= 2(x - y + a)(x - y).\end{aligned}$$

83. $(a + b)^2 - 1 - ab(a + b + 1)$.

$$\begin{aligned}(a + b)^2 - 1 - ab(a + b + 1) &= \{(a + b)^2 - 1\} - ab(a + b + 1)\\ &= (a + b + 1)(a + b - 1) - ab(a + b + 1)\\ &= (a + b + 1)(a + b - ab - 1).\end{aligned}$$

84. $x^3 - x^2 + 3x + 5$.

$$\begin{aligned}x^3 - x^2 + 3x + 5 &= (x^3 - x^2 - 2x) + (5x + 5)\\ &= x(x^2 - x - 2) + 5(x + 1)\\ &= x(x + 1)(x - 2) + 5(x + 1)\\ &= (x + 1)\{x(x - 2) + 5\}\\ &= (x + 1)(x^2 - 2x + 5).\end{aligned}$$

85. $6x^3 - 23x^2 + 16x - 3$.

$$\begin{aligned}6x^3 - 23x^2 + 16x - 3 &= (6x^3 - 23x^2 + 7x) + (9x - 3)\\ &= x(6x^2 - 23x + 7) + 3(3x - 1)\\ &= x(3x - 1)(2x - 7) + 3(3x - 1)\\ &= (3x - 1)\{x(2x - 7) + 3\}\\ &= (3x - 1)(2x^2 - 7x + 3)\\ &= (3x - 1)(2x - 1)(x - 3).\end{aligned}$$

86. $x^2+y^2+z^2-2xy-2xz+2yz.$ (See § 103.)

$$\begin{aligned}x^2+y^2+z^2-2xy-2xz+2yz\\&=(x^2-2xy+y^2)-(2xz-2yz)+z^2\\&=(x-y)^2-2z(x-y)+z^2\\&=(x-y-z)^2.\end{aligned}$$

87. $4a^2b^2-(a^2+b^2-c^2)^2.$

$$\begin{aligned}4a^2b^2-(a^2+b^2-c^2)^2\\&=(2ab+a^2+b^2-c^2)(2ab-a^2-b^2+c^2)\\&=(a^2+2ab+b^2-c^2)(c^2-a^2+2ab-b^2)\\&=\{(a+b)^2-c^2\}\{c^2-(a-b)^2\}\\&=(a+b+c)(a+b-c)(c+a-b)(c-a+b).\end{aligned}$$

Exercise 38. Page 107.

Find the H. C. F. of

1. 120 and 168.

$$\begin{aligned}120&=2\times2\times2\times3\times5\\168&=2\times2\times2\times3\times7\\\therefore\text{ the H. C. F.}&=2\times2\times2\times3\\&=24.\end{aligned}$$

2. $36x^3$ and $27x^4$.

$$\begin{aligned}36x^3&=2\times2\times3\times3\times xxx\\27x^4&=3\times3\times3\times xxxx\\\therefore\text{ the H. C. F.}&=3\times3\times xxx\\&=9x^3.\end{aligned}$$

3. $42a^2x^3$ and $60a^3x^2$.

$$\begin{aligned}42a^2x^3&=2\times3\times7\times aa\times xxx\\60a^3x^2&=2\times2\times3\times5\times aaa\times xx\\\therefore\text{ the H. C. F.}&=2\times3\times aa\times xx\\&=6a^2x^2.\end{aligned}$$

4. $36a^3x^2$ and $28x^3y$.

$$\begin{aligned}36a^3x^2&=2\times2\times3\times3\times aaa\times xx\\28x^3y&=2\times2\times7\times xxx\times y\\\therefore\text{ the H. C. F.}&=2\times2\times xx\\&=4x^2.\end{aligned}$$

5. $48a^2b^3c$ and $60a^3c^3$.

$$\begin{aligned}48a^2b^3c &= 2\times2\times2\times2\times3\times aa\times bbb\times c\\ 60a^3c^3 &= 2\times2\times3\times5\times aaa\times ccc\\ \therefore \text{ the H. C. F.} &= 2\times2\times3\times aa\times c\\ &= 12a^2c.\end{aligned}$$

6. $8(a+b)^2$ and $6(a+b)^3$.

$$\begin{aligned}8(a+b)^2 &= 2\times2\times2\times(a+b)\times(a+b)\\ 6(a+b)^3 &= 2\times3\times(a+b)\times(a+b)\times(a+b)\\ \therefore \text{ the H. C. F.} &= 2\times(a+b)\times(a+b)\\ &= 2(a+b)^2.\end{aligned}$$

7. $12a(x+y)^2$ and $4b(x+y)^3$.

$$\begin{aligned}12a(x+y)^2 &= 2\times2\times3\times a\times(x+y)\times(x+y)\\ 4b(x+y)^3 &= 2\times2\times(x+y)\times(x+y)\times(x+y)\\ \therefore \text{ the H. C. F.} &= 2\times2\times(x+y)\times(x+y)\\ &= 4(x+y)^2.\end{aligned}$$

8. $(x-1)^2(x+2)^2$ and $(x-3)(x+2)^3$.

$$\begin{aligned}(x-1)^2(x+2)^2 &= (x-1)(x-1)(x+2)(x+2)\\ (x-3)(x+2)^3 &= (x-3)(x+2)(x+2)(x+2)\\ \therefore \text{ the H. C. F.} &= (x+2)(x+2)\\ &= (x+2)^2.\end{aligned}$$

9. $24a^2b^3(a+b)$ and $42a^3b(a+b)^2$.

$$\begin{aligned}24a^2b^3(a+b) &= 2\times2\times2\times3\times aa\times bbb\times(a+b)\\ 42a^3b(a+b)^2 &= 2\times3\times7\times aaa\times b\times(a+b)\times(a+b)\\ \therefore \text{ the H. C. F.} &= 2\times3\times aa\times b\times(a+b)\\ &= 6a^2b(a+b).\end{aligned}$$

10. $x^2(x-3)^2$ and x^2-3x.

$$\begin{aligned}x^2(x-3)^2 &= xx(x-3)(x-3)\\ x^2-3x &= x(x-3)\\ \therefore \text{ the H. C. F.} &= x(x-3)\\ &= x^2-3x.\end{aligned}$$

11. x^2-16 and x^2+4x.

$$\begin{aligned}x^2-16 &= (x+4)(x-4)\\ x^2+4x &= x(x+4)\\ \therefore \text{ the H. C. F.} &= x+4.\end{aligned}$$

12. x^2-4x and x^2-6x+8.

$$\begin{aligned}x^2-4x &= x(x-4)\\ x^2-6x+8 &= (x-2)(x-4)\\ \therefore \text{ the H. C. F.} &= x-4.\end{aligned}$$

13. $x^2-7x+12$ and x^2-16.

$$\begin{aligned}x^2-7x+12 &= (x-3)(x-4)\\ x^2-16 &= (x+4)(x-4)\\ \therefore \text{ the H. C. F.} &= x-4.\end{aligned}$$

14. $9x^2 - 4y^2$ and $12x^2 - xy - 6y^2$.

$$\begin{aligned} 9x^2 - 4y^2 &= (3x + 2y)(3x - 2y) \\ 12x^2 - xy - 6y^2 &= (3x + 2y)(4x - 3y) \\ \therefore \text{ the H.C.F.} &= 3x + 2y. \end{aligned}$$

15. $x^2 - 7x - 8$ and $x^2 + 5x + 4$.

$$\begin{aligned} x^2 - 7x - 8 &= (x + 1)(x - 8) \\ x^2 + 5x + 4 &= (x + 1)(x + 4) \\ \therefore \text{ the H.C.F.} &= x + 1. \end{aligned}$$

16. $x^2 + 3xy - 10y^2$ and $x^2 - 2xy - 35y^2$.

$$\begin{aligned} x^2 + 3xy - 10y^2 &= (x - 2y)(x + 5y) \\ x^2 - 2xy - 35y^2 &= (x + 5y)(x - 7y) \\ \therefore \text{ the H.C.F.} &= x + 5y. \end{aligned}$$

17. $x^4 - 2x^3 - 24x^2$ and $6x^5 - 6x^4 - 180x^3$.

$$\begin{aligned} x^4 - 2x^3 - 24x^2 &= x^2(x + 4)(x - 6) \\ 6x^5 - 6x^4 - 180x^3 &= 6x^3(x + 5)(x - 6) \\ \therefore \text{ the H.C.F.} &= x^2(x - 6). \end{aligned}$$

18. $x^3 - 3x^2y$ and $x^3 - 27y^3$.

$$\begin{aligned} x^3 - 3x^2y &= x^2(x - 3y) \\ x^3 - 27y^3 &= (x - 3y)(x^2 + 3xy + 9y^2) \\ \therefore \text{ the H.C.F.} &= x - 3y. \end{aligned}$$

19. $1 + 64x^3$ and $1 - 4x + 16x^2$.

$$\begin{aligned} 1 + 64x^3 &= (1 + 4x)(1 - 4x + 16x^2) \\ 1 - 4x + 16x^2 &= 1 - 4x + 16x^2 \\ \therefore \text{ the H.C.F.} &= 1 - 4x + 16x^2. \end{aligned}$$

20. $x^4 - 81$ and $x^4 + 8x^2 - 9$.

$$\begin{aligned} x^4 - 81 &= (x^2 + 9)(x^2 - 9) \\ &= (x^2 + 9)(x + 3)(x - 3) \\ x^4 + 8x^2 - 9 &= (x^2 + 9)(x^2 - 1) \\ &= (x^2 + 9)(x + 1)(x - 1) \\ \therefore \text{ the H.C.F.} &= x^2 + 9. \end{aligned}$$

21. $x^2 + 2x - 3$ and $x^2 + 7x + 12$.

$$\begin{aligned} x^2 + 2x - 3 &= (x - 1)(x + 3) \\ x^2 + 7x + 12 &= (x + 3)(x + 4) \\ \therefore \text{ the H.C.F.} &= x + 3. \end{aligned}$$

22. x^2-6x+5 and $x^2+3x-40$.

$$\begin{aligned} x^2-6x+5 &= (x-1)(x-5)\\ x^2+3x-40 &= (x-5)(x+8)\\ \therefore \text{ the H. C. F.} &= x-5. \end{aligned}$$

23. $3a^4+15a^3b-72a^2b^2$ and $6a^3-30a^2b+36ab^2$.

$$\begin{aligned} 3a^4+15a^3b-72a^2b^2 &= 3a^2(a^2+5ab-24b^2)\\ &= 3a^2(a-3b)(a+8b)\\ 6a^3-30a^2b+36ab^2 &= 6a(a^2-5ab+6b^2)\\ &= 6a(a-2b)(a-3b)\\ \therefore \text{ the H. C. F.} &= 3a(a-3b)\\ &= 3a^2-9ab. \end{aligned}$$

24. $6x^2y-12xy^2+6y^3$ and $3x^2y^2+9xy^3-12y^4$.

$$\begin{aligned} 6x^2y-12xy^2+6y^3 &= 6y(x^2-2xy+y^2)\\ &= 6y(x-y)(x-y)\\ 3x^2y^2+9xy^3-12y^4 &= 3y^2(x^2+3xy-4y^2)\\ &= 3y^2(x-y)(x+4y)\\ \therefore \text{ the H. C. F.} &= 3y(x-y)\\ &= 3xy-3y^2. \end{aligned}$$

25. $1-16c^4$ and $1+c^2-12c^4$.

$$\begin{aligned} 1-16c^4 &= (1+4c^2)(1-4c^2)\\ &= (1+4c^2)(1+2c)(1-2c)\\ 1+c^2-12c^4 &= (1-3c^2)(1+4c^2)\\ \therefore \text{ the H. C. F.} &= 1+4c^2. \end{aligned}$$

26. $8x^2+2x-1$ and $6x^2+7x+2$.

$$\begin{aligned} 8x^2+2x-1 &= (4x-1)(2x+1)\\ 6x^2+7x+2 &= (3x+2)(2x+1)\\ \therefore \text{ the H. C. F.} &= 2x+1. \end{aligned}$$

27. $6x^2+x-2$ and $12x^2-x-6$.

$$\begin{aligned} 6x^2+x-2 &= (3x+2)(2x-1)\\ 12x^2-x-6 &= (3x+2)(4x-3)\\ \therefore \text{ the H. C. F.} &= 3x+2. \end{aligned}$$

28. $15x^3-19x^2y+6xy^2$ and $10x^4-x^3y-3x^2y^2$.

$$\begin{aligned} 15x^3-19x^2y+6xy^2 &= x(15x^2-19xy+6y^2)\\ &= x(3x-2y)(5x-3y)\\ 10x^4-x^3y-3x^2y^2 &= x^2(10x^2-xy-3y^2)\\ &= x^2(2x+y)(5x-3y)\\ \therefore \text{ the H. C. F.} &= x(5x-3y)\\ &= 5x^2-3xy. \end{aligned}$$

29. $10x^3y + 9x^2y^2 - 9xy^3$ and $4xy^2 + 15y^3 - 4x^2y$.

$$\begin{aligned}
10x^3y + 9x^2y^2 - 9xy^3 &= xy(10x^2 + 9xy - 9y^2) \\
&= xy(5x - 3y)(2x + 3y) \\
4xy^2 + 15y^3 - 4x^2y &= y(4xy + 15y^2 - 4x^2) \\
&= y(5y - 2x)(2x + 3y) \\
\therefore \text{ the H. C. F.} &= y(2x + 3y) \\
&= 2xy + 3y^2.
\end{aligned}$$

Exercise 39. Page 115.

Find by division the H. C. F. of

1. $4x^2 + 3x - 10$, $4x^3 + 7x^2 - 3x - 15$.

$$\begin{array}{r|r|l}
4x^2 + 3x - 10 & 4x^3 + 7x^2 - 3x - 15 & x \\
4x^2 - 5x & 4x^3 + 3x^2 - 10x & \\
\cline{1-2}
8x - 10 & 4x^2 + 7x - 15 & 1 \\
8x - 10 & 4x^2 + 3x - 10 & \\
\cline{1-2}
 & 4x - 5 & x + 2
\end{array}$$

$\therefore$ the H. C. F. $= 4x - 5$.

2. $2x^3 - 6x^2 + 5x - 2$, $8x^3 - 23x^2 + 17x - 6$.

$$\begin{array}{r|r|l}
2x^3 - 6x^2 + 5x - 2 & 8x^3 - 23x^2 + 17x - 6 & 4 \\
2x^3 - 6x^2 + 4x & 8x^3 - 24x^2 + 20x - 8 & \\
\cline{1-2}
x - 2 & x^2 - 3x + 2 & 2x,\ x - 1 \\
 & x^2 - 2x & \\
\cline{2-2}
 & -x + 2 & \\
 & -x + 2 & \\
\cline{2-2}
\end{array}$$

$\therefore$ the H. C. F. $= x - 2$.

3. $20x^3 + 2x^2 - 18x + 48$, $20x^4 - 17x^2 + 48x - 3$.

$$\begin{array}{r|r|l}
20x^3 + 2x^2 - 18x + 48 & 20x^4 - 17x^2 + 48x - 3 & x \\
20x^3 - 30x^2 + 30x & 20x^4 + 2x^3 - 18x^2 + 48x & \\
\cline{1-2}
32x^2 - 48x + 48 & -2x^3 + x^2 - 3 & \\
32x^2 - 48x + 48 & 10 & \\
\cline{2-2}
 & -20x^3 + 10x^2 - 30 & -1 \\
 & -20x^3 - 2x^2 + 18x - 48 & \\
\cline{2-2}
 & 6)12x^2 - 18x + 18 & \\
\cline{2-2}
 & 2x^2 - 3x + 3 & 10x + 16
\end{array}$$

$\therefore$ the H C. F. $= 2x^2 - 3x + 3$.

4. $4x^3-2x^2-16x-91,\ 12x^3-28x^2-37x-42.$

$$\begin{array}{r|r|l}
4x^3-2x^2-16x-91 & 12x^3-28x^2-37x-42 & 3 \\
4x^3-2x^2-42x & 12x^3-6x^2-48x-273 & \\ \cline{1-2}
13)\underline{26x-91} & 11)\underline{-22x^2+11x+231} & \\
2x-7 & -2x^2+x+21 & -2x,\ -x-3 \\
 & -2x^2+7x & \\ \cline{2-2}
 & -6x+21 & \\
 & -6x+21 & \\ \cline{2-2}
\end{array}$$

$\therefore$ the H. C. F. $=2x-7.$

5. $12x^3+4x^2+17x-3,\ 24x^3-52x^2+14x-1.$

$$\begin{array}{r|r|l}
12x^3+4x^2+17x-3 & 24x^3-52x^2+14x-1 & 2 \\
12x^3+4x^2-x & 24x^3+8x^2+34x-6 & \\ \cline{1-2}
3)\underline{18x-3} & -5)\underline{-60x^2-20x+5} & \\
6x-1 & 12x^2+4x-1 & x,\ 2x+1 \\
 & 12x^2-2x & \\ \cline{2-2}
 & 6x-1 & \\
 & 6x-1 & \\ \cline{2-2}
\end{array}$$

$\therefore$ the H. C. F. $=6x-1.$

6. $2x^3+5x^2-2x+3,\ 3x^3+2x^2-17x+12.$

$$\begin{array}{r|r|l}
2x^3+5x^2-2x+3 & 3x^3+2x^2-17x+12 & \\
11 & 2 & \\ \cline{1-2}
22x^3+55x^2-22x+33 & 6x^3+4x^2-34x+24 & 3 \\
22x^3+56x^2-30x & 6x^3+15x^2-6x+9 & \\ \cline{1-2}
-x^2+8x+33 & -11x^2-28x+15 & -2x+1 \\
11 & -11x^2-33x & \\ \cline{1-2}
-11x^2+88x+363 & 5x+15 & \\
-11x^2-28x+15 & 5x+15 & \\ \cline{1-2}
116)\underline{116x+348} & & \\
x+3 & & -11x+5
\end{array}$$

$\therefore$ the H. C. F. $=x+3.$

7. $8x^4-6x^3-x^2+15x-25,\ 4x^3+7x^2-3x-15.$

$$\begin{array}{r|r|l}
4x^3+7x^2-3x-15 & 8x^4-6x^3-x^2+15x-25 & 2x-5\\
4x^3+3x^2-10x & 8x^4+14x^3-6x^2-30x & \\
\cline{1-2}
4x^2+7x-15 & -20x^3+5x^2+45x-25 & \\
4x^2+3x-10 & -20x^3-35x^2+15x+75 & \\
\cline{1-2}
4x-5 & 10\underline{)40x^2+30x-100} & \\
 & 4x^2+3x-10 & x+1\\
 & 4x^2-5x & x+2\\
\cline{2-2}
 & 8x-10 & \\
 & 8x-10 & \\
\cline{2-2}
\end{array}$$

∴ the H.C.F. $=4x-5.$

8. $4x^3-4x^2-5x+3,\ 10x^2-19x+6.$

$$\begin{array}{r|r|l}
10x^2-19x+6 & 4x^3-4x^2-5x+3 & \\
10x^2-15x & 5 & \\
\cline{1-2}
-4x+6 & 20x^3-20x^2-25x+15 & 2x\\
-4x+6 & 20x^3-38x^2+12x & \\
\cline{1-2}
 & 18x^2-37x+15 & \\
 & 5 & \\
\cline{2-2}
 & 90x^2-185x+75 & +9\\
 & 90x^2-171x+54 & \\
\cline{2-2}
 & -7\underline{)-14x+21} & \\
 & 2x-3 & 5x-2
\end{array}$$

∴ the H.C.F. $=2x-3.$

9. $6x^4-13x^3+3x^2+2x,\ 6x^4-10x^3+4x^2-6x+4.$

Remove the factor x from the first expression. Then,

$$\begin{array}{r|r|l}
6x^3-13x^2+3x+2 & 6x^4-10x^3+4x^2-6x+4 & x\\
15 & 6x^4-13x^3+3x^2+2x & \\
\cline{1-2}
90x^3-195x^2+45x+30 & 3x^3+x^2-8x+4 & \\
90x^3-114x^2+36x & 2 & \\
\cline{1-2}
3\underline{)-81x^2+9x+30} & 6x^3+2x^2-16x+8 & +1\\
-27x^2+3x+10 & 6x^3-13x^2+3x+2 & \\
\cline{2-2}
5 & & \\
\cline{1-1}
-135x^2+15x+50 & 15x^2-19x+6 & 6x-9\\
-135x^2+171x-54 & 15x^2-10x & \\
\cline{1-2}
-52\underline{)-156x+104} & -9x+6 & 5x-3\\
3x-2 & -9x+6 & \\
\cline{2-2}
\end{array}$$

∴ the H.C.F. $=3x-2.$

10. $2x^4-3x^3+2x^2-2x-3,\ 4x^4+3x^2+4x-3.$

$$\begin{array}{r|r|l}
2x^4-3x^3+2x^2-2x-3 & 4x^4\qquad\quad+3x^2+4x-3 & 2\\
3 & 4x^4-6x^3+4x^2-4x-6 & \\
\cline{1-2}
6x^4-9x^3+6x^2-6x-9 & 6x^3-x^2+8x+3 & x-4\\
6x^4-x^3+8x^2+3x & 6x^3-3x^2+9x & 3x+1\\
\cline{1-2}
-8x^3-2x^2-9x-9 & 2x^2-x+3 & \\
3 & 2x^2-x+3 & \\
\cline{1-2}
-24x^3-6x^2-27x-27 & & \\
-24x^3+4x^2-32x-12 & & \\
\cline{1-1}
-5)\,\underline{-10x^2+5x-15} & & \\
2x^2-x+3 & &
\end{array}$$

$\therefore$ the H.C.F. $=2x^2-x+3.$

11. $3x^4-x^3-2x^2+2x-8,\ 6x^3+13x^2+3x+20.$

$$\begin{array}{r|r|l}
6x^3+13x^2+3x+20 & 3x^4-x^3-2x^2+2x-8 & \\
6x^3-2x^2+8x & 2 & \\
\cline{1-2}
5)\,\underline{15x^2-5x+20} & 6x^4-2x^3-4x^2+4x-16 & x\\
3x^2-x+4 & 6x^4+13x^3+3x^2+20x & \\
\cline{2-2}
3x^2-x+4 & -15x^3-7x^2-16x-16 & \\
\cline{1-1}
 & 2 & \\
\cline{2-2}
 & -30x^3-14x^2-32x-32 & -5\\
 & -30x^3-65x^2-15x-100 & \\
\cline{2-2}
 & 17)\,\underline{51x^2-17x+68} & \\
 & 3x^2-x+4 & 2x+1
\end{array}$$

$\therefore$ the H.C.F. $=3x^2-x+4.$

12. $3x^5+2x^4+x^2,\ 3x^4+2x^3-3x^2+2x-1.$

Remove the factor x^2 from the first expression. Then,

$$\begin{array}{r|r|l}
3x^3+2x^2\qquad+1 & 3x^4+2x^3-3x^2+2x-1 & x\\
3x^3-x^2+x & 3x^4+2x^3\qquad\quad+x & \\
\cline{1-2}
3x^2-x+1 & -3x^2+x-1 & -x-1\\
3x^2-x+1 & & \\
\cline{1-1}
\end{array}$$

$\therefore$ the H.C.F. $=3x^2-x+1.$

13. $6x^5-9x^4+11x^3+6x^2-10x,\ 4x^5+10x^4+10x^3+4x^2+60x.$

Remove the factor x from the first expression, and $2x$ from the second expression. These factors have x in common. Then,

$$\begin{array}{r|r|l}
2x^4+5x^3+5x^2+2x+30 & 6x^4-9x^3+11x^2+6x-10 & 3 \\
\underline{3\qquad\qquad\qquad\qquad\qquad\qquad} & \underline{6x^4+15x^3+15x^2+6x+90} & \\
6x^4+15x^3+15x^2+6x+90 & -4)\underline{-24x^3-4x^2\qquad\qquad-100} & \\
\underline{6x^4+x^3\qquad\qquad+25x\qquad} & 6x^3+x^2\qquad\qquad+25 & x+7 \\
14x^3+15x^2-19x+90 & \underline{6x^3-9x^2+15x\qquad} & 3x \\
\underline{3\qquad\qquad\qquad\qquad\qquad} & 5)\underline{10x^2-15x+25} & \\
42x^3+45x^2-57x+270 & 2x^2-3x+5 & 1 \\
\underline{42x^3+7x^2\qquad\qquad+175} & & \\
19)\underline{38x^2-57x-95} & & \\
2x^2-3x+5 & & \\
\underline{2x^2-3x+5} & &
\end{array}$$

$\therefore$ the H. C. F. $=x(2x^2-3x+5)=2x^3-3x^2+5x.$

14. $2x^5-11x^2-9,\ 4x^5+11x^4+81.$

$$\begin{array}{r|r|l}
2x^5\qquad-11x^2\qquad-9 & 4x^5+11x^4\qquad\qquad\qquad+81 & 2 \\
\underline{2x^5+4x^3\qquad+18x\qquad} & \underline{4x^5\qquad\qquad-22x^2\qquad-18} & \\
-4x^3-11x^2-18x-9 & 11)\underline{11x^4\qquad+22x^2\qquad+99} & \\
\underline{-4x^3-8x^2-12x\qquad} & x^4\qquad+2x^2\qquad+9 & 2x \\
-3x^2-6x-9 & \underline{4\qquad\qquad\qquad\qquad\qquad} & \\
\underline{-3x^2-6x-9} & 4x^4\qquad+8x^2\qquad+36 & -x \\
 & \underline{4x^4+11x^3+18x^2+9x\qquad} & \\
 & -11x^3-10x^2-9x+36 & \\
 & \underline{4\qquad\qquad\qquad\qquad\qquad} & \\
 & -44x^3-40x^2-36x+144 & +11 \\
 & \underline{-44x^3-121x^2-198x-99} & \\
 & 81)\underline{81x^2+162x+243} & \\
 & x^2+2x+3 & -4x-3
\end{array}$$

$\therefore$ the H. C. F. $=x^2+2x+3.$

15. $x^4 - 4x^3 + 10x^2 - 12x + 9,\ x^4 + 2x^2 + 9.$

$$\begin{array}{r|r|l}
x^4 \quad + 2x^2 \quad + 9 & x^4 - 4x^3 + 10x^2 - 12x + 9 & 1 \\
\underline{x^4 - 2x^3 + 3x^2 \qquad\quad} & \underline{x^4 \qquad + 2x^2 \qquad + 9} & \\
2x^3 - x^2 \qquad + 9 & -4x\overline{)-4x^3 + 8x^2 - 12x} & \\
\underline{2x^3 - 4x^2 + 6x \quad} & x^2 - 2x + 3 & x^2 + 2x + 3 \\
3x^2 - 6x + 9 & & \\
\underline{3x^2 - 6x + 9} & &
\end{array}$$

$\therefore$ the H.C.F. $= x^2 - 2x + 3.$

16. $2x^3 - 3x^2 - 16x + 24,\ 4x^5 + 2x^4 - 28x^3 - 16x^2 - 32x.$

$$\begin{array}{r|r|l}
2x^3 - 3x^2 - 16x + 24 & 2x\overline{)4x^5 + 2x^4 - 28x^3 - 16x^2 - 32x} & \\
\underline{2x^3 \qquad - 16x \qquad} & 2x^4 + x^3 - 14x^2 - 8x - 16 & x + 2 \\
-3x^2 \qquad + 24 & \underline{2x^4 - 3x^3 - 16x^2 + 24x \qquad} & \\
\underline{-3x^2 \qquad + 24} & 4x^3 + 2x^2 - 32x - 16 & \\
 & \underline{4x^3 - 6x^2 - 32x + 48} & \\
 & 8\overline{)8x^2 \qquad - 64} & \\
 & x^2 \qquad - 8 & 2x - 3
\end{array}$$

$\therefore$ the H.C.F. $= x^2 - 8.$

17. $x^4 - x^3 - 14x^2 + x + 1,\ x^5 - 4x^4 - x^3 - 2x^2 + 8x + 2.$

$$\begin{array}{r|r|l}
x^4 - x^3 - 14x^2 + x + 1 & x^5 - 4x^4 - x^3 - 2x^2 + 8x + 2 & x - 3 \\
\underline{2 \qquad\qquad\qquad\qquad} & \underline{x^5 - x^4 - 14x^3 + x^2 + x \quad} & \\
2x^4 - 2x^3 - 28x^2 + 2x + 2 & -3x^4 + 13x^3 - 3x^2 + 7x + 2 & \\
\underline{2x^4 - 9x^3 + 2x^2 + x \quad} & \underline{-3x^4 + 3x^3 + 42x^2 - 3x - 3} & \\
7x^3 - 30x^2 + x + 2 & 5\overline{)10x^3 - 45x^2 + 10x + 5} & \\
\underline{2 \qquad\qquad\qquad} & 2x^3 - 9x^2 + 2x + 1 & x + 7 \\
14x^3 - 60x^2 + 2x + 4 & \underline{2x^3 - 8x^2 - 2x \quad} & 2x - 1 \\
\underline{14x^3 - 63x^2 + 14x + 7} & -x^2 + 4x - 1 & \\
3\overline{)3x^2 - 12x - 3} & \underline{-x^2 + 4x - 1} & \\
x^2 - 4x - 1 & &
\end{array}$$

$\therefore$ the H.C.F. $= x^2 - 4x - 1.$

18. $6x^3-14ax^2+6a^2x-4a^3,\ x^4-ax^3-a^2x^2-a^3x-2a^4.$

$2)\,6x^3-14ax^2+6a^2x-4a^3$	$x^4-ax^3-a^2x^2-a^3x-2a^4$	
$3x^3-7ax^2+3a^2x-2a^3$	3	
2		
	$3x^4-3ax^3-3a^2x^2-3a^3x-6a^4$	x
$6x^3-14ax^2+6a^2x-4a^3$	$3x^4-7ax^3+3a^2x^2-2a^3x$	
$6x^3-9ax^2-6a^2x$		
	$a)\,4ax^3-6a^2x^2-a^3x-6a^4$	
$-5ax^2+12a^2x-4a^3$	$4x^3-6ax^2-a^2x-6a^3$	
2	3	
$-10ax^2+24a^2x-8a^3$	$12x^3-18ax^2-3a^2x-18a^3$	$+4$
$-10ax^2+15a^2x+10a^3$	$12x^3-28ax^2+12a^2x-8a^3$	
$9a^2)\,9a^2x-18a^3$	$5a)\,10ax^2-15a^2x-10a^3$	
$x-2a$	$2x^2-3ax-2a^2$	$3x-5a$
	$2x^2-4ax$	$2x+a$
	$ax-2a^2$	
	$ax-2a^2$	

∴ the H. C. F. $= x-2a$.

19. $2a^4-2a^3-3a^2-2a,\ 3a^4-a^3-2a^2-16a.$

Remove the factor a from each expression. Then,

$2a^3-2a^2-3a-2$	$3a^3-a^2-2a-16$	
2	2	
$4a^3-4a^2-6a-4$	$6a^3-2a^2-4a-32$	3
$4a^3+5a^2-26a$	$6a^3-6a^2-9a-6$	
$-9a^2+20a-4$	$4a^2+5a-26$	$a-9$
4	$4a^2-8a$	$4a+13$
$-36a^2+80a-16$	$13a-26$	
$-36a^2-45a+234$	$13a-26$	
$125)\,125a-250$		
$a-2$		

∴ the H. C. F. $= a(a-2)$.

20. $2x^3+7ax^2+4a^2x-3a^3,\ 4x^3+9ax^2-2a^2x-a^3.$

$2x^3+7ax^2+4a^2x-3a^3$	$4x^3+9ax^2-2a^2x-a^3$	2
$2x^3+4ax^2-2a^2x$	$4x^3+14ax^2+8a^2x-6a^3$	
$3ax^2+6a^2x-3a^3$	$-5a)\,-5ax^2-10a^2x+5a^3$	
$3ax^2+6a^2x-3a^3$	$x^2+2ax-a^2$	$2x+3a$

∴ the H. C. F. $= x^2+2ax-a^2$.

21. $2x^3 - 9ax^2 + 9a^2x - 7a^3,\ 4x^3 - 20ax^2 + 20a^2x - 16a^3.$

$$\begin{array}{r|r|l}
2x^3 - 9ax^2 + 9a^2x - 7a^3 & 4x^3 - 20ax^2 + 20a^2x - 16a^3 & 2 \\
\underline{2x^3 - 2ax^2 + 2a^2x \qquad\quad} & \underline{4x^3 - 18ax^2 + 18a^2x - 14a^3} & \\
-7ax^2 + 7a^2x - 7a^3 & -2a\underline{)-\ 2ax^2 + \ 2a^2x - \ 2a^3} & \\
\underline{-7ax^2 + 7a^2x - 7a^3} & x^2 - \quad ax + \quad a^2 & 2x - 7a
\end{array}$$

$\therefore$ the H. C. F. $= x^2 - ax + a^2.$

22. $2x^4 + 9x^3 + 14x + 3,\ 3x^4 + 14x^3 + 9x + 2.$

$$\begin{array}{r|r|l}
2x^4 + 9x^3 \qquad\quad + \ 14x + \ 3 & 3x^4 + 14x^3 \qquad\quad + \ 9x + 2 & \\
\underline{2x^4 \qquad - 48x^2 - \ 10x \qquad} & \underline{2 \qquad\qquad\qquad\qquad\qquad\quad} & \\
9x^3 + 48x^2 + \ 24x + \ 3 & 6x^4 + 28x^3 \qquad\quad + 18x + 4 & 3 \\
\underline{9x^3 \qquad - 216x - 45} & \underline{6x^4 + 27x^3 \qquad\quad + 42x + 9} & \\
48\underline{)48x^2 + 240x + 48} & x^3 \qquad - 24x - 5 & 2x + 9 \\
x^2 + \quad 5x + \ 1 & \underline{x^3 + 5x^2 + \quad x \quad} & x + 5 \\
 & -5x^2 - 25x - 5 & \\
 & \underline{-5x^2 - 25x - 5} &
\end{array}$$

$\therefore$ the H. C. F. $= x^2 + 5x + 1.$

23. $9x^5 - 7x^3 + 8x^2 + 2x - 4,\ 6x^4 - 7x^3 - 10x^2 + 5x + 2.$

$$\begin{array}{r|r|l}
6x^4 - \ 7x^3 - \ 10x^2 + \ 5x + \ 2 & 9x^5 \qquad\qquad - 7x^3 + \ 8x^2 + \ 2x - \ 4 & \\
\underline{9 \qquad\qquad\qquad\qquad\qquad\qquad} & \underline{2 \qquad\qquad\qquad\qquad\qquad\qquad\quad} & \\
54x^4 - 63x^3 - \ 90x^2 + \ 45x + \ 18 & 18x^5 \qquad\quad -14x^3 + 16x^2 + \ 4x - \ 8 & 3x \\
\underline{54x^4 + 48x^3 - \ 26x^2 - \ 20x \qquad} & \underline{18x^5 - 21x^4 - 30x^3 + 15x^2 + \ 6x \qquad} & \\
-111x^3 - \ 64x^2 + \ 65x + \ 18 & 21x^4 + 16x^3 + \quad x^2 - \ 2x - \ 8 & \\
\underline{9 \qquad\qquad\qquad\qquad\qquad} & \underline{2 \qquad\qquad\qquad\qquad\qquad} & \\
-999x^3 - 576x^2 + 585x + 162 & 42x^4 + 32x^3 + \ 2x^2 - \ 4x - 16 & +\ 7 \\
\underline{-999x^3 - 888x^2 + 481x + 370} & \underline{42x^4 - 49x^3 - 70x^2 + 35x + 14} & \\
104\underline{)312x^2 + 104x - 208} & 3\underline{)81x^3 + 72x^2 - 39x - 30} & \\
3x^2 + \quad x - \ 2 & 27x^3 + 24x^2 - 13x - 10 & 2x - 37 \\
 & \underline{27x^3 + \ 9x^2 - 18x \qquad} & 9x + 5 \\
 & 15x^2 + \ 5x - 10 & \\
 & \underline{15x^2 + \ 5x - 10} &
\end{array}$$

$\therefore$ the H. C. F. $= 3x^2 + x - 2.$

Exercise 40. Page 116.

Find the H. C. F. of

1. $x^2+3x+2,\ x^2+4x+3,\ x^2+6x+5.$

$$\begin{aligned}x^2+3x+2&=(x+1)(x+2)\\x^2+4x+3&=(x+1)(x+3)\\x^2+6x+5&=(x+1)(x+5)\\\therefore \text{ the H. C. F.}&=x+1.\end{aligned}$$

2. $x^2-9x-10,\ x^2-7x-30,\ x^2-11x+10.$

$$\begin{aligned}x^2-9x-10&=(x+1)(x-10)\\x^2-7x-30&=(x+3)(x-10)\\x^2-11x+10&=(x-1)(x-10)\\\therefore \text{ the H. C. F.}&=x-10.\end{aligned}$$

3. $x^3-1,\ x^3-2x^2+1,\ x^3-2x+1.$

$$\begin{aligned}x^3-1&=(x-1)(x^2+x+1)\\x^3-2x^2+1&=(x-1)(x^2-x-1)\\x^3-2x\ \ +1&=(x-1)(x^2+x-1)\\\therefore \text{ the H. C. F.}&=x-1.\end{aligned}$$

4. $6x^2+x-2,\ 2x^2+7x-4,\ 2x^2-7x+3.$

$$\begin{aligned}6x^2+x-2&=(3x+2)(2x-1)\\2x^2+7x-4&=(x+4)(2x-1)\\2x^2-7x+3&=(x-3)(2x-1)\\\therefore \text{ the H. C. F.}&=2x-1.\end{aligned}$$

5. $a^2+2ab+b^2,\ a^2-b^2,\ a^3+2a^2b+2ab^2+b^3.$

$$\begin{aligned}a^2+2ab+b^2&=(a+b)(a+b)\\a^2-b^2&=(a+b)(a-b)\\a^3+2a^2b+2ab^2+b^3&=(a+b)(a^2+ab+b^2)\\\therefore \text{ the H. C. F.}&=a+b.\end{aligned}$$

6. $x^2-5ax+4a^2,\ x^2-3ax+2a^2,\ 3x^2-10ax+7a^2.$

$$\begin{aligned}x^2-5ax+4a^2&=(x-a)(x-4a)\\x^2-3ax+2a^2&=(x-a)(x-2a)\\3x^2-10ax+7a^2&=(x-a)(3x-7a)\\\therefore \text{ the H. C. F.}&=x-a.\end{aligned}$$

7. $x^2+x-6,\ x^3-2x^2-x+2,\ x^3+3x^2-6x-8.$

$$\begin{aligned}
x^2+x-6 &= (x-2)(x+3)\\
x^3-2x^2-x+2 &= (x^3-2x^2)-(x-2)\\
&= x^2(x-2)-(x-2)\\
&= (x^2-1)(x-2)\\
&= (x-1)(x+1)(x-2)\\
x^3+3x^2-6x-8 &= (x-2)(x^2+5x+4)\\
&= (x-2)(x+1)(x+4)
\end{aligned}$$

$\therefore$ the H. C. F. $= x-2.$

8. $x^3+7x^2+5x-1,\ x^2+3x-3x^3-1,\ 3x^3+5x^2+x-1.$

$$\begin{array}{r|r|l}
x^3+7x^2+5x-1 & -3x^3+x^2+3x-1 & -3\\
11 & -3x^3-21x^2-15x+3 & \\
11x^3+77x^2+55x-11 & 2)\underline{22x^2+18x-4} & \\
11x^3+9x^2-2x & 11x^2+9x-2 & x+68\\
68x^2+57x-11 & 11x^2+11x & 11x-2\\
11 & -2x-2 & \\
748x^2+627x-121 & -2x-2 & \\
748x^2+612x-136 & & \\
15)\underline{15x+15} & & \\
x+1 & &
\end{array}$$

$\therefore$ the H. C. F. of x^3+7x^2+5x-1 and $x^2+3x-3x^3-1$ is $x+1$.
Also $3x^3+5x^2+x-1=(x+1)(3x^2+2x-1)$.
$\therefore$ the required H. C. F. is $x+1$.

9. $x^3-6x^2+11x-6,\ x^3-8x^2+19x-12,\ x^3-9x^2+26x-24.$

$$\begin{array}{r|r|l}
x^3-6x^2+11x-6 & x^3-8x^2+19x-12 & 1\\
x^3-4x^2+3x & x^3-6x^2+11x-6 & \\
-2x^2+8x-6 & -2)\underline{-2x^2+8x-6} & \\
-2x^2+8x-6 & x^2-4x+3 & x-2
\end{array}$$

$\therefore$ the H. C. F. of $x^3-6x^2+11x-6$ and $x^3-8x^2+19x-12$ is

x^2-4x+3 or $(x-1)(x-3)$.

Also
$$\begin{aligned}
x^3-9x^2+26x-24 &= (x-3)(x^2-6x+8)\\
&= (x-3)(x-2)(x-4).
\end{aligned}$$

$\therefore$ the required H. C. F. is $x-3$.

Exercise 41. Page 118.

Find the L. C. M. of

1. 24, 32, and 60.

$$\begin{aligned} 24 &= 2\times 2\times 2\times 3 \\ 32 &= 2\times 2\times 2\times 2\times 2 \\ 60 &= 2\times 2\times 3\times 5 \\ \therefore \text{ the L. C. M.} &= 2\times 2\times 2\times 2\times 2\times 3\times 5 \\ &= 480. \end{aligned}$$

2. $24a^2x^3$, $60a^3x^2$, and $32a^2x^2$.

$$\begin{aligned} 24a^2x^3 &= 2\times 2\times 2\times 3\times a^2\times x^3 \\ 60a^3x^2 &= 2\times 2\times 3\times 5\times a^3\times x^2 \\ 32a^2x^2 &= 2\times 2\times 2\times 2\times 2\times a^2\times x^2 \\ \therefore \text{ the L. C. M.} &= 2\times 2\times 2\times 2\times 2\times 3\times 5\times a^3\times x^3 \\ &= 480a^3x^3. \end{aligned}$$

3. $x^2 - 2xy + y^2$ and $x^2 - y^2$.

$$\begin{aligned} x^2 - 2xy + y^2 &= (x-y)(x-y) \\ x^2 - y^2 &= (x+y)(x-y) \\ \therefore \text{ the L. C. M.} &= (x+y)(x-y)(x-y) \\ &= x^3 - x^2y - xy^2 + y^3. \end{aligned}$$

4. $x^2 - 4x + 4$, $x^2 + 4x + 4$, and $x^2 - 4$.

$$\begin{aligned} x^2 - 4x + 4 &= (x-2)(x-2) \\ x^2 + 4x + 4 &= (x+2)(x+2) \\ x^2 - 4 &= (x+2)(x-2) \\ \therefore \text{ the L. C. M.} &= (x-2)(x-2)(x+2)(x+2) \\ &= x^4 - 8x^2 + 16. \end{aligned}$$

5. $x^3 + a^3$ and $x^2 - a^2$.

$$\begin{aligned} x^3 + a^3 &= (x+a)(x^2 - ax + a^2) \\ x^2 - a^2 &= (x+a)(x-a) \\ \therefore \text{ the L. C. M.} &= (x+a)(x-a)(x^2 - ax + a^2) \\ &= x^4 - ax^3 + a^3x - a^4. \end{aligned}$$

6. $x^2 + ax + a^2$, $x^2 - a^2$, and $x^3 - a^3$.

$$\begin{aligned} x^2 + ax + a^2 &= x^2 + ax + a^2 \\ x^2 - a^2 &= (x+a)(x-a) \\ x^3 - a^3 &= (x-a)(x^2 + ax + a^2) \\ \therefore \text{ the L. C. M.} &= (x+a)(x-a)(x^2 + ax + a^2) \\ &= x^4 + ax^3 - a^3x - a^4. \end{aligned}$$

7. $x^2(x-3)^2$ and x^2-5x+6.

$$\begin{aligned} x^2(x-3)^2 &= x^2(x-3)(x-3) \\ x^2-5x+6 &= (x-2)(x-3) \\ \therefore \text{ the L. C. M.} &= x^2(x-2)(x-3)(x-3) \\ &= x^5-8x^4+21x^3-18x^2. \end{aligned}$$

8. $x^2+7x+12$ and x^4-9x^2.

$$\begin{aligned} x^2+7x+12 &= (x+3)(x+4) \\ x^4-9x^2 &= x^2(x+3)(x-3) \\ \therefore \text{ the L. C. M.} &= x^2(x+3)(x-3)(x+4) \\ &= x^5+4x^4-9x^3-36x^2. \end{aligned}$$

9. $x^2-7x+10,\ x^2-4x-5,\ x^2-x-2.$

$$\begin{aligned} x^2-7x+10 &= (x-2)(x-5) \\ x^2-4x-5 &= (x+1)(x-5) \\ x^2-x-2 &= (x+1)(x-2) \\ \therefore \text{ the L. C. M.} &= (x+1)(x-2)(x-5) \\ &= x^3-6x^2+3x+10. \end{aligned}$$

10. $1-3x-4x^2,\ 1-4x-5x^2,\ 1-9x-10x^2.$

$$\begin{aligned} 1-3x-4x^2 &= (1-4x)(1+x) \\ 1-4x-5x^2 &= (1-5x)(1+x) \\ 1-9x-10x^2 &= (1+x)(1-10x) \\ \therefore \text{ the L. C. M.} &= (1+x)(1-4x)(1-5x)(1-10x) \\ &= 1-18x+91x^2-90x^3-200x^4. \end{aligned}$$

11. $6x^2+7xy-3y^2,\ 3x^2+11xy-4y^2,\ 2x^2+11xy+12y^2.$

$$\begin{aligned} 6x^2+7xy-3y^2 &= (2x+3y)(3x-y) \\ 3x^2+11xy-4y^2 &= (x+4y)(3x-y) \\ 2x^2+11xy+12y^2 &= (2x+3y)(x+4y) \\ \therefore \text{ the L. C. M.} &= (2x+3y)(3x-y)(x+4y) \\ &= 6x^3+31x^2y+25xy^2-12y^3. \end{aligned}$$

12. $8-14a+6a^2,\ 4a+4a^2-3a^3,\ 4a^2+2a^3-6a^4.$

$$\begin{aligned} 8-14a+6a^2 &= 2(1-a)(4-3a) \\ 4a+4a^2-3a^3 &= a(2-a)(2+3a) \\ 4a^2+2a^3-6a^4 &= 2a^2(1-a)(2+3a) \\ \therefore \text{ the L. C. M.} &= 2a^2(1-a)(4-3a)(2-a)(2+3a) \\ &= 32a^2-24a^3-56a^4+66a^5-18a^6. \end{aligned}$$

13. $6x^3 + 7x^2 - 3x,\ 3x^2 + 14x - 5,\ 6x^2 + 39x + 45.$

$$\begin{aligned} 6x^3 + 7x^2 - 3x &= x(2x+3)(3x-1)\\ 3x^2 + 14x - 5 &= (3x-1)(x+5)\\ 6x^2 + 39x + 45 &= 3(x+5)(2x+3)\\ \therefore \text{ the L. C. M.} &= 3x(2x+3)(3x-1)(x+5)\\ &= 18x^4 + 111x^3 + 96x^2 - 45x. \end{aligned}$$

14. $6ax + 9bx - 2ay - 3by,\ 6x^2 + 3ax - 2xy - ay.$

$$\begin{aligned} 6ax + 9bx - 2ay - 3by &= 3x(2a+3b) - y(2a+3b)\\ &= (3x-y)(2a+3b)\\ 6x^2 + 3ax - 2xy - ay &= 3x(2x+a) - y(2x+a)\\ &= (3x-y)(2x+a).\\ \therefore \text{ the L. C. M.} &= (3x-y)(2a+3b)(2x+a). \end{aligned}$$

15. $12ax - 9ay - 8xy + 6y^2,\ 6ax + 3ay - 4xy - 2y^2.$

$$\begin{aligned} 12ax - 9ay - 8xy + 6y^2 &= 3a(4x-3y) - 2y(4x-3y)\\ &= (3a-2y)(4x-3y)\\ 6ax + 3ay - 4xy - 2y^2 &= 3a(2x+y) - 2y(2x+y)\\ &= (3a-2y)(2x+y)\\ \therefore \text{ the L. C. M.} &= (3a-2y)(4x-3y)(2x+y). \end{aligned}$$

16. $27x^3 - a^3,\ 6x^2 + ax - a^2,\ 15x^2 - 5ax + 3bx - ab.$

$$\begin{aligned} 27x^3 - a^3 &= (3x-a)(9x^2+3ax+a^2)\\ 6x^2 + ax - a^2 &= (3x-a)(2x+a)\\ 15x^2 - 5ax + 3bx - ab &= 5x(3x-a) + b(3x-a)\\ &= (5x+b)(3x-a)\\ \therefore \text{ the L. C. M.} &= (3x-a)(9x^2+3ax+a^2)(2x+a)(5x+b)\\ &= (27x^3 - a^3)(10x^2 + 2bx + 5ax + ab). \end{aligned}$$

17. $x^3 - 1,\ 2x^2 - x - 1,\ 3x^2 - x - 2.$

$$\begin{aligned} x^3 - 1 &= (x-1)(x^2+x+1)\\ 2x^2 - x - 1 &= (x-1)(2x+1)\\ 3x^2 - x - 2 &= (x-1)(3x+2)\\ \therefore \text{ the L. C. M.} &= (x-1)(x^2+x+1)(2x+1)(3x+2)\\ &= (x^3-1)(6x^2+7x+2). \end{aligned}$$

Exercise 42. Page 119.

Find the L. C. M. of

1. $6x^3-7ax^2-20a^2x,\ 3x^2+ax-4a^2.$

$$\begin{array}{r|r|l}
3x^2+ax-4a^2 & x)\underline{6x^3-7ax^2-20a^2x} & \\
\underline{3x^2+4ax} & 6x^2-7ax-20a^2 & 2 \\
-3ax-4a^2 & \underline{6x^2+2ax-8a^2} & \\
\underline{-3ax-4a^2} & -3a)\underline{-9ax-12a^2} & \\
 & 3x+4a & x-a
\end{array}$$

$\therefore$ the H. C. F. $=3x+4a.$

Hence, $3x^2+ax-4a^2=(3x+4a)(x-a),$

and $6x^3-7ax^2-20a^2x=x(3x+4a)(2x-5a).$

$\therefore$ the L. C. M. $=x(3x+4a)(x-a)(2x-5a).$

2. $3x^3-13x^2+23x-21,\ 6x^3+x^2-44x+21.$

$$\begin{array}{r|r|l}
3x^3-13x^2+23x-21 & 6x^3+x^2-44x+21 & 2 \\
\underline{3x^3-10x^2+7x} & \underline{6x^3-26x^2+46x-42} & \\
-3x^2+16x-21 & 9)\underline{27x^2-90x+63} & \\
\underline{-3x^2+10x-7} & 3x^2-10x+7 & x-1 \\
2)\underline{6x-14} & \underline{3x^2-7x} & x-1 \\
3x-7 & -3x+7 & \\
 & \underline{-3x+7} &
\end{array}$$

$\therefore$ the H. C. F. $=3x-7.$

Hence, $3x^3-13x^2+23x-21=(3x-7)(x^2-2x+3),$

and $6x^3+x^2-44x+21=(3x-7)(2x^2+5x-3).$

$\therefore$ the L. C. M. $=(3x-7)(x^2-2x+3)(2x^2+5x-3).$

3. $3x^3-3x^2y+xy^2-y^3,\ 4x^3-x^2y-3xy^2.$

$$\begin{array}{r|r|l}
x)\underline{4x^3-x^2y-3xy^2} & 3x^3-3x^2y+xy^2-y^3 & \\
4x^2-xy-3y^2 & \underline{4} & \\
\underline{4x^2-4xy} & 12x^3-12x^2y+4xy^2-4y^3 & 3x \\
3xy-3y^2 & \underline{12x^3-3x^2y-9xy^2} & \\
\underline{3xy-3y^2} & -9x^2y+13xy^2-4y^3 & \\
 & \underline{4} & \\
 & -36x^2y+52xy^2-16y^3 & -9y \\
 & \underline{-36x^2y+9xy^2+27y^3} & \\
 & 43xy)\underline{43xy^2-43y^3} & \\
 & x-y & 4x+3y
\end{array}$$

$\therefore$ the H. C. F. $= x - y$.

Hence, $$4x^3 - x^2y - 3xy^2 = x(x - y)(4x + 3y),$$
and $$3x^3 - 3x^2y + xy^2 - y^3 = (x - y)(3x^2 + y^2).$$

$\therefore$ the L. C. M. $= x(x - y)(4x + 3y)(3x^2 + y^2)$.

4. $c^4 - 2c^3 + c,\ 2c^4 - 2c^3 - 2c - 2$.

$$\begin{array}{r|r|l}
c)\underline{c^4 - 2c^3 \qquad + c} & 2)\underline{2c^4 - 2c^3 \qquad\qquad - 2c - 2} & \\
c^3 - 2c^2 \qquad + 1 & c^4 - \ c^3 \qquad\qquad - \ c - 1 & c + 1 \\
\underline{c^3 - \ c^2 - c \qquad} & \underline{c^4 - 2c^3 \qquad\qquad + \ c \qquad} & \\
- \ c^2 + c + 1 & c^3 \qquad - 2c - 1 & \\
\underline{- \ c^2 + c + 1} & \underline{c^3 - 2c^2 \qquad + 1} & \\
 & 2)\underline{2c^2 - 2c - 2} & \\
 & c^2 - \ c - 1 & c - 1
\end{array}$$

$\therefore$ the H. C. F. $= c^2 - c - 1$.

Hence, $$c^4 - 2c^3 + c = c(c^2 - c - 1)(c - 1),$$
and $$2c^4 - 2c^3 - 2c - 2 = 2(c^2 - c - 1)(c^2 + 1).$$

$\therefore$ the L. C. M. $= 2c(c^2 - c - 1)(c - 1)(c^2 + 1)$.

5. $x^3 - 8x + 3,\ x^6 - 3x^5 + 21x - 8$.

$$\begin{array}{r|r|l}
x^3 \qquad -8x+3 & x^6-3x^5 \qquad\qquad\qquad\qquad + 21x- 8 & x^3-3x^2+8x-27 \\
\underline{x^3-3x^2+ \ x \qquad} & \underline{x^6 \qquad -8x^4+ 3x^3 \qquad\qquad\qquad\qquad} & \\
3x^2-9x+3 & -3x^5+8x^4- 3x^3 \qquad + 21x- 8 & \\
\underline{3x^2-9x+3} & \underline{-3x^5 \qquad +24x^3- 9x^2 \qquad\qquad} & \\
 & 8x^4-27x^3+ 9x^2+ 21x- 8 & \\
 & \underline{8x^4 \qquad -64x^2+ 24x \qquad} & \\
 & -27x^3+73x^2- \ 3x- 8 & \\
 & \underline{-27x^3 \qquad +216x-81} & \\
 & 73)\underline{73x^2-219x+73} & \\
 & x^2- \ 3x+ \ 1 & x+3
\end{array}$$

$\therefore$ the H. C. F. $= x^2 - 3x + 1$.

Hence, $$x^3 - 8x + 3 = (x^2 - 3x + 1)(x + 3),$$
and $$x^6 - 3x^5 + 21x - 8 = (x^2 - 3x + 1)(x^4 - x^2 - 3x - 8).$$

$\therefore$ the L. C. M. $= (x + 3)(x^2 - 3x + 1)(x^4 - x^2 - 3x - 8)$.

6. $a^3 - 6a^2x + 12ax^2 - 8x^3,\ 2a^2 - 8ax + 8x^2.$

$$\begin{array}{r|r|l}
2)\,2a^2 - 8ax + 8x^2 & a^3 - 6a^2x + 12ax^2 - 8x^3 & a - 2x \\
a^2 - 4ax + 4x^2 & a^3 - 4a^2x + 4ax^2 \phantom{{}- 8x^3} & \\
\cline{2-2}
 & -2a^2x + 8ax^2 - 8x^3 & \\
 & -2a^2x + 8ax^2 - 8x^3 & \\
\cline{2-2}
\end{array}$$

$\therefore$ the H. C. F. $= a^2 - 4ax + 4x^2.$

Hence, $\quad 2a^2 - 8ax + 8x^2 = 2(a^2 - 4ax + 4x^2,$

and $\quad a^3 - 6a^2x + 12ax^2 - 8x^3 = (a^2 - 4ax + 4x^2)(a - 2x).$

$\therefore$ the L. C. M. $= 2(a^2 - 4ax + 4x^2)(a - 2x)$

$= 2a^3 - 12a^2x + 24ax^2 - 16x^3.$

7. $2x^3 + x^2 - 12x + 9,\ 2x^3 - 7x^2 + 12x - 9.$

$$\begin{array}{r|r|l}
2x^3 + x^2 - 12x + 9 & 2x^3 - 7x^2 + 12x - 9 & 1 \\
2 \phantom{{}+ x^2 - 12x + 9} & 2x^3 + x^2 - 12x + 9 & \\
\cline{1-2}
4x^3 + 2x^2 - 24x + 18 & -2)\,-8x^2 + 24x - 18 & \\
4x^3 - 12x^2 + 9x \phantom{{}+ 18} & 4x^2 - 12x + 9 & x + 7 \\
\cline{1-1}
14x^2 - 33x + 18 & 4x^2 - 6x \phantom{{}+ 9} & 2x - 3 \\
\cline{2-2}
2 \phantom{{}- 33x + 18} & -6x + 9 & \\
\cline{1-1}
28x^2 - 66x + 36 & -6x + 9 & \\
\cline{2-2}
28x^2 - 84x + 63 & & \\
\cline{1-1}
9)\,18x - 27 & & \\
2x - 3 & & \\
\end{array}$$

$\therefore$ the H. C. F. $= 2x - 3.$

Hence, $\quad 2x^3 + x^2 - 12x + 9 = (2x - 3)(x^2 + 2x - 3),$

and $\quad 2x^3 - 7x^2 + 12x - 9 = (2x - 3)(x^2 - 2x + 3).$

$\therefore$ the L. C. M. $= (2x - 3)(x^2 + 2x - 3)(x^2 - 2x + 3).$

8. $7x^3 - 2x^2 - 5,\ 7x^3 + 12x^2 + 10x + 5.$

$$\begin{array}{r|r|l}
7x^3 - 2x^2 \phantom{{}+ 5x} - 5 & 7x^3 + 12x^2 + 10x + 5 & 1 \\
7x^3 + 5x^2 + 5x \phantom{{}- 5} & 7x^3 - 2x^2 \phantom{{}+ 10x} - 5 & \\
\cline{1-2}
-7x^2 - 5x - 5 & 2)\,14x^2 + 10x + 10 & \\
-7x^2 - 5x - 5 & 7x^2 + 5x + 5 & x - 1 \\
\cline{1-1}
\end{array}$$

$\therefore$ the H. C. F. $= 7x^2 + 5x + 5$.

Hence, $\quad 7x^3 - 2x^2 - 5 = (7x^2 + 5x + 5)(x - 1)$,

and $\quad 7x^3 + 12x^2 + 10x + 5 = (7x^2 + 5x + 5)(x + 1)$.

$$\therefore \text{ the L. C. M.} = (7x^2 + 5x + 5)(x - 1)(x + 1) = (7x^2 + 5x + 5)(x^2 - 1).$$

9. $x^4 - 13x^2 + 36,\ x^4 - x^3 - 7x^2 + x + 6$.

$$\begin{array}{r|r|l}
x^4 \qquad - 13x^2 \qquad + 36 & x^4 - x^3 - 7x^2 + x + 6 & 1 \\
x^4 - 6x^3 - x^2 + 30x & x^4 \qquad - 13x^2 \qquad + 36 & \\
\cline{1-2}
6x^3 - 12x^2 - 30x + 36 & -x^3 + 6x^2 + x - 30 & -x - 6 \\
6x^3 - 36x^2 - 6x + 180 & -x^3 + x^2 + 6x & -x + 5 \\
\cline{1-2}
24\underline{)24x^2 - 24x - 144} & 5x^2 - 5x - 30 & \\
x^2 - x - 6 & 5x^2 - 5x - 30 & \\
\cline{2-2}
\end{array}$$

$\therefore$ the H. C. F. $= x^2 - x - 6$.

Hence, $\quad x^4 - 13x^2 + 36 = (x^2 - x - 6)(x^2 + x - 6)$,

and $\quad x^4 - x^3 - 7x^2 + x + 6 = (x^2 - x - 6)(x^2 - 1)$.

$\therefore$ the L. C. M. $= (x^2 - x - 6)(x^2 + x - 6)(x^2 - 1)$.

10. $2x^3 + 3x^2 - 7x - 10,\ 4x^3 - 4x^2 - 9x + 5$.

$$\begin{array}{r|r|l}
2x^3 + 3x^2 - 7x - 10 & 4x^3 - 4x^2 - 9x + 5 & 2 \\
2x^3 - x^2 - 5x & 4x^3 + 6x^2 - 14x - 20 & \\
\cline{1-2}
4x^2 - 2x - 10 & -5\underline{)-10x^2 + 5x + 25} & \\
4x^2 - 2x - 10 & 2x^2 - x - 5 & x + 2 \\
\cline{1-1}
\end{array}$$

$\therefore$ the H. C. F. $= 2x^2 - x - 5$.

Hence, $\quad 2x^3 + 3x^2 - 7x - 10 = (2x^2 - x - 5)(x + 2)$,

and $\quad 4x^3 - 4x^2 - 9x + 5 = (2x^2 - x - 5)(2x - 1)$.

$\therefore$ the L. C. M. $= (2x^2 - x - 5)(x + 2)(2x - 1)$.

11. $12x^3 - x^2 - 30x - 16,\ 6x^3 - 2x^2 - 13x - 6$.

$$\begin{array}{r|r|l}
6x^3 - 2x^2 - 13x - 6 & 12x^3 - x^2 - 30x - 16 & 2 \\
6x^3 - 8x^2 - 8x & 12x^3 - 4x^2 - 26x - 12 & \\
\cline{1-2}
6x^2 - 5x - 6 & 3x^2 - 4x - 4 & 2x + 2 \\
6x^2 - 8x - 8 & 3x^2 + 2x & x - 2 \\
\cline{1-2}
3x + 2 & -6x - 4 & \\
 & -6x - 4 & \\
\cline{2-2}
\end{array}$$

$\therefore$ the H. C. F. $= 3x + 2$.

Hence, $$6x^3 - 2x^2 - 13x - 6 = (3x + 2)(2x^2 - 2x - 3),$$
and $$12x^3 - x^2 - 30x - 16 = (3x + 2)(4x^2 - 3x - 8).$$

$\therefore$ the L. C. M. $= (3x + 2)(2x^2 - 2x - 3)(4x^2 - 3x - 8)$.

12. $6x^3 + x^2 - 5x - 2,\ 6x^3 + 5x^2 - 3x - 2$.

$$\begin{array}{r|r|l}
6x^3 + x^2 - 5x - 2 & 6x^3 + 5x^2 - 3x - 2 & 1 \\
6x^3 + 3x^2 & 6x^3 + x^2 - 5x - 2 & \\
\hline
-2x^2 - 5x - 2 & 2x\overline{)4x^2 + 2x} & \\
-2x^2 - x & 2x\ + 1 & 3x^2 - x - 2 \\
\hline
-4x - 2 & & \\
-4x - 2 & & \\
\hline
\end{array}$$

$\therefore$ the H. C. F. $= 2x + 1$.

Hence, $$6x^3 + x^2 - 5x - 2 = (2x + 1)(3x^2 - x - 2),$$
and $$6x^3 + 5x^2 - 3x - 2 = (2x + 1)(3x^2 + x - 2).$$

$\therefore$ the L. C. M. $= (2x + 1)(3x^2 - x - 2)(3x^2 + x - 2)$.

13 $x^3 - 9x^2 + 26x - 24,\ x^3 - 12x^2 + 47x - 60$.

$$\begin{array}{r|r|l}
x^3 - 9x^2 + 26x - 24 & x^3 - 12x^2 + 47x - 60 & 1 \\
x^3 - 7x^2 + 12x & x^3 - 9x^2 + 26x - 24 & \\
\hline
-2x^2 + 14x - 24 & -3\overline{)-\ 3x^2 + 21x - 36} & \\
-2x^2 + 14x - 24 & x^2 - 7x + 12 & x - 2 \\
\hline
\end{array}$$

$\therefore$ the H. C. F. $= x^2 - 7x + 12$.

Hence, $$x^3 - 9x^2 + 26x - 24 = (x^2 - 7x + 12)(x - 2),$$
and $$x^3 - 12x^2 + 47x - 60 = (x^2 - 7x + 12)(x - 5).$$

$\therefore$ the L. C. M. $= (x^2 - 7x + 12)(x - 2)(x - 5)$.

Exercise 43. Page 121.

Reduce to lowest terms:

1. $\dfrac{6ab^2}{9a^2b}$.

$$\frac{6ab^2}{9a^2b} = \frac{2 \times 3ab^2}{3 \times 3a^2b} = \frac{2b}{3a}.$$

2. $\dfrac{3ab^2c}{15a^2b^2c^2}$.

$$\frac{3ab^2c}{15a^2b^2c^2} = \frac{3ab^2c}{3 \times 5a^2b^2c^2} = \frac{1}{5ac}.$$

3. $\dfrac{26\,x^2y^3}{39\,xy^5}.$

$$\frac{26\,x^2y^3}{39\,xy^5}=\frac{2\times 13\,x^2y^3}{3\times 13\,xy^5}=\frac{2\,x}{3\,y^2}.$$

4. $\dfrac{42\,m^2b}{49\,mn^2}.$

$$\frac{42\,m^2b}{49\,mn^2}=\frac{2\times 3\times 7\,m^2b}{7\times 7\,mn^2}=\frac{6\,mb}{7\,n^2}.$$

5. $\dfrac{30\,xy^3z^4}{18\,x^2y^2z^2}.$

$$\frac{30\,xy^3z^4}{18\,x^2y^2z^2}=\frac{5\times 6\,xy^3z^4}{3\times 6\,x^2y^2z^2}=\frac{5\,yz^2}{3\,x}.$$

6. $\dfrac{21\,m^2n^2}{28\,m^2p}.$

$$\frac{21\,m^2n^2}{28\,m^2p}=\frac{3\times 7\,m^2n^2}{4\times 7\,m^2p}=\frac{3\,n^2}{4\,p}.$$

7. $\dfrac{34\,ax^3y^2}{51\,a^2xy^7}.$

$$\frac{34\,ax^3y^2}{51\,a^2xy^7}=\frac{2\times 17\,ax^3y^2}{3\times 17\,a^2xy^7}=\frac{2\,x^2}{3\,ay^5}.$$

8. $\dfrac{35\,a^6b^4c^2}{5\,a^3b^3c}.$

$$\frac{35\,a^6b^4c^2}{5\,a^3b^3c}=\frac{5\times 7\,a^6b^4c^2}{5\,a^3b^3c}=7\,a^3bc.$$

9. $\dfrac{58\,ab^2c^3}{87\,a^4b^3c^2}.$

$$\frac{58\,ab^2c^3}{87\,a^4b^3c^2}=\frac{2\times 29\,ab^2c^3}{3\times 29\,a^4b^3c^2}=\frac{2\,c}{3\,a^3b}.$$

10. $\dfrac{9\,xy-12\,y^2}{12\,x^2-16\,xy}.$

$$\frac{9\,xy-12\,y^2}{12\,x^2-16\,xy}=\frac{3\,y(3\,x-4\,y)}{4\,x(3\,x-4\,y)}=\frac{3\,y}{4\,x}.$$

11. $\dfrac{4\,a^2-9\,c^2}{4\,a^2+6\,ac}.$

$$\frac{4\,a^2-9\,c^2}{4\,a^2+6\,ac}=\frac{(2\,a+3\,c)(2\,a-3\,c)}{2\,a(2\,a+3\,c)}=\frac{2\,a-3\,c}{2\,a}.$$

12. $\dfrac{3\,a^2+6\,a}{a^2+4\,a+4}.$

$$\frac{3\,a^2+6\,a}{a^2+4\,a+4}=\frac{3\,a(a+2)}{(a+2)(a+2)}=\frac{3\,a}{a+2}.$$

13. $\dfrac{b^2-5\,b}{b^2-4\,b-5}.$

$$\frac{b^2-5\,b}{b^2-4\,b-5}=\frac{b(b-5)}{(b+1)(b-5)}=\frac{b}{b+1}.$$

14. $\dfrac{20(a^3-c^3)}{4(a^2+ac+c^2)}.$

$$\frac{20(a^3-c^3)}{4(a^2+ac+c^2)}=\frac{4\times 5(a-c)(a^2+ac+c^2)}{4(a^2+ac+c^2)}=5(a-c).$$

15. $\dfrac{x^3+y^3}{x^2+2xy+y^2}$.

$$\frac{x^3+y^3}{x^2+2xy+y^2}=\frac{(x+y)(x^2-xy+y^2)}{(x+y)(x+y)}=\frac{x^2-xy+y^2}{x+y}.$$

16. $\dfrac{x^3-27}{x^2+2x-15}$.

$$\frac{x^3-27}{x^2+2x-15}=\frac{(x-3)(x^2+3x+9)}{(x-3)(x+5)}=\frac{x^2+3x+9}{x+5}.$$

17. $\dfrac{x^2-8x+15}{2x^2-13x+21}$.

$$\frac{x^2-8x+15}{2x^2-13x+21}=\frac{(x-3)(x-5)}{(2x-7)(x-3)}=\frac{x-5}{2x-7}.$$

18. $\dfrac{x^2-x-20}{2x^2-7x-15}$.

$$\frac{x^2-x-20}{2x^2-7x-15}=\frac{(x+4)(x-5)}{(2x+3)(x-5)}=\frac{x+4}{2x+3}.$$

19. $\dfrac{4x^2+12ax+9a^2}{8x^3+27a^3}$.

$$\frac{4x^2+12ax+9a^2}{8x^3+27a^3}=\frac{(2x+3a)(2x+3a)}{(2x+3a)(4x^2-6ax+9a^2)}$$
$$=\frac{2x+3a}{4x^2-6ax+9a^2}.$$

20. $\dfrac{x^2-y^2-2yz-z^2}{x^2+2xy+y^2-z^2}$.

$$\frac{x^2-y^2-2yz-z^2}{x^2+2xy+y^2-z^2}=\frac{x^2-(y^2+2yz+z^2)}{(x^2+2xy+y^2)-z^2}$$
$$=\frac{(x-y-z)(x+y+z)}{(x+y+z)(x+y-z)}=\frac{x-y-z}{x+y-z}.$$

21. $\dfrac{x^4+x^2y^2+y^4}{x^3-y^3}$.

$$\frac{x^4+x^2y^2+y^4}{x^3-y^3}=\frac{(x^2+xy+y^2)(x^2-xy+y^2)}{(x-y)(x^2+xy+y^2)}=\frac{x^2-xy+y^2}{x-y}.$$

22. $\dfrac{2a^2+17a+21}{3a^2+26a+35}$.

$$\frac{2a^2+17a+21}{3a^2+26a+35}=\frac{(2a+3)(a+7)}{(3a+5)(a+7)}=\frac{2a+3}{3a+5}.$$

23. $\dfrac{(a+b)^2-c^2}{(a+b+c)^2}$.

$$\frac{(a+b)^2-c^2}{(a+b+c)^2}=\frac{(a+b+c)(a+b-c)}{(a+b+c)(a+b+c)}=\frac{a+b-c}{a+b+c}.$$

24. $\dfrac{x^2-y^2}{y-x}$.

$$\frac{x^2-y^2}{y-x}=\frac{(x+y)(x-y)}{y-x}=-(x+y).$$

25. $\dfrac{(x+a)^2-b^2}{(x+b)^2-a^2}$.

$$\frac{(x+a)^2-b^2}{(x+b)^2-a^2}=\frac{(x+a+b)(x+a-b)}{(x+b+a)(x+b-a)}=\frac{x+a-b}{x+b-a}.$$

26. $\dfrac{(a+b)^2-(c+d)^2}{(a+c)^2-(b+d)^2}$.

$$\frac{(a+b)^2-(c+d)^2}{(a+c)^2-(b+d)^2}=\frac{(a+b+c+d)(a+b-c-d)}{(a+c+b+d)(a+c-b-d)}=\frac{a+b-c-d}{a+c-b-d}.$$

27. $\dfrac{(a+c)^2-b^2}{4a^2c^2-(a^2+c^2-b^2)^2}$.

$$\begin{aligned}\frac{(a+c)^2-b^2}{4a^2c^2-(a^2+c^2-b^2)^2}&=\frac{(a+c+b)(a+c-b)}{(2ac+a^2+c^2-b^2)(2ac-a^2-c^2+b^2)}\\&=\frac{(a+c+b)(a+c-b)}{[(a^2+2ac+c^2)-b^2][b^2-(a^2-2ac+c^2)]}\\&=\frac{(a+c+b)(a+c-b)}{(a+c+b)(a+c-b)(b+a-c)(b-a+c)}\\&=\frac{1}{(a+b-c)(b-a+c)}.\end{aligned}$$

Reduce by finding the H. C. F. of the terms:

28. $\dfrac{x^3-6x-4}{3x^3-8x+8}.$

$$\begin{array}{r|r|l}
x^3 \qquad\quad -6x-4 & 3x^3-\ \ 8x+\ \ 8 & 3 \\
x^3+2x^2 \qquad\qquad\ \ & 3x^3-18x-12 & \\
\hline
-2x^2-6x-4 & 10\underline{)10x+20} & \\
-2x^2-4x \quad\ \ & x+\ \ 2 & x^2-2x-2 \\
\cline{1-1}
-2x-4 & & \\
-2x-4 & & \\
\cline{1-1}
\end{array}$$

$\therefore$ the H. C. F. $= x+2$.

$$\therefore\ \frac{x^3-6x-4}{3x^3-8x+8}=\frac{(x+2)(x^2-2x-2)}{(x+2)(3x^2-6x+4)}=\frac{x^2-2x-2}{3x^2-6x+4}.$$

29. $\dfrac{x^3-3x+2}{x^3+4x^2-5}.$

$$\begin{array}{r|r|l}
x^3 \qquad\qquad -\ \ 3x+\ \ 2 & x^3+4x^2 \qquad\ -5 & 1 \\
4 \qquad\qquad\qquad\qquad\quad\ & x^3 \qquad\quad -3x+2 & \\
\hline
4x^3 \qquad\quad\ -12x+\ \ 8 & 4x^2+3x-7 & x-3 \\
4x^3+\ \ 3x^2-\ \ 7x \qquad & 4x^2-4x \quad\ \ & 4x+7 \\
\hline
-\ \ 3x^2-\ \ 5x+\ \ 8 & 7x-7 & \\
4 \qquad\qquad\quad\ \ & 7x-7 & \\
\cline{1-2}
-12x^2-20x+32 & & \\
-12x^2-\ \ 9x+21 & & \\
\cline{1-1}
-11\underline{)-11x+11} & & \\
x-\ \ 1 & &
\end{array}$$

$\therefore$ the H. C. F. $= x-1$.

$$\therefore\ \frac{x^3-3x+2}{x^3+4x^2-5}=\frac{(x-1)(x^2+x-2)}{(x-1)(x^2+5x+5)}=\frac{x^2+x-2}{x^2+5x+5}.$$

30. $\dfrac{3x^3 - x^2 - x - 1}{3x^3 - 4x^2 - x + 2}.$

$$\begin{array}{r|r|l}
3x^3 - x^2 - x - 1 & 3x^3 - 4x^2 - x + 2 & 1 \\
\underline{3x^3 \qquad\; - 3x \phantom{{}-1}} & \underline{3x^3 - x^2 - x - 1} & \\
-x^2 + 2x - 1 & -3)\underline{-3x^2 \qquad + 3} & \\
\underline{-x^2 \qquad + 1} & x^2 \qquad - 1 & 3x - 1 \\
2)\underline{2x - 2} & \underline{x^2 - x \qquad\;} & x + 1 \\
x - 1 & x - 1 & \\
 & \underline{x - 1} &
\end{array}$$

∴ the H. C. F. $= x - 1$.

$$\therefore \frac{3x^3 - x^2 - x - 1}{3x^3 - 4x^2 - x + 2} = \frac{(x-1)(3x^2 + 2x + 1)}{(x-1)(3x^2 - x - 2)} = \frac{3x^2 + 2x + 1}{3x^2 - x - 2}.$$

31. $\dfrac{x^4 - 13x^2 + 36}{x^4 - x^3 - 7x^2 + x + 6}.$

$$\begin{array}{r|r|l}
x^4 \qquad\quad - 13x^2 \qquad\quad + 36 & x^4 - x^3 - 7x^2 + x + 6 & 1 \\
\underline{x^4 - 6x^3 - x^2 + 30x \phantom{{}+ 136}} & \underline{x^4 \qquad - 13x^2 \qquad + 36} & \\
6x^3 - 12x^2 - 30x + 36 & -x^3 + 6x^2 + x - 30 & -x - 6 \\
\underline{6x^3 - 36x^2 - 6x + 180} & \underline{-x^3 + x^2 + 6x \phantom{{}-30}} & -x + 5 \\
24)\underline{24x^2 - 24x - 144} & 5x^2 - 5x - 30 & \\
x^2 - x - 6 & \underline{5x^2 - 5x - 30} &
\end{array}$$

∴ the H. C. F. $= x^2 - x - 6$.

$$\therefore \frac{x^4 - 13x^2 + 36}{x^4 - x^3 - 7x^2 + x + 6} = \frac{(x^2 - x - 6)(x^2 + x - 6)}{(x^2 - x - 6)(x^2 - 1)} = \frac{x^2 + x - 6}{x^2 - 1}.$$

32. $\dfrac{3x^3 + 17x^2 + 22x + 8}{6x^3 + 25x^2 + 23x + 6}.$

$$\begin{array}{r|r|l}
3x^3 + 17x^2 + 22x + 8 & 6x^3 + 25x^2 + 23x + 6 & 2 \\
\underline{3 \qquad\qquad\qquad\qquad\qquad\quad} & \underline{6x^3 + 34x^2 + 44x + 16} & \\
9x^3 + 51x^2 + 66x + 24 & -9x^2 - 21x - 10 & -x - 10 \\
\underline{9x^3 + 21x^2 + 10x \phantom{{}+ 124}} & \underline{-9x^2 - 6x \phantom{{}-10}} & -3x - 5 \\
30x^2 + 56x + 24 & -15x - 10 & \\
\underline{3 \qquad\qquad\qquad\quad} & \underline{-15x - 10} & \\
90x^2 + 168x + 72 & & \\
\underline{90x^2 + 210x + 100} & & \\
-14)\underline{-42x - 28} & & \\
3x + 2 & &
\end{array}$$

$\therefore$ the H. C. F. $= 3x + 2$.

$$\therefore \frac{3x^3 + 17x^2 + 22x + 8}{6x^3 + 25x^2 + 23x + 6} = \frac{(3x+2)(x^2+5x+4)}{(3x+2)(2x^2+7x+3)} = \frac{x^2+5x+4}{2x^2+7x+3}.$$

33. $\dfrac{x^3 - 3x^2 - 15x + 25}{x^3 + 7x^2 + 5x - 25}$.

$$\begin{array}{r|r|l}
x^3 - 3x^2 - 15x + 25 & x^3 + 7x^2 + 5x - 25 & 1 \\
x^3 + 2x^2 - 5x & x^3 - 3x^2 - 15x + 25 & \\
\cline{1-2}
-5x^2 - 10x + 25 & 10\overline{)10x^2 + 20x - 50} & \\
-5x^2 - 10x + 25 & x^2 + 2x - 5 & x - 5 \\
\cline{1-1}
\end{array}$$

$\therefore$ the H. C. F. $= x^2 + 2x - 5$.

$$\therefore \frac{x^3 - 3x^2 - 15x + 25}{x^3 + 7x^2 + 5x - 25} = \frac{(x^2 + 2x - 5)(x - 5)}{(x^2 + 2x - 5)(x + 5)} = \frac{x - 5}{x + 5}.$$

34. $\dfrac{2x^3 + x^2 - 8x + 3}{3x^3 + 8x^2 + x - 2}$.

$$\begin{array}{r|r|l}
2x^3 + x^2 - 8x + 3 & 3x^3 + 8x^2 + x - 2 & \\
2x^3 + 4x^2 - 2x & 2 & \\
\cline{1-2}
-3x^2 - 6x + 3 & 6x^3 + 16x^2 + 2x - 4 & 3 \\
-3x^2 - 6x + 3 & 6x^3 + 3x^2 - 24x + 9 & \\
\cline{1-2}
 & 13\overline{)13x^2 + 26x - 13} & \\
 & x^2 + 2x - 1 & 2x - 3
\end{array}$$

$\therefore$ the H. C. F. $= x^2 + 2x - 1$.

$$\therefore \frac{2x^3 + x^2 - 8x + 3}{3x^3 + 8x^2 + x - 2} = \frac{(x^2 + 2x - 1)(2x - 3)}{(x^2 + 2x - 1)(3x + 2)} = \frac{2x - 3}{3x + 2}.$$

35. $\dfrac{x^3 + 4x^2 - 8x + 24}{x^4 - x^3 + 8x - 8}$.

$$\begin{array}{r|r|l}
x^3 + 4x^2 - 8x + 24 & x^4 - x^3 + 8x - 8 & x - 5 \\
x^3 - 2x^2 + 4x & x^4 + 4x^3 - 8x^2 + 24x & \\
\cline{1-2}
6x^2 - 12x + 24 & -5x^3 + 8x^2 - 16x - 8 & \\
6x^2 - 12x + 24 & -5x^3 - 20x^2 + 40x - 120 & \\
\cline{1-2}
 & 28\overline{)28x^2 - 56x + 112} & \\
 & x^2 - 2x + 4 & x + 6
\end{array}$$

$\therefore$ the H. C. F. $= x^2 - 2x + 4$.

$$\therefore \frac{x^3 + 4x^2 - 8x + 24}{x^4 - x^3 + 8x - 8} = \frac{(x^2 - 2x + 4)(x + 6)}{(x^2 - 2x + 4)(x^2 + x - 2)} = \frac{x + 6}{x^2 + x - 2}.$$

Exercise 44 Page 124.

Reduce to integral or mixed expressions:

1. $\dfrac{4x^2+12x+3}{4x}.$

$$\frac{4x^2+12x+3}{4x}=\frac{4x^2}{4x}+\frac{12x}{4x}+\frac{3}{4x}$$
$$=x+3+\frac{3}{4x}.$$

2. $\dfrac{3x^2-9x-2}{3x}.$

$$\frac{3x^2-9x-2}{3x}=\frac{3x^2}{3x}-\frac{9x}{3x}-\frac{2}{3x}$$
$$=x-3-\frac{2}{3x}.$$

3. $\dfrac{x^2+y^2}{x+y}.$

$$\begin{array}{r|l} x^2+y^2 & x+y \\ \cline{2-2} x^2+xy & x-y \\ \hline -xy+y^2 & \\ -xy-y^2 & \\ \hline 2y^2 & \end{array}$$

$$\therefore\ \frac{x^2+y^2}{x+y}=x-y+\frac{2y^2}{x+y}.$$

4. $\dfrac{x^2+y^2}{x-y}.$

$$\begin{array}{r|l} x^2+y^2 & x-y \\ \cline{2-2} x^2-xy & x+y \\ \hline xy+y^2 & \\ xy-y^2 & \\ \hline 2y^2 & \end{array}$$

$$\therefore\ \frac{x^2+y^2}{x-y}=x+y+\frac{2y^2}{x-y}.$$

5. $\dfrac{x^3+y^3}{x-y}.$

$$\begin{array}{r|l} x^3+y^3 & x-y \\ \cline{2-2} x^3-x^2y & x^2+xy+y^2 \\ \hline x^2y+y^3 & \\ x^2y-xy^2 & \\ \hline xy^2+y^3 & \\ xy^2-y^3 & \\ \hline 2y^3 & \end{array}$$

$$\therefore\ \frac{x^3+y^3}{x-y}=x^2+xy+y^2+\frac{2y^3}{x-y}.$$

6. $\dfrac{x^3-y^3}{x+y}.$

$$\begin{array}{r|l} x^3-y^3 & x+y \\ \cline{2-2} x^3+x^2y & x^2-xy+y^2 \\ \hline -x^2y-y^3 & \\ -x^2y-xy^2 & \\ \hline xy^2-y^3 & \\ xy^2+y^3 & \\ \hline -2y^3 & \end{array}$$

$$\therefore\ \frac{x^3-y^3}{x+y}=x^2-xy+y^2-\frac{2y^3}{x+y}.$$

7. $\frac{x^4+81}{x+3}$.

$$\begin{array}{l|l}
x^4+81 & x+3 \\ \cline{2-2}
x^4+3x^3 & x^3-3x^2+9x-27 \\ \cline{1-1}
-3x^3+81 & \\
-3x^3-9x^2 & \\ \cline{1-1}
9x^2+81 & \\
9x^2+27x & \\ \cline{1-1}
-27x+81 & \\
-27x-81 & \\ \cline{1-1}
162 &
\end{array}$$

$$\therefore \frac{x^4+81}{x+3}=x^3-3x^2+9x-27+\frac{162}{x+3}.$$

8. $\frac{2a^2-ab-b^2}{a+b}$.

$$\begin{array}{l|l}
2a^2-ab-b^2 & a+b \\ \cline{2-2}
2a^2+2ab & 2a-3b \\ \cline{1-1}
-3ab-b^2 & \\
-3ab-3b^2 & \\ \cline{1-1}
2b^2 &
\end{array}$$

$$\therefore \frac{2a^2-ab-b^2}{a+b}=2a-3b+\frac{2b^2}{a+b}.$$

9. $\frac{3x^2+2x+1}{x+4}$.

$$\begin{array}{l|l}
3x^2+2x+1 & x+4 \\ \cline{2-2}
3x^2+12x & 3x-10 \\ \cline{1-1}
-10x+1 & \\
-10x-40 & \\ \cline{1-1}
41 &
\end{array}$$

$$\therefore \frac{3x^2+2x+1}{x+4}=3x-10+\frac{41}{x+4}.$$

10. $\frac{a^3+2x^3}{a+2x}$.

$$\begin{array}{l|l}
a^3+2x^3 & a+2x \\ \cline{2-2}
a^3+2a^2x & a^2-2ax+4x^2 \\ \cline{1-1}
-2a^2x+2x^3 & \\
-2a^2x-4ax^2 & \\ \cline{1-1}
4ax^2+2x^3 & \\
4ax^2+8x^3 & \\ \cline{1-1}
-6x^3 &
\end{array}$$

$$\therefore \frac{a^3+2x^3}{a+2x}=a^2-2ax+4x^2-\frac{6x^3}{a+2x}.$$

11. $\frac{4x^2-3x-54}{x-4}$.

$$\begin{array}{l|l}
4x^2-3x-54 & x-4 \\ \cline{2-2}
4x^2-16x & 4x+13 \\ \cline{1-1}
13x-54 & \\
13x-52 & \\ \cline{1-1}
-2 &
\end{array}$$

$$\therefore \frac{4x^2-3x-54}{x-4}=4x+13-\frac{2}{x-4}.$$

12. $\frac{3x^3-x^2-2}{x^2+x-3}$.

$$\begin{array}{l|l}
3x^3-x^2-2 & x^2+x-3 \\ \cline{2-2}
3x^3+3x^2-9x & 3x-4 \\ \cline{1-1}
-4x^2+9x-2 & \\
-4x^2-4x+12 & \\ \cline{1-1}
13x-14 &
\end{array}$$

$$\therefore \frac{3x^3-x^2-2}{x^2+x-3}=3x-4+\frac{13x-14}{x^2+x-3}.$$

13. $\dfrac{4x^2 + 6ax + 9a^2}{2x - 3a}$.

$$\begin{array}{r|l}
4x^2 + 6ax + 9a^2 & 2x - 3a \\ \cline{2-2}
4x^2 - 6ax \phantom{{}+9a^2} & 2x + 6a \\ \cline{1-1}
12ax + 9a^2 & \\
12ax - 18a^2 & \\ \cline{1-1}
27a^2 &
\end{array}$$

$$\therefore \frac{4x^2 + 6ax + 9a^2}{2x - 3a} = 2x + 6a + \frac{27a^2}{2x - 3a}.$$

14. $\dfrac{a^3 + 4a^2 - 5}{a^2 + a - 2}$.

$$\begin{array}{r|l}
a^3 + 4a^2 \phantom{{}-2a} - 5 & a^2 + a - 2 \\ \cline{2-2}
a^3 + a^2 - 2a \phantom{{}-5} & a + 3 \\ \cline{1-1}
3a^2 + 2a - 5 & \\
3a^2 + 3a - 6 & \\ \cline{1-1}
- a + 1 &
\end{array}$$

$$\therefore \frac{a^3 + 4a^2 - 5}{a^2 + a - 2} = a + 3 - \frac{a - 1}{a^2 + a - 2}.$$

Exercise 45. Page 125.

Reduce to a fraction:

1. $x + 1 + \dfrac{3x - 4}{2x}$.

$$x + 1 + \frac{3x - 4}{2x} = \frac{2x(x + 1) + 3x - 4}{2x}$$
$$= \frac{2x^2 + 2x + 3x - 4}{2x}$$
$$= \frac{2x^2 + 5x - 4}{2x}.$$

2. $x + 1 - \dfrac{2x - 3}{2x}$.

$$x + 1 - \frac{2x - 3}{2x} = \frac{2x(x + 1) - (2x - 3)}{2x}$$
$$= \frac{2x^2 + 2x - 2x + 3}{2x}$$
$$= \frac{2x^2 + 3}{2x}.$$

3. $a+b-\dfrac{2ab}{a+b}$.

$$a+b-\frac{2ab}{a+b}=\frac{(a+b)(a+b)-2ab}{a+b}$$
$$=\frac{a^2+2ab+b^2-2ab}{a+b}$$
$$=\frac{a^2+b^2}{a+b}.$$

4. $x-1-\dfrac{2}{x-2}$.

$$x-1-\frac{2}{x-2}=\frac{(x-2)(x-1)-2}{x-2}$$
$$=\frac{x^2-3x+2-2}{x-2}$$
$$=\frac{x^2-3x}{x-2}.$$

5. $1-\dfrac{a-b}{a+b}$.

$$1-\frac{a-b}{a+b}=\frac{a+b-(a-b)}{a+b}$$
$$=\frac{2b}{a+b}.$$

6. $5a-\dfrac{1+5a^2}{a}$.

$$5a-\frac{1+5a^2}{a}=\frac{5a^2-(1+5a^2)}{a}$$
$$=-\frac{1}{a}.$$

7. $a+x-\dfrac{a^2+x^2}{a+x}$.

$$a+x-\frac{a^2+x^2}{a+x}=\frac{(a+x)(a+x)-(a^2+x^2)}{a+x}$$
$$=\frac{a^2+2ax+x^2-a^2-x^2}{a+x}.$$
$$=\frac{2ax}{a+x}.$$

8. $x+4-\frac{x-12}{x-3}$.

$$\begin{aligned} x+4-\frac{x-12}{x-3} &= \frac{(x-3)(x+4)-(x-12)}{x-3} \\ &= \frac{x^2+x-12-x+12}{x-3} \\ &= \frac{x^2}{x-3}. \end{aligned}$$

9. $a^2-ax+x^2-\frac{x^3}{a+x}$.

$$\begin{aligned} a^2-ax+x^2-\frac{x^3}{a+x} &= \frac{(a+x)(a^2-ax+x^2)-x^3}{a+x} \\ &= \frac{(a^3+x^3)-x^3}{a+x} \\ &= \frac{a^3}{a+x}. \end{aligned}$$

10. $a^2+ax+x^2-\frac{a^3}{a-x}$.

$$\begin{aligned} a^2+ax+x^2-\frac{a^3}{a-x} &= \frac{(a-x)(a^2+ax+x^2)-a^3}{a-x} \\ &= \frac{(a^3-x^3)-a^3}{a-x} \\ &= -\frac{x^3}{a-x}. \end{aligned}$$

11. $\frac{a-3x}{4}-a+2x$.

$$\begin{aligned} \frac{a-3x}{4}-a+2x &= \frac{a-3x-4(a-2x)}{4} \\ &= \frac{a-3x-4a+8x}{4} \\ &= \frac{5x-3a}{4}. \end{aligned}$$

12. $3a-2b-\frac{3a^2-2b^2}{a+b}$.

$$\begin{aligned} 3a-2b-\frac{3a^2-2b^2}{a+b} &= \frac{(a+b)(3a-2b)-(3a^2-2b^2)}{a+b} \\ &= \frac{3a^2+ab-2b^2-3a^2+2b^2}{a+b} \\ &= \frac{ab}{a+b}. \end{aligned}$$

13. $2x-7-\dfrac{21-13x}{x-3}$.

$$2x-7-\frac{21-13x}{x-3}=\frac{(x-3)(2x-7)-(21-13x)}{x-3}$$
$$=\frac{2x^2-13x+21-21+13x}{x-3}$$
$$=\frac{2x^2}{x-3}.$$

14. $5x-3+\dfrac{3x+21}{x+7}$.

$$5x-3+\frac{3x+21}{x+7}=5x-3+3$$
$$=5x.$$

Exercise 46. Page 127.

Express with lowest common denominator:

1. $\dfrac{a-2x}{3a}$, $\dfrac{3x^2-2ax}{9ax}$.

The L. C. D. is $9ax$.
The respective quotients are $3x$ and 1.
The respective products are $3ax-6x^2$ and $3x^2-2ax$.
Hence the required fractions are $\dfrac{3ax-6x^2}{9ax}$ and $\dfrac{3x^2-2ax}{9ax}$.

2. $\dfrac{1}{x+2}$, $\dfrac{2}{x+3}$.

The L. C. D. is $(x+2)(x+3)$.
The respective quotients are $x+3$ and $x+2$.
The respective products are $x+3$ and $2x+4$.
Hence the required fractions are $\dfrac{x+3}{(x+2)(x+3)}$ and $\dfrac{2x+4}{(x+2)(x+3)}$.

3. $\dfrac{a}{x-a}$, $\dfrac{a^2}{x^2-a^2}$.

The L. C. D. is x^2-a^2.
The respective quotients are $x+a$ and 1.
The respective products are $ax+a^2$ and a^2.
Hence the required fractions are $\dfrac{ax+a^2}{x^2-a^2}$ and $\dfrac{a^2}{x^2-a^2}$.

4. $\frac{4a^2+c^2}{4a^2-c^2}, \frac{2a+c}{2a-c}.$

The L. C. D. is $4a^2-c^2$.

The respective quotients are 1 and $2a+c$.

The respective products are $4a^2+c^2$ and $4a^2+4ac+c^2$.

Hence the required fractions are $\frac{4a^2+c^2}{4a^2-c^2}$ and $\frac{4a^2+4ac+c^2}{4a^2-c^2}$.

5. $\frac{x^2+y^2}{25x^2-4y^2}, \frac{1}{5x+2y}.$

The L. C. D. is $25x^2-4y^2$.

The respective quotients are 1 and $5x-2y$.

The respective products are x^2+y^2 and $5x-2y$.

Hence the required fractions are $\frac{x^2+y^2}{25x^2-4y^2}$ and $\frac{5x-2y}{25x^2-4y^2}$.

6. $\frac{x+2}{x-2}, \frac{x-2}{x+2}.$

The L. C. D. is x^2-4.

The respective quotients are $x+2$ and $x-2$.

The respective products are $(x+2)^2$ and $(x-2)^2$.

Hence the required fractions are $\frac{x^2+4x+4}{x^2-4}$ and $\frac{x^2-4x+4}{x^2-4}$.

7. $\frac{1}{x+y}, \frac{1}{x-y}, \frac{2}{x^2-y^2}.$

The L. C. D. is x^2-y^2.

The respective quotients are $x-y$, $x+y$, and 1.

The respective products are $x-y$, $x+y$, and 2.

Hence the required fractions are $\frac{x-y}{x^2-y^2}$, $\frac{x+y}{x^2-y^2}$, and $\frac{2}{x^2-y^2}$.

8. $\frac{1}{1+2x}, \frac{1}{1-4x^2}, \frac{3}{1-2x}.$

The L. C. D. is $1-4x^2$.

The respective quotients are $1-2x$, 1, and $1+2x$.

The respective products are $1-2x$, 1, and $3+6x$.

Hence the required fractions are $\frac{1-2x}{1-4x^2}$, $\frac{1}{1-4x^2}$, and $\frac{3+6x}{1-4x^2}$

9. $\frac{5}{9-x^2}, \frac{7}{3+x}, \frac{3}{3-x}.$

The L. C. D. is $9-x^2$.

The respective quotients are 1, $3-x$, and $3+x$.

The respective products are 5, $21-7x$, and $9+3x$.

Hence the required fractions are $\frac{5}{9-x^2}$, $\frac{21-7x}{9-x^2}$, and $\frac{9+3x}{9-x^2}$.

10. $\frac{1}{x^2-9x+18}$, $\frac{1}{x^2-10x+24}$.

$$\frac{1}{x^2-9x+18}=\frac{1}{(x-3)(x-6)},\quad \frac{1}{x^2-10x+24}=\frac{1}{(x-4)(x-6)}.$$

The L. C. D. is $(x-3)(x-4)(x-6)$.

The respective quotients are $x-4$ and $x-3$.

The respective products are $x-4$ and $x-3$.

Hence the required fractions are

$$\frac{x-4}{(x-3)(x-4)(x-6)} \text{ and } \frac{x-3}{(x-3)(x-4)(x-6)}.$$

Exercise 47. Page 129.

Simplify:

1. $\frac{x-1}{2}-\frac{x-3}{5}-\frac{x-7}{10}+\frac{x-2}{5}$.

The L. C. D. is 10.

The respective multipliers are 5, 2, 1, and 2.

$$\begin{aligned}\therefore\ 5(x-1) &= 5x-5 = \text{1st numerator.}\\ -2(x-3) &= -2x+6 = \text{2d numerator.}\\ -(x-7) &= -x+7 = \text{3d numerator.}\\ 2(x-2) &= \underline{2x-4} = \text{4th numerator.}\\ & 4x+4 = \text{the sum of the numerators.}\end{aligned}$$

$\therefore$ the sum of the fractions is $\frac{4x+4}{10}=\frac{2x+2}{5}$.

2. $\frac{2x-1}{3}-\frac{x+7}{6}+\frac{x-4}{4}-\frac{x-3}{2}$.

The L. C. D. is 12.

The respective multipliers are 4, 2, 3, and 6.

$$\begin{aligned}\therefore\ 4(2x-1) &= 8x-4 = \text{1st numerator.}\\ -2(x+7) &= -2x-14 = \text{2d numerator.}\\ 3(x-4) &= 3x-12 = \text{3d numerator.}\\ -6(x-3) &= \underline{-6x+18} = \text{4th numerator.}\\ & 3x-12 = \text{the sum of the numerators.}\end{aligned}$$

$\therefore$ the sum of the fractions is $\frac{3x-12}{12}=\frac{x-4}{4}$.

3. $\frac{7x-5}{8}-\frac{3x+2}{3}+\frac{x+1}{4}-\frac{5x-10}{12}.$

The L. C. D. is 24.

The respective multipliers are 3, 8, 6, and 2.

$$\begin{array}{rrl}
\therefore\ 3(7x-5) = & 21x-15 = & \text{1st numerator.} \\
-8(3x+2) = & -24x-16 = & \text{2d numerator.} \\
6(x+1) = & 6x+6 = & \text{3d numerator.} \\
-2(5x-10) = & \underline{-10x+20} = & \text{4th numerator.} \\
 & -7x-5 = & \text{the sum of the numerators.}
\end{array}$$

$\therefore$ the sum of the fractions is $\frac{-7x-5}{24}=-\frac{7x+5}{24}.$

4. $\frac{2x+3}{9}+\frac{x-2}{6}-\frac{5x+4}{12}-\frac{2x-4}{3}.$

The L. C. D is 36.

The respective multipliers are 4, 6, 3, and 12.

$$\begin{array}{rrl}
\therefore\ 4(2x+3) = & 8x+12 = & \text{1st numerator.} \\
6(x-2) = & 6x-12 = & \text{2d numerator.} \\
-3(5x+4) = & -15x-12 = & \text{3d numerator.} \\
-12(2x-4) = & \underline{-24x+48} = & \text{4th numerator.} \\
 & -25x+36 = & \text{the sum of the numerators.}
\end{array}$$

$\therefore$ the sum of the fractions is $\frac{-25x+36}{36}=\frac{36-25x}{36}.$

5. $\frac{2x+3}{2x}+\frac{x+3}{4x}-\frac{18x+5}{8x^2}-\frac{x-3}{x}.$

The L. C. D. is $8x^2$.

The respective multipliers are $4x$, $2x$, 1, and $8x$.

$$\begin{array}{rrl}
\therefore\ 4x(2x+3) = & 8x^2+12x = & \text{1st numerator.} \\
2x(x+3) = & 2x^2+6x = & \text{2d numerator.} \\
-(18x+5) = & -18x-5 = & \text{3d numerator.} \\
-8x(x-3) = & \underline{-8x^2+24x} = & \text{4th numerator.} \\
 & 2x^2+24x-5 = & \text{the sum of the numerators.}
\end{array}$$

$\therefore$ the sum of the fractions is $\frac{2x^2+24x-5}{8x^2}.$

6. $\frac{x}{2}-\frac{2x-11}{3}-\frac{x+3}{4}+\frac{x-7}{6}-\frac{x-1}{12}.$

The L. C. D. is 12.

The respective multipliers are 6, 4, 3, 2, and 1.

$$\begin{array}{rrl}
\therefore\ 6x & & = \text{1st numerator.}\\
-4(2x-11) = & -8x+44 & = \text{2d numerator.}\\
-3(x+3) = & -3x-9 & = \text{3d numerator.}\\
2(x-7) = & 2x-14 & = \text{4th numerator.}\\
-(x-1) = & -x+1 & = \text{5th numerator.}\\
\hline
 & -4x+22 & = \text{the sum of the numerators.}
\end{array}$$

$\therefore$ the sum of the fractions is $\frac{-4x+22}{12}=\frac{-2x+11}{6}=\frac{11-2x}{6}.$

7. $\frac{4a^2}{b^2}-\frac{a+b}{ab}+\frac{4b^2}{a^2}+\frac{a^2b+ab^2-4a^4}{a^2b^2}.$

The L. C. D. is a^2b^2.

The respective multipliers are a^2, ab, b^2, and 1.

$$\begin{array}{rlrl}
\therefore\ 4a^4 & & & = \text{1st numerator.}\\
-ab(a+b) = & -a^2b-ab^2 & & = \text{2d numerator.}\\
 & & 4b^4 & = \text{3d numerator.}\\
-4a^4 & +a^2b+ab^2 & & = \text{4th numerator.}\\
\hline
 & & 4b^4 & = \text{the sum of the numerators.}
\end{array}$$

$\therefore$ the sum of the fractions is $\frac{4b^4}{a^2b^2}=\frac{4b^2}{a^2}.$

8. $\frac{5x-11}{4}-\frac{x-1}{10}+\frac{11x-1}{12}-\frac{12x-5}{3}.$

The L. C. D. is 60.

The respective multipliers are 15, 6, 5, and 20.

$$\begin{array}{rrl}
\therefore\ 15(5x-11) = & 75x-165 & = \text{1st numerator.}\\
-6(x-1) = & -6x+6 & = \text{2d numerator.}\\
5(11x-1) = & 55x-5 & = \text{3d numerator.}\\
-20(12x-5) = & -240x+100 & = \text{4th numerator.}\\
\hline
 & -116x-64 & = \text{the sum of the numerators.}
\end{array}$$

$\therefore$ the sum of the fractions is

$$\frac{-116x-64}{60}=\frac{-29x-16}{15}=-\frac{29x+16}{15}.$$

9. $\frac{x+1}{2}+\frac{3x-4}{5}+\frac{1}{4}-\frac{6x+7}{8}.$

The L. C. D. is 40.
The respective multipliers are 20, 8, 10, and 5.

$$\begin{aligned}
\therefore\ 20(x+1) &= 20x+20 = \text{1st numerator.}\\
8(3x-4) &= 24x-32 = \text{2d numerator.}\\
&10 = \text{3d numerator.}\\
-5(6x+7) &= \underline{-30x-35} = \text{4th numerator.}\\
&14x-37 = \text{the sum of the numerators.}
\end{aligned}$$

$\therefore$ the sum of the fractions is $\frac{14x-37}{40}.$

10. $\frac{2x-6}{5x}-\frac{8x-4}{15x}+\frac{56x-48}{45x}.$

The L. C. D. is $45x$.
The respective multipliers are 9, 3, and 1.

$$\begin{aligned}
\therefore\ 9(2x-6) &= 18x-54 = \text{1st numerator.}\\
-3(8x-4) &= -24x+12 = \text{2d numerator.}\\
&\underline{56x-48} = \text{3d numerator.}\\
&50x-90 = \text{the sum of the numerators.}
\end{aligned}$$

$\therefore$ the sum of the fractions is $\frac{50x-90}{45x}=\frac{10x-18}{9x}.$

11. $\frac{11xy+2}{x^2y^2}-\frac{5y^2-3}{xy^3}-\frac{6x^2-5}{x^3y}.$

The L. C. D. is x^3y^3.
The respective multipliers are xy, x^2, and y^2.

$$\begin{aligned}
\therefore\ xy(11xy+2) &= 11x^2y^2 \qquad\qquad\qquad +2xy = \text{1st numerator.}\\
-x^2(5y^2-3) &= -\ 5x^2y^2+3x^2 \qquad\qquad\quad = \text{2d numerator.}\\
-y^2(6x^2-5) &= \underline{-\ 6x^2y^2 \qquad\quad +5y^2 \qquad\ } = \text{3d numerator.}\\
&3x^2+5y^2+2xy = \text{the sum of the numerators.}
\end{aligned}$$

$\therefore$ the sum of the fractions is $\frac{3x^2+5y^2+2xy}{x^3y^3}.$

12. $\frac{3}{2x^2y}+\frac{1}{6y^2z}-\frac{1}{2xz^2}+\frac{2x-z}{4x^2z^2}+\frac{y-6z}{4x^2yz}.$

The L. C. D. is $12x^2y^2z^2$.

The respective multipliers are $6yz^2$, $2x^2z$, $6xy^2$, $3y^2$, $3yz$.

$$\begin{aligned}
\therefore\ 18yz^2 &= \text{1st numerator.}\\
2x^2z &= \text{2d numerator.}\\
-\ 6xy^2 &= \text{3d numerator.}\\
3y^2(2x-z) = 6xy^2 - \ 3y^2z &= \text{4th numerator.}\\
3yz(y-6z) = \underline{3y^2z - 18yz^2} &= \text{5th numerator.}\\
2x^2z &= \text{the sum of the numerators.}
\end{aligned}$$

$\therefore$ the sum of the fractions is $\frac{2x^2z}{12x^2y^2z^2}=\frac{1}{6y^2z}.$

Exercise 48. Page 130.

Simplify:

1. $\frac{1}{x+6}+\frac{1}{x-5}.$

The L. C. D. is $(x+6)(x-5)$.

$$\begin{aligned}
\therefore\ x-5 &= \text{1st numerator.}\\
\underline{x+6} &= \text{2d numerator.}\\
2x+1 &= \text{the sum of the numerators.}
\end{aligned}$$

$\therefore$ the sum of the fractions is $\frac{2x+1}{(x+6)(x-5)}=\frac{2x+1}{x^2+x-30}.$

2. $\frac{1}{1+x}-\frac{1}{1-x}.$

The L. C. D. is $(1+x)(1-x)$.

$$\begin{aligned}
\therefore\ 1-\ x &= \text{1st numerator.}\\
-(1+x) = \underline{-1-\ x} &= \text{2d numerator.}\\
-2x &= \text{the sum of the numerators.}
\end{aligned}$$

$\therefore$ the sum of the fractions is $-\frac{2x}{(1+x)(1-x)}=\frac{2x}{x^2-1}.$

3. $\frac{1}{1+x}-\frac{2}{1-x^2}.$

The L. C. D. is $1-x^2$.

$$\begin{aligned}
1-x &= \text{1st numerator.}\\
\underline{-2\quad} &= \text{2d numerator.}\\
-1-x &= \text{the sum of the numerators.}
\end{aligned}$$

$\therefore$ the sum of the fractions is $\frac{-1-x}{1-x^2}=\frac{1+x}{x^2-1}=\frac{1}{x-1}.$

4. $\dfrac{x+y}{x-y} - \dfrac{x^2-y^2}{(x-y)^2}$.

The L. C. D. is $(x-y)^2$.

$$\begin{aligned} \therefore\ (x-y)(x+y) &= x^2-y^2 = \text{1st numerator.}\\ -(x^2-y^2) &= \underline{-x^2+y^2} = \text{2d numerator.}\\ &\phantom{=-x^2+{}}0 \quad\ = \text{the sum of the numerators.} \end{aligned}$$

$\therefore$ the sum of the fractions is 0.

5. $\dfrac{x-y}{x+y} - \dfrac{(x-y)^2}{(x+y)^2}$.

The L. C. D. is $(x+y)^2$.

$$\begin{aligned} \therefore\ (x+y)(x-y) &= x^2 \phantom{{}+2xy} - y^2 = \text{1st numerator.}\\ -(x-y)^2 &= \underline{-x^2+2xy-y^2} = \text{2d numerator.}\\ &\phantom{=-x^2+{}}2xy-2y^2 = \text{the sum of the numerators.} \end{aligned}$$

$\therefore$ the sum of the fractions is $\dfrac{2xy-2y^2}{(x+y)^2}$.

6. $\dfrac{1}{2a(a+b)} + \dfrac{1}{2a(a-b)}$.

The L. C. D. is $2a(a+b)(a-b)$.

$$\begin{aligned} \therefore\ a-b &= \text{1st numerator.}\\ \underline{a+b} &= \text{2d numerator.}\\ 2a\quad &= \text{the sum of the numerators.} \end{aligned}$$

$\therefore$ the sum of the fractions is

$$\frac{2a}{2a(a+b)(a-b)} = \frac{1}{(a+b)(a-b)} = \frac{1}{a^2-b^2}.$$

7. $\dfrac{1+x}{1+x+x^2} - \dfrac{1-x}{1-x+x^2}$.

The L. C. D. is $(1+x+x^2)(1-x+x^2)$.

$$\begin{aligned} \therefore\ (1+x)(1-x+x^2) &= 1+x^3 = \text{1st numerator.}\\ -(1-x)(1+x+x^2) &= \underline{-1+x^3} = \text{2d numerator.}\\ &\phantom{=-1+{}}2x^3 = \text{the sum of the numerators.} \end{aligned}$$

$\therefore$ the sum of the fractions is $\dfrac{2x^3}{(1+x+x^2)(1-x+x^2)} = \dfrac{2x^3}{1+x^2+x^4}$.

8. $\dfrac{1}{a-3c}-\dfrac{(a-3c)^2}{a^3-27c^3}.$

The L. C. D. is a^3-27c^3.

$$\begin{aligned}
\therefore\ a^2+3ac+9c^2 &= \text{1st numerator.}\\
-(a-3c)^2=\underline{-a^2+6ac-9c^2} &= \text{2d numerator.}\\
9ac \quad &= \text{the sum of the numerators.}
\end{aligned}$$

$\therefore$ the sum of the fractions is $\dfrac{9ac}{a^3-27c^3}$.

9. $\dfrac{x+y}{x-y}-\dfrac{x-y}{x+y}-\dfrac{4xy}{x^2-y^2}.$

The L. C. D. is x^2-y^2.

$$\begin{aligned}
\therefore\ (x+y)(x+y) &= x^2+2xy+y^2 = \text{1st numerator.}\\
-(x-y)(x-y) &= -x^2+2xy-y^2 = \text{2d numerator.}\\
& \quad\ \underline{\quad -4xy \quad} = \text{3d numerator.}\\
& \qquad\quad 0 \qquad = \text{the sum of the numerators.}
\end{aligned}$$

$\therefore$ the sum of the fractions is 0.

10. $\dfrac{x}{a-x}+\dfrac{x}{a+x}+\dfrac{2x^2}{a^2+x^2}.$

The L. C. D. is $(a-x)(a+x)(a^2+x^2)$.

$$\begin{aligned}
\therefore\ (a+x)(a^2+x^2)x &= a^3x+a^2x^2+ax^3+x^4 = \text{1st numerator.}\\
(a-x)(a^2+x^2)x &= a^3x-a^2x^2+ax^3-x^4 = \text{2d numerator.}\\
(a-x)(a+x)2x^2 &= \underline{\qquad\quad 2a^2x^2 \qquad\quad -2x^4} = \text{3d numerator.}\\
& \quad 2a^3x+2a^2x^2+2ax^3-2x^4 = \text{the sum of the numerators.}
\end{aligned}$$

$\therefore$ the sum of the fractions is

$$\frac{2a^3x+2a^2x^2+2ax^3-2x^4}{(a-x)(a+x)(a^2+x^2)}=\frac{2a^3x+2a^2x^2+2ax^3-2x^4}{a^4-x^4}.$$

11. $\dfrac{1}{x}-\dfrac{2}{x+1}+\dfrac{1}{x+2}.$

The L. C. D. is $x(x+1)(x+2)$.

$$\begin{aligned}
\therefore\ (x+1)(x+2) &= x^2+3x+2 = \text{1st numerator.}\\
-2x(x+2) &= -2x^2-4x \quad = \text{2d numerator.}\\
x(x+1) &= \underline{\ x^2+\ x \qquad} = \text{3d numerator.}\\
& \qquad\qquad\qquad 2 = \text{the sum of the numerators.}
\end{aligned}$$

$\therefore$ the sum of the fractions is $\dfrac{2}{x(x+1)(x+2)}=\dfrac{2}{x^3+3x^2+2x}$.

12. $\dfrac{6-2a}{9-a^2}-\dfrac{2}{3+a}-\dfrac{1}{3-a}.$

The L. C. D. is $9-a^2$.

$$\begin{array}{rrl}
 & \therefore\ 6-2a &= \text{1st numerator.}\\
-2(3-a) = & \underline{-6+2a} &= \text{2d numerator.}\\
 & 0 &= \text{the sum of the first two numerators.}
\end{array}$$

$\therefore$ the sum of the first two fractions is 0.

$\therefore$ the sum of all the fractions is the last fraction $-\dfrac{1}{3-a}$, or $\dfrac{1}{a-3}$.

13. $\dfrac{1}{x+2y}+\dfrac{1}{x-2y}-\dfrac{x}{x^2-4y^2}.$

The L. C. D. is x^2-4y^2.

$$\begin{array}{rl}
\therefore\ x-2y &= \text{1st numerator.}\\
x+2y &= \text{2d numerator.}\\
\underline{-x\qquad} &= \text{3d numerator.}\\
x &= \text{the sum of the numerators.}
\end{array}$$

$\therefore$ the sum of the fractions is $\dfrac{x}{x^2-4y^2}$.

14. $\dfrac{1}{y-1}-\dfrac{2}{y}+\dfrac{1}{y+1}-\dfrac{1}{y^2-1}.$

The L. C. D. is $y(y^2-1)$.

$$\begin{array}{rrl}
\therefore\ y(y+1) = & y^2+y &= \text{1st numerator.}\\
-2(y^2-1) = & -2y^2+2 &= \text{2d numerator.}\\
y(y-1) = & y^2-y &= \text{3d numerator.}\\
 & \underline{-y} &= \text{4th numerator.}\\
 & 2-y &= \text{the sum of the numerators.}
\end{array}$$

$\therefore$ the sum of the fractions is $\dfrac{2-y}{y(y^2-1)}$.

15. $\dfrac{x}{x-1}-1-\dfrac{1}{x^2-x}+\dfrac{1}{x}.$

The L. C. D. is x^2-x.

$$\begin{array}{rl}
\therefore\ x^2 &= \text{1st numerator.}\\
-x^2+x &= \text{2d numerator.}\\
-1 &= \text{3d numerator.}\\
\underline{x-1} &= \text{4th numerator.}\\
2x-2 &= \text{the sum of the numerators.}
\end{array}$$

$\therefore$ the sum of the fractions is $\dfrac{2x-2}{x^2-x}=\dfrac{2(x-1)}{x(x-1)}=\dfrac{2}{x}$.

16. $\dfrac{b}{a+b}-\dfrac{ab}{(a+b)^2}-\dfrac{ab^2}{(a+b)^3}.$

The L. C. D. is $(a+b)^3$.

$$\begin{array}{rrl}
\therefore\ b(a+b)^2 = & a^2b+2ab^2+b^3 = & \text{1st numerator.}\\
-ab(a+b) = & -a^2b-\quad ab^2 \qquad = & \text{2d numerator.}\\
 & -\quad ab^2 \qquad = & \text{3d numerator.}\\ \hline
 & b^3 = & \text{the sum of the numerators.}
\end{array}$$

$\therefore$ the sum of the fractions is $\dfrac{b^3}{(a+b)^3}.$

17. $\dfrac{x+y}{y}-\dfrac{2x}{x+y}-\dfrac{x^2(x-y)}{y(x^2-y^2)}=\dfrac{x+y}{y}-\dfrac{2x}{x+y}-\dfrac{x^2}{y(x+y)}.$

The L. C. D. is $y(x+y)$.

$$\begin{array}{rrl}
\therefore\ (x+y)^2 = & x^2+2xy+y^2 = & \text{1st numerator.}\\
 & -2xy \qquad = & \text{2d numerator.}\\
 & -x^2 \qquad\qquad = & \text{3d numerator.}\\ \hline
 & y^2 = & \text{the sum of the numerators.}
\end{array}$$

$\therefore$ the sum of the fractions is $\dfrac{y^2}{y(x+y)}=\dfrac{y}{x+y}.$

18. $\dfrac{3}{x-a}+\dfrac{4a}{(x-a)^2}-\dfrac{5a^2}{(x-a)^3}.$

The L. C. D. is $(x-a)^3$.

$$\begin{array}{rrl}
\therefore\ 3(x-a)^2 = & 3x^2-6ax+3a^2 = & \text{1st numerator.}\\
4a(x-a) = & 4ax-4a^2 = & \text{2d numerator.}\\
 & -5a^2 = & \text{3d numerator.}\\ \hline
 & 3x^2-2ax-6a^2 = & \text{the sum of the numerators.}
\end{array}$$

$\therefore$ the sum of the fractions is $\dfrac{3x^2-2ax-6a^2}{(x-a)^3}.$

19. $\dfrac{1}{x-2a}+\dfrac{a^2}{x^3-8a^3}-\dfrac{x+a}{x^2+2ax+4a^2}.$

The L. C. D. is x^3-8a^3.

$$\begin{array}{rrl}
 & \therefore\ x^2+2ax+4a^2 = & \text{1st numerator.}\\
 & a^2 = & \text{2d numerator.}\\
-(x+a)(x-2a) = & -x^2+\quad ax+2a^2 = & \text{3d numerator.}\\ \hline
 & 3ax+7a^2 = & \text{the sum of the numerators.}
\end{array}$$

$\therefore$ the sum of the fractions is $\dfrac{3ax+7a^2}{x^3-8a^3}.$

20. $\dfrac{x^2-2x+3}{x^3+1}+\dfrac{x-2}{x^2-x+1}-\dfrac{1}{x+1}.$

The L. C. D. is x^3+1.

$$\begin{array}{rrl}
\therefore & x^2-2x+3 & = \text{1st numerator.} \\
(x+1)(x-2) = & x^2-x-2 & = \text{2d numerator.} \\
& -x^2+x-1 & = \text{3d numerator.} \\
\hline
& x^2-2x & = \text{the sum of the numerators.}
\end{array}$$

$\therefore$ the sum of the fractions is $\dfrac{x^2-2x}{x^3+1}.$

21. $\dfrac{1}{x-3}+\dfrac{x-1}{x^2+3x+9}+\dfrac{x^2+x-3}{x^3-27}.$

The L. C. D. is x^3-27.

$$\begin{array}{rrl}
\therefore & x^2+3x+9 & = \text{1st numerator.} \\
(x-3)(x-1) = & x^2-4x+3 & = \text{2d numerator.} \\
& x^2+x-3 & = \text{3d numerator.} \\
\hline
& 3x^2+9 & = \text{the sum of the numerators.}
\end{array}$$

$\therefore$ the sum of the fractions is $\dfrac{3x^2+9}{x^3-27}.$

22. $$\begin{aligned}\frac{x^2+8x+15}{x^2+7x+10}-\frac{x-1}{x-2} &= \frac{(x+3)(x+5)}{(x+2)(x+5)}-\frac{x-1}{x-2} \\ &= \frac{x+3}{x+2}-\frac{x-1}{x-2}.\end{aligned}$$

The L. C. D. is $(x+2)(x-2)=x^2-4$.

$$\begin{array}{rrl}
\therefore\ (x-2)(x+3) = & x^2+x-6 & = \text{1st numerator.} \\
-(x+2)(x-1) = & -x^2-x+2 & = \text{2d numerator.} \\
\hline
& -4 & = \text{the sum of the numerators.}
\end{array}$$

$\therefore$ the sum of the fractions is $\dfrac{-4}{x^2-4}=\dfrac{4}{4-x^2}.$

23. $$\begin{aligned}\frac{x^2-5ax+6a^2}{x^2-8ax+15a^2}-\frac{x-7a}{x-5a} &= \frac{(x-2a)(x-3a)}{(x-3a)(x-5a)}-\frac{x-7a}{x-5a} \\ &= \frac{x-2a}{x-5a}-\frac{x-7a}{x-5a} \\ &= \frac{x-2a-x+7a}{x-5a} \\ &= \frac{5a}{x-5a}.\end{aligned}$$

24. $\frac{1}{x-2}+\frac{1}{x^2-3x+2}-\frac{2}{x^2-4x+3}$

$$=\frac{1}{x-2}+\frac{1}{(x-1)(x-2)}-\frac{2}{(x-1)(x-3)}.$$

The L. C. D. is $(x-1)(x-2)(x-3)$.

$$\begin{aligned} \therefore (x-1)(x-3) = x^2-4x+3 &= \text{1st numerator.}\\ x-3 &= \text{2d numerator.}\\ -2(x-2) = \quad -2x+4 &= \text{3d numerator.}\\ \hline x^2-5x+4 &= \text{the sum of the numerators.} \end{aligned}$$

$\therefore$ the sum of the fractions is

$$\frac{x^2-5x+4}{(x-1)(x-2)(x-3)}=\frac{(x-1)(x-4)}{(x-1)(x-2)(x-3)}=\frac{x-4}{(x-2)(x-3)}.$$

25. $\frac{1}{a^2-7a+12}+\frac{2}{a^2-4a+3}-\frac{3}{a^2-5a+4}.$

$$=\frac{1}{(a-3)(a-4)}+\frac{2}{(a-1)(a-3)}-\frac{3}{(a-1)(a-4)}.$$

The L. C. D. is $(a-1)(a-3)(a-4)$.

$$\begin{aligned} \therefore\ a-1 &= \text{1st numerator.}\\ 2(a-4) = \quad 2a-8 &= \text{2d numerator.}\\ -3(a-3) = -3a+9 &= \text{3d numerator.}\\ \hline 0 \quad &= \text{the sum of the numerators.} \end{aligned}$$

$\therefore$ the sum of the fractions is 0.

26.

$$\frac{3}{10a^2+a-3}-\frac{4}{2a^2+7a-4}=\frac{3}{(2a-1)(5a+3)}-\frac{4}{(2a-1)(a+4)}.$$

The L. C. D. is $(2a-1)(5a+3)(a+4)$.

$$\begin{aligned} \therefore\ 3(a+4) = \quad 3a+12 &= \text{1st numerator.}\\ -4(5a+3) = -20a-12 &= \text{2d numerator.}\\ \hline -17a \quad &= \text{the sum of the numerators.} \end{aligned}$$

$\therefore$ the sum of the fractions is $\frac{-17a}{(2a-1)(5a+3)(a+4)}$.

27. $\frac{3}{2-x-6x^2}-\frac{1}{1-x-2x^2}=\frac{3}{(2+3x)(1-2x)}-\frac{1}{(1+x)(1-2x)}.$

The L. C. D. is $(2+3x)(1-2x)(1+x)$.

$\therefore\ 3(1+x) = 3+3x$ = 1st numerator.
$-(2+3x) = -2-3x$ = 2d numerator.
1 = the sum of the numerators.

$\therefore$ the sum of the fractions is $\dfrac{1}{(2+3x)(1-2x)(1+x)}$.

Exercise 49. Page 134.

Simplify:

1. $\dfrac{x^2}{x^2-1}+\dfrac{x}{x+1}-\dfrac{x}{1-x}=\dfrac{x^2}{x^2-1}+\dfrac{x}{x+1}+\dfrac{x}{x-1}$.

The L. C. D. is x^2-1.

$\therefore\ x^2$ = 1st numerator.
$x(x-1) = x^2-x$ = 2d numerator.
$x(x+1) = x^2+x$ = 3d numerator.
$3x^2$ = the sum of the numerators.

$\therefore$ the sum of the fractions is $\dfrac{3x^2}{x^2-1}$.

2. $\dfrac{a}{a-x}+\dfrac{3a}{a+x}+\dfrac{2ax}{x^2-a^2}=\dfrac{a}{a-x}+\dfrac{3a}{a+x}-\dfrac{2ax}{a^2-x^2}$.

The L. C. D. is a^2-x^2.

$\therefore\ a(a+x) = a^2+ax$ = 1st numerator.
$3a(a-x) = 3a^2-3ax$ = 2d numerator.
$-2ax$ = 3d numerator.
$4a^2-4ax$ = the sum of the numerators.

$\therefore$ the sum of the fractions is $\dfrac{4a^2-4ax}{a^2-x^2}=\dfrac{4a(a-x)}{(a+x)(a-x)}=\dfrac{4a}{a+x}$.

3. $\dfrac{3}{2a-3}-\dfrac{2}{3+2a}+\dfrac{15}{9-4a^2}=\dfrac{3}{2a-3}-\dfrac{2}{2a+3}-\dfrac{15}{4a^2-9}$.

The L. C. D. is $4a^2-9$.

$\therefore\ 3(2a+3) = 6a+9$ = 1st numerator.
$-2(2a-3) = -4a+6$ = 2d numerator.
-15 = 3d numerator.
$2a$ = the sum of the numerators.

$\therefore$ the sum of the fractions is $\dfrac{2a}{4a^2-9}$.

4. $$\begin{aligned}\frac{a-b}{b}+\frac{2a}{a-b}+\frac{a^3+a^2b}{b^3-a^2b}&=\frac{a-b}{b}+\frac{2a}{a-b}+\frac{a^2(a+b)}{b(b-a)(b+a)}\\&=\frac{a-b}{b}+\frac{2a}{a-b}+\frac{a^2}{b(b-a)}\\&=\frac{a-b}{b}+\frac{2a}{a-b}-\frac{a^2}{b(a-b)}.\end{aligned}$$

The L. C. D. is $b(a-b)$.

$$\begin{array}{rll}\therefore (a-b)^2 = & a^2-2ab+b^2 & = \text{1st numerator.}\\ & \phantom{a^2-{}}2ab & = \text{2d numerator.}\\ & -a^2 & = \text{3d numerator.}\\ \hline & \phantom{a^2-2ab+{}}b^2 & = \text{the sum of the numerators.}\end{array}$$

$\therefore$ the sum of the fractions is $\frac{b^2}{b(a-b)}=\frac{b}{a-b}$.

5. $$\frac{3}{x}+\frac{5}{1-2x}-\frac{2x-7}{4x^2-1}=\frac{3}{x}+\frac{5}{1-2x}+\frac{2x-7}{1-4x^2}.$$

The L. C. D. is $x(1-4x^2)$.

$$\begin{array}{rll}\therefore 3(1-4x^2) = & 3 \qquad\qquad -12x^2 & = \text{1st numerator.}\\ 5x(1+2x) = & \qquad 5x+10x^2 & = \text{2d numerator.}\\ x(2x-7) = & \qquad -7x+2x^2 & = \text{3d numerator.}\\ \hline & 3-2x & = \text{the sum of the numerators.}\end{array}$$

$\therefore$ the sum of the fractions is $\frac{3-2x}{x(1-4x^2)}=\frac{2x-3}{x(4x^2-1)}$.

6. $$\frac{1}{(x+a)^2}+\frac{1}{(a-x)^2}+\frac{1}{x^2-a^2}=\frac{1}{(x+a)^2}+\frac{1}{(x-a)^2}+\frac{1}{x^2-a^2}.$$

The L. C. D. is $(x+a)^2(x-a)^2$.

$$\begin{array}{rll}\therefore (x-a)^2 = & x^2-2ax+a^2 & = \text{1st numerator.}\\ (x+a)^2 = & x^2+2ax+a^2 & = \text{2d numerator.}\\ (x+a)(x-a) = & x^2 \qquad\quad -a^2 & = \text{3d numerator.}\\ \hline & 3x^2 \qquad\quad +a^2 & = \text{the sum of the numerators.}\end{array}$$

$\therefore$ the sum of the fractions is $\frac{3x^2+a^2}{(x+a)^2(x-a)^2}=\frac{3x^2+a^2}{x^4-2a^2x^2+a^4}$.

7. $$\frac{1}{x-y}+\frac{x-y}{x^2+xy+y^2}-\frac{xy-2x^2}{y^3-x^3}=\frac{1}{x-y}+\frac{x-y}{x^2+xy+y^2}+\frac{xy-2x^2}{x^3-y^3}.$$

The L. C. D. is x^3-y^3.

$$\begin{array}{rrl}
 & \therefore\ x^2 + xy + y^2 & = \text{1st numerator.} \\
(x-y)^2 = & x^2 - 2xy + y^2 & = \text{2d numerator.} \\
 & -2x^2 + xy & = \text{3d numerator.} \\
\hline
 & 2y^2 & = \text{the sum of the numerators.}
\end{array}$$

$\therefore$ the sum of the fractions is $\dfrac{2y^2}{x^3 - y^3}$.

8. $$\frac{1}{(x-2)(x-3)} + \frac{2}{(x-1)(3-x)} + \frac{1}{(x-1)(x-2)}$$
$$= \frac{1}{(x-2)(x-3)} - \frac{2}{(x-1)(x-3)} + \frac{1}{(x-1)(x-2)}.$$

The L. C. D. is $(x-1)(x-2)(x-3)$.

$$\begin{array}{rl}
\therefore\ x - 1 & = \text{1st numerator.} \\
-2(x-2) = -2x + 4 & = \text{2d numerator.} \\
x - 3 & = \text{3d numerator.} \\
\hline
0 & = \text{the sum of the numerators.}
\end{array}$$

$\therefore$ the sum of the fractions is 0.

9. $$\frac{bc}{(c-a)(a-b)} + \frac{ac}{(a-b)(b-c)} + \frac{ab}{(b-c)(c-a)}.$$

The L. C. D. is $(a-b)(b-c)(c-a)$.

$$\begin{array}{rl}
\therefore\ bc(b-c) = bc^2 - bc^2 & = \text{1st numerator.} \\
ac(c-a) = ac^2 - a^2c & = \text{2d numerator.} \\
ab(a-b) = a^2b - ab^2 & = \text{3d numerator.} \\
\hline
a^2b - ab^2 + b^2c - bc^2 + ac^2 - a^2c & = \text{the sum of the numerators.}
\end{array}$$

$\therefore$ the sum of the fractions is

$$\frac{a^2b - ab^2 + b^2c - bc^2 + ac^2 - a^2c}{(a-b)(b-c)(c-a)} = -\frac{(a-b)(b-c)(c-a)}{(a-b)(b-c)(c-a)} = -1.$$

See School Algebra, page 133, Example (2).

10. $$\frac{b+c}{(a-b)(a-c)} + \frac{a+c}{(b-c)(b-a)} + \frac{a+b}{(c-a)(c-b)}$$
$$= -\frac{b+c}{(a-b)(c-a)} - \frac{a+c}{(b-c)(a-b)} - \frac{a+b}{(c-a)(b-c)}.$$

The L. C. D. is $(a-b)(b-c)(c-a)$.

$$\begin{aligned}
\therefore\ -(b-c)(b+c) &= c^2-b^2 = \text{1st numerator.}\\
-(c-a)(a+c) &= a^2-c^2 = \text{2d numerator.}\\
-(a-b)(a+b) &= \underline{b^2-a^2} = \text{3d numerator.}\\
&\quad\ 0 \qquad = \text{the sum of the numerators.}
\end{aligned}$$

$\therefore$ the sum of the fractions is 0.

11. $$\frac{3}{(a-b)(b-c)}-\frac{4}{(b-a)(c-a)}-\frac{6}{(a-c)(c-b)}$$
$$=\frac{3}{(a-b)(b-c)}+\frac{4}{(a-b)(c-a)}-\frac{6}{(c-a)(b-c)}.$$

The L.C.D. is $(a-b)(b-c)(c-a)$.

$$\begin{aligned}
\therefore\ 3(c-a) &= \quad 3c-3a = \text{1st numerator.}\\
4(b-c) &= \quad 4b-4c = \text{2d numerator.}\\
-6(a-b) &= \underline{-6a+6b} = \text{3d numerator.}\\
&\quad -9a+10b-c = \text{the sum of the numerators.}
\end{aligned}$$

$\therefore$ the sum of the fractions is $\dfrac{-9a+10b-c}{(a-b)(b-c)(c-a)}$.

12. $$\frac{1}{x(x-y)(x-z)}+\frac{1}{y(y-x)(y-z)}-\frac{1}{xyz}$$
$$=\frac{1}{x(x-y)(x-z)}-\frac{1}{y(x-y)(y-z)}-\frac{1}{xyz}.$$

The L.C.D. is $xyz(x-y)(y-z)(x-z)$.

$$\begin{aligned}
\therefore\ yz(y-z) &= \quad y^2z-yz^2 = \text{1st numerator.}\\
-xz(x-z) &= -x^2z+xz^2 = \text{2d numerator.}\\
-(x-y)(y-z)(x-z) &= \underline{x^2z-x^2y+xy^2-xz^2+yz^2-y^2z} = \text{3d numerator.}\\
&\quad -x^2y+xy^2 \qquad = \text{the sum of the numerators.}
\end{aligned}$$

$\therefore$ the sum of the fractions is

$$\frac{xy^2-x^2y}{xyz(x-y)(y-z)(x-z)}=\frac{xy(y-x)}{xyz(x-y)(y-z)(x-z)}$$
$$=\frac{-1}{z(y-z)(x-z)}=\frac{1}{z(y-z)(z-x)}.$$

13. $$\frac{a^2-bc}{(a-b)(a-c)}+\frac{b^2+ac}{(b-a)(b+c)}+\frac{c^2+ab}{(c-a)(c+b)}$$
$$=\frac{bc-a^2}{(a-b)(c-a)}-\frac{b^2+ac}{(a-b)(b+c)}+\frac{c^2+ab}{(c-a)(b+c)}.$$

The L.C.D. is $(a-b)(c-a)(b+c)$.

$$\begin{array}{rcl}
\therefore\ (bc-a^2)(b+c) &=& b^2c+bc^2-a^2b-a^2c = \text{1st numerator.}\\
-(b^2+ac)(c-a) &=& -b^2c+ab^2+a^2c-ac^2 = \text{2d numerator.}\\
(c^2+ab)(a-b) &=& \underline{-bc^2-ab^2+a^2b+ac^2} = \text{3d numerator.}\\
&& 0 \qquad\qquad = \text{the sum of the numerators.}
\end{array}$$

$\therefore$ the sum of the fractions is 0.

Exercise 50. Page 137.

Simplify:

1.
$$\frac{12x^2}{7y^2}\times\frac{14xy}{9x^2z^3}=\frac{12\times 14x^3y}{7\times 9x^2y^2z^3}$$
$$=\frac{8x}{3yz^3}.$$

2.
$$\frac{3a^2b^2c^3}{4amn}\times\frac{20m^2n^3}{21a^4c^5}=\frac{3\times 20a^2b^2c^3m^2n^3}{4\times\ 21a^5c^5mn}$$
$$=\frac{5b^2mn^2}{7a^3c^2}.$$

3.
$$\frac{6a^2b^2c^5}{7mxy}\times\frac{5m^2x^3}{3a^3c^6}=\frac{6\times 5a^2b^2c^5m^2x^3}{7\times 3a^3c^6mxy}$$
$$=\frac{10b^2mx^2}{7acy}.$$

4.
$$\frac{9m^2n^2}{8x^3y^3}\times\frac{4x^2y^2}{15mn}=\frac{9\times 4m^2n^2x^2y^2}{8\times 15mnx^3y^3}$$
$$=\frac{3mn}{10xy}.$$

5.
$$\frac{16a^4b^2}{21x^3y^2}\div\frac{4a^3b}{3x^2y}=\frac{16a^4b^2}{21x^3y^2}\times\frac{3x^2y}{4a^3b}$$
$$=\frac{4ab}{7xy}.$$

6.
$$\frac{7xy}{12z^2}\div\frac{35xz}{36yz^2}=\frac{7xy}{12z^2}\times\frac{36yz^2}{35xz}$$
$$=\frac{3y^2}{5z}.$$

7.
$$\frac{x^2-y^2}{x^2+y^2}\times\frac{4x}{x+y}=\frac{(x+y)(x-y)}{x^2+y^2}\times\frac{4x}{x+y}$$
$$=\frac{4x(x-y)}{x^2+y^2}.$$

8.
$$\begin{aligned}\frac{3x^2-x}{a}\times\frac{2a}{2x^2-4x}&=\frac{x(3x-1)}{a}\times\frac{2a}{2x(x-2)}\\&=\frac{3x-1}{x-2}.\end{aligned}$$

9.
$$\begin{aligned}\frac{3x^2}{5x-10}\times\frac{3x-6}{4x^3}&=\frac{3x^2}{5(x-2)}\times\frac{3(x-2)}{4x^3}\\&=\frac{9}{20x}.\end{aligned}$$

10.
$$\begin{aligned}\frac{8a^3}{a^3-b^3}\div\frac{4a^2}{a^2+ab+b^2}&=\frac{8a^3}{a^3-b^3}\times\frac{a^2+ab+b^2}{4a^2}\\&=\frac{8a^3}{(a-b)(a^2+ab+b^2)}\times\frac{a^2+ab+b^2}{4a^2}\\&=\frac{2a}{a-b}.\end{aligned}$$

11.
$$\begin{aligned}\frac{x^2+y^2}{x^2-y^2}\div\frac{3x^2+3y^2}{x+y}&=\frac{x^2+y^2}{x^2-y^2}\times\frac{x+y}{3x^2+3y^2}\\&=\frac{x^2+y^2}{(x+y)(x-y)}\times\frac{x+y}{3(x^2+y^2)}\\&=\frac{1}{3(x-y)}.\end{aligned}$$

12.
$$\begin{aligned}\frac{ab-b^2}{a(a+b)}\div\frac{b^2}{a(a^2-b^2)}&=\frac{ab-b^2}{a(a+b)}\times\frac{a(a^2-b^2)}{b^2}\\&=\frac{b(a-b)}{a(a+b)}\times\frac{a(a+b)(a-b)}{b^2}\\&=\frac{(a-b)^2}{b}.\end{aligned}$$

13.
$$\begin{aligned}\frac{a^2-4x^2}{a^2+4ax}\div\frac{a^2-2ax}{ax+4x^2}&=\frac{a^2-4x^2}{a^2+4ax}\times\frac{ax+4x^2}{a^2-2ax}\\&=\frac{(a-2x)(a+2x)}{a(a+4x)}\times\frac{x(a+4x)}{a(a-2x)}\\&=\frac{x(a+2x)}{a^2}.\end{aligned}$$

14.
$$\begin{aligned}\frac{x^4-y^4}{(x-y)^2}\div\frac{x^2+xy}{x-y}&=\frac{x^4-y^4}{(x-y)^2}\times\frac{x-y}{x^2+xy}\\&=\frac{(x^2+y^2)(x+y)(x-y)}{(x-y)^2}\times\frac{x-y}{x(x+y)}\\&=\frac{x^2+y^2}{x}.\end{aligned}$$

15.
$$\frac{x^3+a^3}{x^2-9a^2}\times\frac{x+3a}{x+a}=\frac{(x+a)(x^2-ax+a^2)}{(x+3a)(x-3a)}\times\frac{x+3a}{x+a}$$
$$=\frac{x^2-ax+a^2}{x-3a}.$$

16.
$$\frac{a^3-b^3}{a^3+b^3}\div\frac{a-b}{a^2-ab+b^2}=\frac{(a-b)(a^2+ab+b^2)}{(a+b)(a^2-ab+b^2)}\times\frac{a^2-ab+b^2}{a-b}$$
$$=\frac{a^2+ab+b^2}{a+b}.$$

17.
$$\frac{x^2-1}{x^2-4x-5}\times\frac{x^2-25}{x^2+2x-3}=\frac{(x+1)(x-1)}{(x-5)(x+1)}\times\frac{(x+5)(x-5)}{(x-1)(x+3)}$$
$$=\frac{x+5}{x+3}.$$

18.
$$\left(1-\frac{y^4}{x^4}\right)\div\frac{x^2+y^2}{xy}=\frac{x^4-y^4}{x^4}\times\frac{xy}{x^2+y^2}$$
$$=\frac{(x^2-y^2)y}{x^3}.$$

19.
$$\left(\frac{4x^2}{y^2}-1\right)\left(\frac{2x}{2x-y}-1\right)=\frac{4x^2-y^2}{y^2}\times\frac{2x-2x+y}{2x-y}$$
$$=\frac{4x^2-y^2}{y^2}\times\frac{y}{2x-y}$$
$$=\frac{2x+y}{y}.$$

20.
$$\left(\frac{8x^3}{y^3}-1\right)\left(\frac{4x^2+2xy}{4x^2+2xy+y^2}-1\right)=\frac{8x^3-y^3}{y^3}\times\frac{4x^2+2xy-4x^2-2xy-y^2}{4x^2+2xy+y^2}$$
$$=\frac{8x^3-y^3}{y^3}\times\frac{-y^2}{4x^2+2xy+y^2}$$
$$=\frac{(2x-y)(4x^2+2xy+y^2)}{y^3}\times\frac{-y^2}{4x^2+2xy+y^2}$$
$$=-\frac{2x-y}{y}$$
$$=\frac{y-2x}{y}.$$

21. $$\left(x-\frac{xy-y^2}{x+y}\right)\left(x-\frac{xy^2-y^3}{x^2+y^2}\right)\div\left(1-\frac{xy-y^2}{x^2}\right)$$

$$=\frac{x(x+y)-xy+y^2}{x+y}\times\frac{x(x^2+y^2)-xy^2+y^3}{x^2+y^2}\div\frac{x^2-xy+y^2}{x^2}$$

$$=\frac{x^2+y^2}{x+y}\times\frac{x^3+y^3}{x^2+y^2}\div\frac{x^2-xy+y^2}{x^2}$$

$$=\frac{x^2+y^2}{x+y}\times\frac{(x+y)(x^2-xy+y^2)}{x^2+y^2}\times\frac{x^2}{x^2-xy+y^2}$$

$$=x^2.$$

22. $$\frac{8a^2b}{c}\times\frac{c^2d}{8a^3}\times\frac{4ab}{cd}\times\frac{bcd-cd^2}{4(b^2-bd)}.$$

$$=\frac{8a^2b}{c}\times\frac{c^2d}{8a^3}\times\frac{4ab}{cd}\times\frac{cd(b-d)}{4b(b-d)}$$

$$=bcd.$$

23. $$\frac{y(x^3-y^3)}{x(x+y)}\times\frac{(x^2-y^2)^2}{x^2+xy+y^2}\div\frac{(x-y)^3}{(x+y)^2}$$

$$=\frac{y(x-y)(x^2+xy+y^2)}{x(x+y)}\times\frac{(x-y)^2(x+y)^2}{x^2+xy+y^2}\times\frac{(x+y)^2}{(x-y)^3}$$

$$=\frac{y(x+y)^3}{x}.$$

24. $$\left[\frac{(a+b)^2-c^2}{a^2+ab-ac}\div\frac{(a+c)^2-b^2}{a}\right]\times\frac{(a-b)^2-c^2}{ab-b^2-bc}$$

$$=\left[\frac{(a+b+c)(a+b-c)}{a(a+b-c)}\div\frac{(a+c+b)(a+c-b)}{a}\right]\times\frac{(a-b+c)(a-b-c)}{b(a-b-c)}$$

$$=\left[\frac{a+b+c}{a}\div\frac{(a+c+b)(a+c-b)}{a}\right]\times\frac{a-b+c}{b}$$

$$=\frac{a+b+c}{a}\times\frac{a}{(a+c+b)(a+c-b)}\times\frac{a-b+c}{b}$$

$$=\frac{1}{b}.$$

25. $\frac{(a+b)^2-c^2}{a^2-(b-c)^2}\times\frac{c^2-(a-b)^2}{c^2-(a+b)^2}\times\frac{c-a-b}{ac-a^2+ab}$

$=\frac{(a+b+c)(a+b-c)}{(a+b-c)(a-b+c)}\times\frac{(c+a-b)(c-a+b)}{(c+a+b)(c-a-b)}\times\frac{c-a-b}{a(c-a+b)}$

$=\frac{a+b+c}{a-b+c}\times\frac{(c+a-b)(c-a+b)}{(c+a+b)(c-a-b)}\times\frac{c-a-b}{a(c-a+b)}$

$=\frac{1}{a}.$

26. $\frac{(x-a)^2-b^2}{(x-b)^2-a^2}\times\frac{x^2-(b-a)^2}{x^2-(a-b)^2}\times\frac{ax+a^2-ab}{bx-ab+b^2}$

$=\frac{(x-a+b)(x-a-b)}{(x-b+a)(x-b-a)}\times\frac{(x+b-a)(x-b+a)}{(x+a-b)(x-a+b)}\times\frac{a(x+a-b)}{b(x-a+b)}$

$=\frac{x-a+b}{x-b+a}\times\frac{x-b+a}{x+a-b}\times\frac{a(x+a-b)}{b(x-a+b)}$

$=\frac{a}{b}.$

27. $\frac{a^2-2ab+b^2-c^2}{a^2+2ab+b^2-c^2}\times\frac{a+b-c}{a-b+c}$

$=\frac{(a-b)^2-c^2}{(a+b)^2-c^2}\times\frac{a+b-c}{a-b+c}$

$=\frac{(a-b+c)(a-b-c)}{(a+b+c)(a+b-c)}\times\frac{a+b-c}{a-b+c}$

$=\frac{a-b-c}{a+b+c}.$

28. $\left[\frac{x^2+(x+1)^2}{x(x+1)}\div\frac{(x+1)^2-x^2}{x^2+x}\right]\times\frac{2x+1}{x}$

$=\frac{x^2+(x+1)^2}{x(x+1)}\times\frac{x(x+1)}{2x+1}\times\frac{2x+1}{x}$

$=\frac{x^2+(x+1)^2}{x}.$

29. $\frac{2ax^3+2a^3x}{(x-a)^2(x+a)^2}\times\frac{x^2-a^2}{2(x^2+a^2)}\times\frac{x+a}{ax}$

$=\frac{2ax(x^2+a^2)}{(x-a)^2(x+a)^2}\times\frac{(x+a)(x-a)}{2(x^2+a^2)}\times\frac{x+a}{ax}$

$=\frac{1}{x-a}.$

30. $$\frac{a^2-b^2-c^2-2bc}{a^2-ab-ac}\times\left(1-\frac{2c}{a+b+c}\right)$$
$$=\frac{a^2-(b+c)^2}{a(a-b-c)}\times\frac{a+b+c-2c}{a+b+c}$$
$$=\frac{(a+b+c)(a-b-c)}{a(a-b-c)}\times\frac{a+b-c}{a+b+c}$$
$$=\frac{a+b-c}{a}.$$

31. $$\frac{a^4+a^2b^2+b^4}{a^6-b^6}\times\frac{a+b}{a^3+b^3}\times\frac{a^2-b^2}{a}$$
$$=\frac{(a^2+ab+b^2)(a^2-ab+b^2)}{(a^3+b^3)(a^3-b^3)}\times\frac{a+b}{a^3+b^3}\times\frac{a^2-b^2}{a}$$
$$=\frac{(a^2+ab+b^2)(a^2-ab+b^2)}{(a+b)(a^2-ab+b^2)(a-b)(a^2+ab+b^2)}$$
$$\times\frac{a+b}{(a+b)(a^2-ab+b^2)}\times\frac{(a+b)(a-b)}{a}$$
$$=\frac{1}{a(a^2-ab+b^2)}.$$

32. $$\frac{x^2+7xy+12y^2}{x^2+5xy+6y^2}\times\frac{x^2+xy-2y^2}{x^2+3xy-4y^2}$$
$$=\frac{(x+3y)(x+4y)}{(x+2y)(x+3y)}\times\frac{(x-y)(x+2y)}{(x-y)(x+4y)}$$
$$=1.$$

Exercise 51. Page 141.

Simplify:

1. $$\frac{\frac{x}{m}+\frac{y}{m}}{\frac{z}{m}}=\left(\frac{x}{m}+\frac{y}{m}\right)\frac{m}{z}=\frac{x+y}{m}\times\frac{m}{z}=\frac{x+y}{z}.$$

2. $$\frac{x+\frac{y}{z}}{z-\frac{y}{x}}=\frac{\frac{xz+y}{z}}{\frac{xz-y}{x}}=\frac{xz+y}{z}\times\frac{x}{xz-y}=\frac{x^2z+xy}{xz^2-yz}.$$

3. $$\frac{\frac{ab}{c}-3d}{3c-\frac{ab}{d}}=\frac{\frac{ab-3cd}{c}}{\frac{3cd-ab}{d}}=\frac{ab-3cd}{c}\times\frac{d}{3cd-ab}=-\frac{d}{c}.$$

4. $$\frac{1+\frac{1}{x-1}}{1-\frac{1}{x+1}}=\frac{\frac{x-1+1}{x-1}}{\frac{x+1-1}{x+1}}=\frac{\frac{x}{x-1}}{\frac{x}{x+1}}=\frac{x}{x-1}\times\frac{x+1}{x}=\frac{x+1}{x-1}.$$

5. $$\frac{1+\frac{y}{x-y}}{1-\frac{y}{x+y}}=\frac{\frac{x-y+y}{x-y}}{\frac{x+y-y}{x+y}}=\frac{\frac{x}{x-y}}{\frac{x}{x+y}}=\frac{x}{x-y}\times\frac{x+y}{x}=\frac{x+y}{x-y}.$$

6. $$\frac{m+\frac{mn}{m-n}}{m-\frac{mn}{m+n}}=\frac{\frac{m^2-mn+mn}{m-n}}{\frac{m^2+mn-mn}{m+n}}=\frac{\frac{m^2}{m-n}}{\frac{m^2}{m+n}}=\frac{m^2}{m-n}\times\frac{m+n}{m^2}=\frac{m+n}{m-n}.$$

7. $$\frac{3}{a+1}-\frac{2a-1}{a^2+\frac{a}{2}-\frac{1}{2}}=\frac{3}{a+1}-\frac{2a-1}{\frac{2a^2+a-1}{2}}$$
$$=\frac{3}{a+1}-\frac{2(2a-1)}{2a^2+a-1}$$
$$=\frac{3}{a+1}-\frac{2(2a-1)}{(2a-1)(a+1)}$$
$$=\frac{3}{a+1}-\frac{2}{a+1}=\frac{1}{a+1}.$$

8. $$\frac{\frac{2m+n}{m+n}-1}{1-\frac{n}{m+n}}.$$

Multiply numerator and denominator by $m+n$. Then,

$$\frac{\frac{2m+n}{m+n}-1}{1-\frac{n}{m+n}}=\frac{2m+n-m-n}{m+n-n}=\frac{m}{m}=1.$$

9. $$\frac{\frac{x^3+y^3}{x^2-y^2}}{\frac{x^2-xy+y^2}{x-y}}.$$

Multiply numerator and denominator by x^2-y^2. Then,

$$\frac{\frac{x^3+y^3}{x^2-y^2}}{\frac{x^2-xy+y^2}{x-y}}=\frac{x^3+y^3}{(x^2-xy+y^2)(x+y)}=\frac{x^3+y^3}{x^3+y^3}=1.$$

10. $$\frac{\dfrac{1}{a-b}-\dfrac{a}{a^2-b^2}}{\dfrac{a}{ab+b^2}-\dfrac{b}{a^2+ab}}.$$

Multiply numerator and denominator by $ab(a^2-b^2)$. Then,

$$\frac{\dfrac{1}{a-b}-\dfrac{a}{a^2-b^2}}{\dfrac{a}{ab+b^2}-\dfrac{b}{a^2+ab}}=\frac{ab(a+b)-a^2b}{a^2(a-b)-b^2(a-b)}$$

$$=\frac{a^2b+ab^2-a^2b}{a^3-a^2b-ab^2+b^3}=\frac{ab^2}{a^3-a^2b-ab^2+b^3}.$$

11. $$\frac{\dfrac{1}{ab}+\dfrac{1}{ac}+\dfrac{1}{bc}}{\dfrac{a^2-(b+c)^2}{ab}}.$$

Multiply numerator and denominator by abc. Then,

$$\frac{\dfrac{1}{ab}+\dfrac{1}{ac}+\dfrac{1}{bc}}{\dfrac{a^2-(b+c)^2}{ab}}=\frac{c+b+a}{c(a^2-(b+c)^2)}$$

$$=\frac{a+b+c}{c(a+b+c)(a-b-c)}=\frac{1}{c(a-b-c)}.$$

12. $$\frac{\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}}{\dfrac{a}{b}+\dfrac{b}{c}+\dfrac{c}{a}}.$$

Multiply numerator and denominator by abc. Then,

$$\frac{\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}}{\dfrac{a}{b}+\dfrac{b}{c}+\dfrac{c}{a}}=\frac{bc+ac+ab}{a^2c+ab^2+bc^2}.$$

13. $$\frac{\dfrac{1}{ab}-\dfrac{1}{ac}-\dfrac{1}{bc}}{\dfrac{a^2-(b-c)^2}{a}}.$$

Multiply numerator and denominator by abc. Then,

$$\frac{\dfrac{1}{ab}-\dfrac{1}{ac}-\dfrac{1}{bc}}{\dfrac{a^2-(b-c)^2}{a}}=\frac{c-b-a}{bc(a^2-(b-c)^2)}=\frac{c-b-a}{bc(a+b-c)(a-b+c)}$$

$$=-\frac{a+b-c}{bc(a+b-c)(a-b+c)}=-\frac{1}{bc(a-b+c)}.$$

14. $$1+\cfrac{x}{1+x+\cfrac{2x^2}{1-x}}=1+\cfrac{x}{\cfrac{(1+x)(1-x)+2x^2}{1-x}}=1+\cfrac{x}{\cfrac{1-x^2+2x^2}{1-x}}$$

$$=1+\cfrac{x}{\cfrac{1+x^2}{1-x}}=1+\frac{x(1-x)}{1+x^2}$$

$$=\frac{1+x^2+x(1-x)}{1+x^2}=\frac{1+x}{1+x^2}.$$

15. $$\cfrac{1}{a+\cfrac{1}{1+\cfrac{a+1}{3-a}}}=\cfrac{1}{a+\cfrac{1}{\cfrac{3-a+a+1}{3-a}}}=\cfrac{1}{a+\cfrac{1}{\cfrac{4}{3-a}}}$$

$$=\cfrac{1}{a+\cfrac{3-a}{4}}=\cfrac{1}{\cfrac{4a+3-a}{4}}=\frac{4}{3a+3}.$$

16. $$\cfrac{x+y}{x+y+\cfrac{1}{x-y+\cfrac{1}{x+y}}}=\cfrac{x+y}{x+y+\cfrac{1}{\cfrac{(x+y)(x-y)+1}{x+y}}}$$

$$=\cfrac{x+y}{x+y+\cfrac{1}{\cfrac{x^2-y^2+1}{x+y}}}=\cfrac{x+y}{x+y+\cfrac{x+y}{x^2-y^2+1}}$$

$$=\cfrac{x+y}{\cfrac{(x+y)(x^2-y^2+1)+x+y}{x^2-y^2+1}}$$

$$=\frac{(x+y)(x^2-y^2+1)}{(x+y)(x^2-y^2+2)}=\frac{x^2-y^2+1}{x^2-y^2+2}.$$

17. $$\cfrac{x+\cfrac{1}{y}}{x+\cfrac{1}{y+\cfrac{1}{z}}}-\frac{1}{y(xyz+x+z)}=\cfrac{\cfrac{xy+1}{y}}{x+\cfrac{z}{yz+1}}-\frac{1}{y(xyz+x+z)}$$

$$=\frac{(xy+1)(yz+1)}{y(xyz+x+z)}-\frac{1}{y(xyz+x+z)}$$

$$=\frac{xy^2z+xy+yz+1-1}{y(xyz+x+z)}$$

$$=\frac{xy^2z+xy+yz}{y(xyz+x+z)}=\frac{y(xyz+x+z)}{y(xyz+x+z)}$$

$$=1.$$

18. $$\frac{3abc}{bc+ac+ab}-\frac{\frac{a-1}{a}+\frac{b-1}{b}+\frac{c-1}{c}}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}.$$

Multiply numerator and denominator of the second fraction by *abc*. Then,

$$\frac{3abc}{bc+ac+ab}-\frac{\frac{a-1}{a}+\frac{b-1}{b}+\frac{c-1}{c}}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}$$
$$=\frac{3abc}{bc+ac+ab}-\frac{(a-1)bc+(b-1)ac+(c-1)ab}{bc+ac+ab}$$
$$=\frac{3abc}{bc+ac+ab}-\frac{abc-bc+abc-ac+abc-ab}{bc+ac+ab}$$
$$=\frac{3abc-(3abc-bc-ac-ab)}{bc+ac+ab}$$
$$=\frac{bc+ac+ab}{bc+ac+ab}$$
$$=1.$$

19. $$\left(\frac{m+n}{m-n}+\frac{m^2+n^2}{m^2-n^2}\right)\div\left(\frac{m-n}{m+n}-\frac{m^2+n^2}{m^2-n^2}\right)$$
$$=\left(\frac{(m+n)^2}{m^2-n^2}+\frac{m^2+n^2}{m^2-n^2}\right)\div\left(\frac{(m-n)^2}{m^2-n^2}-\frac{m^2+n^2}{m^2-n^2}\right)$$
$$=\frac{m^2+2mn+n^2+m^2+n^2}{m^2-n^2}\div\frac{m^2-2mn+n^2-m^2-n^2}{m^2-n^2}$$
$$=\frac{2m^2+2mn+2n^2}{m^2-n^2}\div\frac{-2mn}{m^2-n^2}$$
$$=\frac{2(m^2+mn+n^2)}{m^2-n^2}\times\frac{m^2-n^2}{-2mn}$$
$$=\frac{m^2+mn+n^2}{-mn}$$
$$=-\frac{m^2+mn+n^2}{mn}$$

20. $\left(\dfrac{x-y}{x+y}-\dfrac{x^3-y^3}{x^3+y^3}\right)\times\left(\dfrac{x+y}{x-y}+\dfrac{x^2+y^2}{x^2-y^2}\right).$

$$\begin{aligned}\frac{x-y}{x+y}-\frac{x^3-y^3}{x^3+y^3}&=\frac{(x-y)(x^2-xy+y^2)}{(x+y)(x^2-xy+y^2)}-\frac{x^3-y^3}{x^3+y^3}\\&=\frac{x^3-2x^2y+2xy^2-y^3}{x^3+y^3}-\frac{x^3-y^3}{x^3+y^3}\\&=\frac{x^3-2x^2y+2xy^2-y^3-x^3+y^3}{x^3+y^3}\\&=\frac{2xy^2-2x^2y}{x^3+y^3}.\end{aligned}$$

$$\begin{aligned}\frac{x+y}{x-y}+\frac{x^2+y^2}{x^2-y^2}&=\frac{(x+y)^2}{x^2-y^2}+\frac{x^2+y^2}{x^2-y^2}\\&=\frac{x^2+2xy+y^2+x^2+y^2}{x^2-y^2}\\&=\frac{2x^2+2xy+2y^2}{x^2-y^2}.\end{aligned}$$

$$\begin{aligned}\therefore\ \left(\frac{x-y}{x+y}-\frac{x^3-y^3}{x^3+y^3}\right)\times\left(\frac{x+y}{x-y}+\frac{x^2+y^2}{x^2-y^2}\right)&=\frac{2xy^2-2x^2y}{x^3+y^3}\times\frac{2x^2+2xy+2y^2}{x^2-y^2}\\&=\frac{2xy(y-x)}{x^3+y^3}\times\frac{2(x^2+xy+y^2)}{(x-y)(x+y)}\\&=-\frac{4xy(x^2+xy+y^2)}{(x^3+y^3)(x+y)}.\end{aligned}$$

21. $$\begin{aligned}\frac{\dfrac{1}{a}+\dfrac{1}{b+c}}{\dfrac{1}{a}-\dfrac{1}{b+c}}\left(1+\frac{b^2+c^2-a^2}{2bc}\right)&=\frac{b+c+a}{b+c-a}\times\frac{2bc+b^2+c^2-a^2}{2bc}\\&=\frac{b+c+a}{b+c-a}\times\frac{(b+c)^2-a^2}{2bc}\\&=\frac{b+c+a}{b+c-a}\times\frac{(b+c+a)(b+c-a)}{2bc}\\&=\frac{(b+c+a)^2}{2bc}.\end{aligned}$$

Exercise 52. Page 142.

1. Find the value of $\sqrt{a^3+b^3+c^3}-(a-b-c)^2$, when $a=2$, $b=-2$ and $c=4$.

$$\begin{aligned}\sqrt{a^3+b^3+c^3}-(a-b-c)^2&=\sqrt{8-8+64}-(2+2-4)^2\\&=\sqrt{64}\\&=8.\end{aligned}$$

2. Reduce to lowest terms $\dfrac{3x^3+10x^2+7x-2}{3x^3+13x^2+17x+6}$.

$$\begin{array}{r|r|l}
3x^3+10x^2+7x-2 & 3x^3+13x^2+17x+6 & 1 \\
\underline{3x^3+10x^2+8x} & \underline{3x^3+10x^2+7x-2} & \\
-\quad x-2 & 3x^2+10x+8 & x \\
 & \underline{3x^2+6x} & -3x-4 \\
 & 4x+8 & \\
 & \underline{4x+8} &
\end{array}$$

∴ the H. C. F. is $x+2$.

Hence $$\begin{aligned}\frac{3x^3+10x^2+7x-2}{3x^3+13x^2+17x+6}&=\frac{(x+2)(3x^2+4x-1)}{(x+2)(3x^2+7x+3)}\\&=\frac{3x^2+4x-1}{3x^2+7x+3}.\end{aligned}$$

Simplify:

3. $\dfrac{1}{(x-3)(x-2)}-\dfrac{x-4}{(x-1)(x-3)}+\dfrac{x-3}{(x-1)(x-2)}$.

The L. C. D. is $(x-1)(x-2)(x-3)$.

$$\begin{aligned}\therefore\ x-1&=\text{1st numerator.}\\-(x-4)(x-2)=-x^2+6x-8&=\text{2d numerator.}\\(x-3)(x-3)=\underline{\ \ x^2-6x+9}&=\text{3d numerator.}\\x\qquad&=\text{the sum of the numerators.}\end{aligned}$$

∴ the sum of the fractions is $\dfrac{x}{(x-1)(x-2)(x-3)}$.

4. $$\frac{x^2+yz}{(x-y)(x-z)}+\frac{y^2+xz}{(y-z)(y-x)}+\frac{z^2+xy}{(z-x)(z-y)}$$
$$=-\frac{x^2+yz}{(x-y)(z-x)}-\frac{y^2+xz}{(y-z)(x-y)}-\frac{z^2+xy}{(z-x)(y-z)}.$$

The L. C. D. is $(x-y)(y-z)(z-x)$.

$$\therefore -(y-z)(x^2+yz)=-x^2y-y^2z+x^2z+yz^2=\text{1st numerator.}$$
$$-(z-x)(y^2+xz)=-y^2z+xy^2-xz^2+x^2z=\text{2d numerator.}$$
$$-(x-y)(z^2+xy)=-xz^2-x^2y+yz^2+xy^2=\text{3d numerator.}$$
$$-2x^2y+2xy^2-2y^2z+2yz^2+2x^2z-2xz^2=\text{the sum of the numerators.}$$

∴ the sum of the fraction is

$$\frac{2xy^2-2x^2y+2yz^2-2y^2z+2x^2z-2xz^2}{(x-y)(y-z)(z-x)}$$
$$=\frac{(2xy^2-2x^2y)+(2x^2z-2y^2z)+(2yz^2-2xz^2)}{(x-y)(y-z)(x-z)}$$
$$=\frac{2xy(y-x)+2z(x^2-y^2)+2z^2(y-x)}{(x-y)(y-z)(z-x)}$$
$$=\frac{(-2xy+2xz+2yz-2z^2)(x-y)}{(x-y)(y-z)(z-x)}$$
$$=\frac{2(x-z)(z-y)(x-y)}{(x-y)(y-z)(z-x)}$$
$$=2.$$

5. $$\left(\frac{x}{1+x}+\frac{1-x}{x}\right)\div\left(\frac{x}{1+x}-\frac{1-x}{x}\right)$$
$$=\frac{x^2+(1-x)(1+x)}{(1+x)x}\div\frac{x^2-(1+x)(1-x)}{(1+x)x}$$
$$=\frac{x^2+1-x^2}{(1+x)x}\times\frac{(1+x)x}{x^2-1+x^2}$$
$$=\frac{1}{(1+x)x}\times\frac{(1+x)x}{2x^2-1}=\frac{1}{2x^2-1}.$$

6. $$\frac{1}{c}\left(\frac{1}{x-c}+\frac{1}{x+2c}\right)-\frac{3}{x^2+cx-2c^2}$$
$$=\frac{1}{c}\left(\frac{x+2c+x-c}{(x-c)(x+2c)}\right)-\frac{3}{x^2+cx-2c^2}$$
$$=\frac{2x+c}{c(x^2+cx-2c^2)}-\frac{3}{x^2+cx-2c^2}$$

$$=\frac{2x+c-3c}{c(x^2+cx-2c^2)}=\frac{2x-2c}{c(x^2+cx-2c^2)}$$
$$=\frac{2(x-c)}{c(x-c)(x+2c)}=\frac{2}{c(x+2c)}.$$

7. $\frac{x^4-y^4}{x^2y^2}\left(\frac{x^2}{x^2-y^2}-1+\frac{y^2}{x^2+y^2}\right)$

$$=\frac{x^4-y^4}{x^2y^2}\left(\frac{x^2(x^2+y^2)-(x^2+y^2)(x^2-y^2)+y^2(x^2-y^2)}{(x^2+y^2)(x^2-y^2)}\right)$$
$$=\frac{x^4-y^4}{x^2y^2}\left(\frac{x^4+x^2y^2-x^4+y^4+x^2y^2-y^4}{x^4-y^4}\right)$$
$$=\frac{x^4-y^4}{x^2y^2}\left(\frac{2x^2y^2}{x^4-y^4}\right)$$
$$=2.$$

8. $\frac{3(x^2+x-2)}{x^2-x-2}-\frac{3(x^2-x-2)}{x^2+x-2}-\frac{8x}{x^2-4}$

$$=\frac{3(x^2+x-2)}{(x-2)(x+1)}-\frac{3(x^2-x-2)}{(x+2)(x-1)}-\frac{8x}{(x+2)(x-2)}.$$

$\therefore$ The L. C. D. is $(x+1)(x-1)(x+2)(x-2)$

$$\begin{array}{rrrrrr}
\therefore\ 3(x^2+x-2)(x-1)(x+2) = & 3x^4 & +6x^3 & -9x^2 & -12x & +12 \\
-3(x^2-x-2)(x+1)(x-2) = & -3x^4 & +6x^3 & +9x^2 & -12x & -12 \\
8x(x+1)(x-1) = & & -8x^3 & & +8x & \\
\hline
 & & 4x^3 & & -16x &
\end{array}$$

$\therefore$ the sum of the fractions is

$$\frac{4x^3-16x}{(x+1)(x-1)(x+2)(x-2)}=\frac{4x(x^2-4)}{(x^2-1)(x^2-4)}=\frac{4x}{x^2-1}.$$

9. $\left(\frac{x+2y}{x+y}+\frac{x}{y}\right)\div\left(\frac{x}{y}+2-\frac{x}{x+y}\right)$

$$=\frac{(x+2y)y+x(x+y)}{y(x+y)}\div\frac{x(x+y)+2y(x+y)-xy}{y(x+y)}$$
$$=\frac{xy+2y^2+x^2+xy}{y(x+y)}\div\frac{x^2+xy+2xy+2y^2-xy}{y(x+y)}$$
$$=\frac{2y^2+x^2+2xy}{y(x+y)}\div\frac{x^2+2xy+2y^2}{y(x+y)}$$
$$=\frac{x^2+2xy+2y^2}{y(x+y)}\times\frac{y(x+y)}{x^2+2xy+2y^2}=1.$$

10. $$\left(1-\frac{4}{x-1}+\frac{12}{x-3}\right)\left(1+\frac{4}{x+1}-\frac{12}{x+3}\right)$$
$$=\frac{(x-1)(x-3)-4(x-3)+12(x-1)}{(x-1)(x-3)}$$
$$\times\frac{(x+1)(x+3)+4(x+3)-12(x+1)}{(x+1)(x+3)}$$
$$=\frac{x^2-4x+3-4x+12+12x-12}{(x-1)(x-3)}\times\frac{x^2+4x+3+4x+12-12x-12}{(x+1)(x+3)}$$
$$=\frac{x^2+4x+3}{(x-1)(x-3)}\times\frac{x^2-4x+3}{(x+1)(x+3)}$$
$$=\frac{(x+1)(x+3)}{(x-1)(x-3)}\times\frac{(x-1)(x-3)}{(x+1)(x+3)}=1.$$

11. $$\left[\frac{x^2-xy+y^2}{x^2+xy+y^2}\times\frac{x^3-y^3}{x^3+y^3}\right]\div\frac{(y-x)^2}{(x+y)^2}$$
$$=\left[\frac{x^2-xy+y^2}{x^2+xy+y^2}\times\frac{(x-y)(x^2+xy+y^2)}{(x+y)(x^2-xy+y^2)}\right]\div\frac{(y-x)^2}{(x-y)^2}$$
$$=\frac{x-y}{x+y}\div\frac{(x-y)^2}{(x+y)^2}=\frac{x-y}{x+y}\times\frac{(x+y)^2}{(x-y)^2}=\frac{x+y}{x-y}.$$

12. $$\frac{2a-b-c}{(a-b)(a-c)}+\frac{2b-c-a}{(b-c)(b-a)}+\frac{2c-a-b}{(c-a)(c-b)}$$
$$=-\frac{2a-b-c}{(a-b)(c-a)}-\frac{2b-c-a}{(b-c)(a-b)}-\frac{2c-a-b}{(c-a)(b-c)}.$$

The L. C. D. is $(a-b)(b-c)(c-a)$.

$$\begin{aligned}\therefore\ -(b-c)(2a-b-c)&=-2ab+b^2+2ac-c^2=\text{1st numerator.}\\ -(c-a)(2b-c-a)&=-2bc+c^2+2ab-a^2=\text{2d numerator.}\\ -(a-b)(2c-a-b)&=\underline{-2ac+a^2+2bc-b^2}=\text{3d numerator.}\\ &\qquad\qquad 0\qquad\quad=\text{the sum of the numerators.}\end{aligned}$$

$\therefore$ The sum of the fractions is 0.

13. $$\frac{\dfrac{1}{x+1}-\dfrac{2}{(x+2)(x+1)}}{\dfrac{1}{x+2}-\dfrac{1}{(x+1)(x+2)}}=\frac{\dfrac{x+2}{(x+2)(x+1)}-\dfrac{2}{(x+2)(x+1)}}{\dfrac{x+1}{(x+1)(x+2)}-\dfrac{1}{(x+1)(x+2)}}$$
$$=\frac{\dfrac{x+2-2}{(x+1)(x+2)}}{\dfrac{x+1-1}{(x+1)(x+2)}}$$
$$=\frac{x}{(x+1)(x+2)}\times\frac{(x+1)(x+2)}{x}=1.$$

14. $$\frac{\frac{1}{a^2}+\frac{1}{b^2}}{\frac{1}{a^2}-\frac{1}{b^2}} \div \frac{\frac{1}{a}+\frac{1}{b}}{\frac{1}{a}-\frac{1}{b}} = \frac{b^2+a^2}{b^2-a^2} \div \frac{b+a}{b-a} = \frac{b^2+a^2}{b^2-a^2} \times \frac{b-a}{b+a}$$
$$= \frac{b^2+a^2}{(b+a)(b+a)}$$
$$= \frac{b^2+a^2}{(b+a)^2}.$$

15. $$\left(1-\frac{1-x}{1+x}+\frac{1+2x^2}{1-x^2}\right)\left(\frac{x+1}{2x+1}\right)$$
$$=\left(\frac{1-x^2}{1-x^2}-\frac{(1-x)^2}{1-x^2}+\frac{1+2x^2}{1-x^2}\right)\left(\frac{x+1}{2x+1}\right)$$
$$=\frac{1-x^2-1+2x-x^2+1+2x^2}{1-x^2}\times\frac{x+1}{2x+1}$$
$$=\frac{1+2x}{1-x^2}\times\frac{x+1}{2x+1}=\frac{1}{1-x}.$$

16. $$\frac{x^3-8y^3}{x(x-y)}\times\frac{x^2-xy+y^2}{x^2+2xy+4y^2}\times\frac{x(x^2-y^2)}{x^3+y^3}$$
$$=\frac{(x-2y)(x^2+2xy+4y^2)}{x(x-y)}\times\frac{x^2-xy+y^2}{x^2+2xy+4y^2}\times\frac{x(x+y)(x-y)}{(x+y)(x^2-xy+y^2)}$$
$$=x-2y.$$

17. $$\frac{2}{a}+\frac{2}{b}+\frac{2}{c}-\frac{b+c-a}{bc}-\frac{c+a-b}{ac}-\frac{a+b-c}{ab}$$
$$=\frac{2bc}{abc}+\frac{2ac}{abc}+\frac{2ab}{abc}-\frac{a(b+c-a)}{abc}-\frac{b(c+a-b)}{abc}-\frac{c(a+b-c)}{abc}$$
$$=\frac{2bc+2ac+2ab-ab-ac+a^2-bc-ab+b^2-ac-bc+c^2}{abc}$$
$$=\frac{a^2+b^2+c^2}{abc}.$$

Exercise 53. Page 145.

Solve:

1. $$\frac{3x-1}{4}-\frac{2x+1}{3}-\frac{4x-5}{5}=4.$$

Multiply through by 60.

$$45x-15-40x-20-48x+60=240$$
$$-43x=215$$
$$x=-5$$

2. $2-\frac{7x-1}{6}=3x-\frac{19x+3}{4}.$

Multiply through by 12.

$$\begin{aligned} 24-14x+2&=36x-57x-9\\ 7x&=-35\\ x&=-5 \end{aligned}$$

3. $\frac{5x+1}{3}+\frac{19x+7}{9}-\frac{3x-1}{2}=\frac{7x-1}{6}.$

Multiply through by 18.

$$\begin{aligned} 30x+6+38x+14-27x+9&=21x-3\\ 20x&=-32\\ x&=-\tfrac{8}{5} \end{aligned}$$

4. $4+\frac{x}{7}=\frac{3x-2}{2}-\frac{11x+2}{14}-\frac{2-7x}{3}.$

Multiply through by 42.

$$\begin{aligned} 168+6x&=63x-42-33x-6-28+98x\\ -122x&=-244\\ x&=2 \end{aligned}$$

5. $\frac{6x-4}{3}-2=\frac{18-4x}{3}+x.$

Multiply through by 3.

$$\begin{aligned} 6x-4-6&=18-4x+3x\\ 7x&=28\\ \therefore\ x&=4 \end{aligned}$$

6. $\frac{6x+7}{9}+\frac{7x-13}{27}=\frac{2x+4}{3}.$

Multiply through by 27.

$$\begin{aligned} 18x+21+7x-13&=18x+36\\ 7x&=28\\ \therefore\ x&=4 \end{aligned}$$

7. $\frac{9x+5}{14}+\frac{8x-7}{14}=\frac{36x+15}{56}+\frac{41}{56}.$

Multiply through by 56.

$$\begin{aligned} 36x+20+32x-28&=36x+15+41\\ 32x&=64\\ \therefore\ x&=2 \end{aligned}$$

8. $\frac{x+3}{2}-\frac{x-2}{3}=\frac{3x-5}{12}+\frac{1}{4}.$

Multiply through by 12.

$$6x+18-4x+8=3x-5+3$$
$$-x=-28$$
$$x=28$$

9. $11-\left(\frac{3x-1}{4}+\frac{2x+1}{3}\right)=10-\left(\frac{2x-5}{3}+\frac{7x-1}{8}\right).$

Multiply through by 24.

$$264-(18x-6+16x+8)=240-(16x-40+21x-3)$$
$$264-18x+6-16x-8=240-16x+40-21x+3$$
$$3x=21$$
$$\therefore\ x=7$$

10. $\frac{7x-4}{9}+\frac{3x-1}{5}-\frac{5(x-1)}{6}=\frac{3(3x-1)}{20}+\frac{x}{7}.$

Multiply through by $2\times2\times3\times3\times5\times7$ or 1260.

$$980x-560+756x-252-1050x+1050=567x-189+180x$$
$$-61x=-427$$
$$x=7$$

11. $6x-\frac{27x-1}{4}-\frac{2(4x-1)}{5}=\frac{9x-5}{4}-\frac{11x-2}{3}+22.$

Multiply through by 60.

$$360x-405x+15-96x+24=135x-75-220x+40+1320$$
$$-56x=1246$$
$$\therefore\ x=-\tfrac{1246}{56}=-\tfrac{89}{4}$$

12. $\frac{10x+11}{6}-\frac{12x-13}{3}-4=\frac{7-6x}{4}.$

Multiply through by 12.

$$20x+22-48x+52-48=21-18x$$
$$-10x=-5$$
$$\therefore\ x=\tfrac{1}{2}$$

13. $\dfrac{3}{y-4}+\dfrac{5}{2(y-4)}+\dfrac{9}{2(y-4)}=\dfrac{1}{2}.$

Add the second and third terms together.

$$\frac{3}{y-4}+\frac{14}{2(y-4)}=\frac{1}{2}$$
$$\frac{3}{y-4}+\frac{7}{y-4}=\frac{1}{2}$$
$$\frac{10}{y-4}=\frac{1}{2}$$
$$20=y-4$$
$$y=24$$

14. $\dfrac{2}{x-1}-\dfrac{5}{2(x-1)}=\dfrac{8}{3(x-1)}-\dfrac{x}{x-1}+\dfrac{5}{18}.$

Multiply through by 18.

$$\frac{36}{x-1}-\frac{45}{x-1}=\frac{48}{x-1}-\frac{18x}{x-1}+5$$
$$\frac{36-45-48+18x}{x-1}=5$$
$$\frac{18x-57}{x-1}=5$$
$$18x-57=5x-5$$
$$13x=52$$
$$x=4$$

15. $\dfrac{2x-3}{2x-4}-6=\dfrac{x+5}{3x-6}-\dfrac{11}{2}.$

Transpose -6 and $\dfrac{x+5}{3x-6}$.

$$\frac{2x-3}{2x-4}-\frac{x+5}{3x-6}=6-\frac{11}{2}$$
$$\frac{2x-3}{2(x-2)}-\frac{x+5}{3(x-2)}=\frac{1}{2}$$

Multiply through by 6.

$$\frac{6x-9}{x-2}-\frac{2x+10}{x-2}=3$$
$$\frac{6x-9-2x-10}{x-2}=3$$
$$\frac{4x-19}{x-2}=3$$
$$4x-19=3x-6$$
$$\therefore x=13$$

16. $\dfrac{10-7x}{6-7x}=\dfrac{5x-4}{5x}.$

Multiply through by $5x(6-7x)$.

$$5x(10-7x)=(5x-4)(6-7x)$$
$$50x-35x^2=58x-24-35x^2$$
$$-8x=-24$$
$$\therefore x=3$$

17. $\dfrac{5+8x}{3+2x}=\dfrac{45-8x}{13-2x}.$

Multiply through by $(3+2x)(13-2x)$.

$$(5+8x)(13-2x)=(3+2x)(45-8x)$$
$$65+94x-16x^2=135+66x-16x^2$$
$$28x=70$$
$$\therefore x=\tfrac{5}{2}$$

18. $\frac{5x-1}{2x+3}=\frac{5x-3}{2x-3}.$

Multiply through by

$$(2x+3)(2x-3).$$

$$(2x-3)(5x-1)=(2x+3)(5x-3)$$
$$10x^2-17x+3=10x^2+9x-9$$
$$-26x=-12$$
$$\therefore\ x=\tfrac{6}{13}$$

19. $\frac{x}{3}-\frac{x^2-5x}{3x-7}=\frac{2}{3}.$

Multiply through by $3(3x-7)$.

$$x(3x-7)-3(x^2-5x)=2(3x-7)$$
$$3x^2-7x-3x^2+15x=6x-14$$
$$2x=-14$$
$$\therefore\ x=-7$$

20. $\frac{2x+1}{2x-1}-\frac{8}{4x^2-1}=\frac{2x-1}{2x+1}.$

Multiply through by $4x^2-1$.

$$(2x+1)^2-8=(2x-1)^2$$
$$4x^2+4x+1-8=4x^2-4x+1$$
$$8x=8$$
$$\therefore\ x=1$$

21. $\frac{5-2x}{x-1}-\frac{2-7x}{x+1}=\frac{5x^2+4}{x^2-1}.$

Multiply through by x^2-1.

$$(x+1)(5-2x)-(x-1)(2-7x)=5x^2+4$$
$$5+3x-2x^2+2-9x+7x^2=5x^2+4$$
$$-6x=-3$$
$$\therefore\ x=\tfrac{1}{2}$$

22. $\frac{6}{x+2}-\frac{x+2}{x-2}+\frac{x^2}{x^2-4}=0.$

Multiply through by x^2-4.

$$6(x-2)-(x+2)^2+x^2=0$$
$$6x-12-x^2-4x-4+x^2=0$$
$$2x=16$$
$$\therefore\ x=8$$

23. $\frac{4}{1+x}+\frac{x+1}{1-x}-\frac{x^2-3}{1-x^2}=0.$

Multiply through by $1-x^2$.

$$4(1-x)+(x+1)^2-x^2+3=0$$
$$4-4x+x^2+2x+1-x^2+3=0$$
$$-2x=-8$$
$$\therefore\ x=4$$

24. $\frac{2x+1}{3x-3}=\frac{7x-1}{6x+6}-\frac{2x^2-3x-45}{4x^2-4}.$

Multiply through by $12(x^2-1)$.

$$4(x+1)(2x+1)=2(x-1)(7x-1)-3(2x^2-3x-45)$$
$$8x^2+12x+4=14x^2-16x+2-6x^2+9x+135$$
$$19x=133$$
$$\therefore\ x=7$$

25. $\frac{x^2-x+1}{x-1}+\frac{x^2+x+1}{x+1}=2x.$

Multiply through by x^2-1.

$$(x+1)(x^2-x+1)+(x-1)(x^2+x+1)=2x(x^2-1)$$
$$x^3+1+x^3-1=2x^3-2x$$
$$2x=0$$
$$\therefore\ x=0$$

26. $\frac{9x+5}{6(x-1)}+\frac{3x^2-51x-71}{18(x^2-1)}=\frac{15x-7}{9(x+1)}.$

Multiply through by $18(x^2-1)$.

$$3(x+1)(9x+5)+3x^2-51x-71=2(x-1)(15x-7)$$
$$27x^2+42x+15+3x^2-51x-71=30x^2-44x+14$$
$$35x=70$$
$$\therefore x=2$$

27. $\frac{4}{x+2}+\frac{7}{x+3}-\frac{37}{x^2+5x+6}=0.$

Multiply through by x^2+5x+6.

$$4(x+3)+7(x+2)-37=0$$
$$4x+12+7x+14-37=0$$
$$11x=11$$
$$\therefore x=1$$

28. $\frac{1}{x-1}-\frac{1}{x-2}=\frac{1}{x-3}-\frac{1}{x-4}.$

$$\frac{x-2-x+1}{(x-1)(x-2)}=\frac{x-4-x+3}{(x-3)(x-4)}$$
$$-\frac{1}{(x-1)(x-2)}=-\frac{1}{(x-3)(x-4)}$$
$$(x-3)(x-4)=(x-1)(x-2)$$
$$x^2-7x+12=x^2-3x+2$$
$$-4x=-10$$
$$\therefore x=\tfrac{5}{2}$$

Exercise 54. Page 147.

Solve:

1. $\frac{4x+3}{10}-\frac{2x-5}{5x-1}=\frac{2x-1}{5}.$

Multiply by 10.

$$4x+3-\frac{10(2x-5)}{5x-1}=4x-2$$
$$-\frac{10(2x-5)}{5x-1}=-5$$

Divide by -5 and multiply by $5x-1$.

$$4x-10=5x-1$$
$$\therefore x=-9$$

2. $\frac{9x+20}{36}=\frac{4x-12}{5x-4}+\frac{x}{4}.$

Multiply by 36.

$$9x+20=\frac{36(4x-12)}{5x-4}+9x$$
$$20=\frac{36(4x-12)}{5x-4}$$

Divide by 4 and multiply by $5x-4$.

$$25x-20=36x-108$$
$$11x=88$$
$$\therefore x=8$$

3. $\dfrac{10x+17}{18}-\dfrac{12x+2}{13x-16}=\dfrac{5x-4}{9}.$

Multiply by 18.

$$10x+17-\frac{18(12x+2)}{13x-16}=10x-8$$
$$-\frac{18(12x+2)}{13x-16}=-25$$
$$216x+36=325x-400$$
$$-109x=-436$$
$$\therefore\ x=4$$

4. $\dfrac{6x+7}{9}+\dfrac{7x-13}{6x+3}=\dfrac{2x+4}{3}.$

Multiply by 9.

$$6x+7+\frac{9(7x-13)}{6x+3}=6x+12$$
$$\frac{9(7x-13)}{6x+3}=5$$
$$63x-117=30x+15$$
$$33x=132$$
$$\therefore\ x=4$$

5. $\dfrac{6x+7}{15}-\dfrac{2x-2}{7x-6}=\dfrac{2x+1}{5}.$

Multiply by 15.

$$6x+7-\frac{15(2x-2)}{7x-6}=6x+3$$
$$\frac{15(2x-2)}{7x-6}=4$$
$$30x-30=28x-24$$
$$2x=6$$
$$\therefore\ x=3$$

6. $\dfrac{6x+1}{15}-\dfrac{2x-4}{7x-16}=\dfrac{2x-1}{5}.$

Multiply by 15.

$$6x+1-\frac{15(2x-4)}{7x-16}=6x-3$$
$$\frac{15(2x-4)}{7x-16}=4$$
$$30x-60=28x-64$$
$$2x=-4$$
$$\therefore\ x=-2$$

7. $\dfrac{11x-13}{14}-\dfrac{22x-75}{28}=\dfrac{13x+7}{2(3x+7)}.$

Multiply by 28.

$$22x-26-22x+75=\frac{14(13x+7)}{3x+7}$$
$$49=\frac{14(13x+7)}{3x+7}$$
$$7=\frac{2(13x+7)}{3x+7}$$
$$21x+49=26x+14$$
$$-5x=-35$$
$$\therefore\ x=7$$

8. $\dfrac{2x+8\frac{1}{2}}{9}-\dfrac{13x-2}{17x-32}+\dfrac{x}{3}=\dfrac{7x}{12}-\dfrac{x+16}{36}.$

Multiply by 36.

$$8x+34-\frac{36(13x-2)}{17x-32}+12x=21x-x-16$$

$$-\frac{36(13x-2)}{17x-32}=-50$$
$$36(13x-2)=50(17x-32)$$
$$468x-72=850x-1600$$
$$1528=382x$$
$$\therefore\ x=4$$

9. $\dfrac{6-5x}{15}-\dfrac{7-2x^2}{14(x-1)}=\dfrac{1+3x}{21}-\dfrac{2x-2\frac{1}{5}}{6}+\dfrac{1}{105}.$

Multiply by 210.

$$84-70x-\frac{15(7-2x^2)}{x-1}=10+30x-70x+77+2$$
$$-\frac{15(7-2x^2)}{x-1}=30x+5$$
$$\frac{3(7-2x^2)}{x-1}=-6x-1$$
$$21-6x^2=-6x^2+5x+1$$
$$20=5x$$
$$\therefore\ x=4$$

10. $\dfrac{2x-5}{5}+\dfrac{x-3}{2x-15}=\dfrac{4x-3}{10}-1\frac{1}{10}.$

Multiply by 10.

$$4x-10+\frac{10(x-3)}{2x-15}=4x-3-11$$
$$\frac{10(x-3)}{2x-15}=-4$$
$$10x-30=-8x+60$$
$$18x=90$$
$$\therefore\ x=5$$

Exercise 55. Page 148.

Solve:

1. $ax+2b=3bx+4a.$

$$ax-3bx=4a-2b$$
$$(a-3b)x=4a-2b$$
$$\therefore\ x=\frac{4a-2b}{a-3b}$$

2. $x^2+b^2=(a-x)^2.$

$$x^2+b^2=a^2-2ax+x^2$$
$$2ax=a^2-b^2$$
$$\therefore\ x=\frac{a^2-b^2}{2a}$$

3. $(a+x)(b+x)=x(x-c)$.

$$ab+ax+bx+x^2=x^2-cx$$
$$ax+bx+cx=-ab$$
$$(a+b+c)x=-ab$$
$$\therefore\ x=-\frac{ab}{a+b+c}$$

4. $(x-a)(x+b)=(x-b)(x-c)$.

$$x^2-ax+bx-ab=x^2-bx-cx+bc$$
$$-ax+2bx+cx=bc+ab$$
$$(2b+c-a)x=b(a+c)$$
$$\therefore\ x=\frac{b(a+c)}{2b+c-a}$$

5. $\dfrac{x}{a+ax}=\dfrac{b}{c+cx}$.

$$\frac{x}{a(1+x)}=\frac{b}{c(1+x)}$$

Multiply by $1+x$.

$$\frac{x}{a}=\frac{b}{c}$$
$$\therefore\ x=\frac{ab}{c}$$

6. $\dfrac{c+d}{ab+bx}=\dfrac{m-x}{an+nx}$.

$$\frac{c+d}{b(a+x)}=\frac{m-x}{n(a+x)}$$

Multiply by $a+x$.

$$\frac{c+d}{b}=\frac{m-x}{n}$$
$$cn+dn=bm-bx$$
$$bx=bm-cn-dn$$
$$\therefore\ x=\frac{bm-cn-dn}{b}$$

7. $\dfrac{x+2}{x-2}=\dfrac{m+n}{m-n}$.

$$(m-n)(x+2)=(m+n)(x-2)$$
$$mx-nx+2m-2n=mx+nx-2m-2n$$
$$-2nx=-4m$$
$$\therefore\ x=\frac{2m}{n}$$

8. $\dfrac{m+n}{2+x}=\dfrac{m-n}{2-x}$.

$$(m+n)(2-x)=(m-n)(2+x)$$
$$2m+2n-mx-nx=2m-2n+mx-nx$$
$$-2mx=-4n$$
$$\therefore\ x=\frac{2n}{m}$$

9. $\dfrac{a+bx}{a+b}=\dfrac{c+dx}{c+d}$.

$$(a+bx)(c+d)=(c+dx)(a+b)$$
$$ac+ad+bcx+bdx=ac+bc+adx+bdx$$
$$(bc-ad)x=bc-ad$$
$$\therefore\ x=1.$$

10. $\dfrac{6x+a}{4x+b}=\dfrac{3x-b}{2x-a}$.

$$(2x-a)(6x+a)=(4x+b)(3x-b)$$
$$12x^2-4ax-a^2=12x^2-bx-b^2$$
$$(b-4a)x=a^2-b^2$$
$$\therefore\ x=\frac{a^2-b^2}{b-4a}$$

11. $\dfrac{x}{a}+\dfrac{x}{b}+\dfrac{x}{c}=d$.

$$x\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=d$$
$$x\left(\frac{bc+ac+ab}{abc}\right)=d$$
$$\therefore\ x=\frac{abcd}{ab+bc+ac}$$

12. $\dfrac{ax-b}{c}-\dfrac{bx+c}{a}=abc.$

$$\begin{aligned}a(ax-b)-c(bx+c)&=a^2bc^2\\a^2x-ab-bcx-c^2&=a^2bc^2\\(a^2-bc)x&=ab+c^2+a^2bc^2\\\therefore\ x&=\frac{ab+c^2+a^2bc^2}{a^2-bc}\end{aligned}$$

13. $\dfrac{5ax}{a-b}-3a=8x.$

$$\begin{aligned}5ax-3a(a-b)&=8x(a-b)\\5ax-3a^2+3ab&=8ax-8bx\\(8b-3a)x&=3a^2-3ab\\\therefore\ x&=\frac{3a^2-3ab}{8b-3a}\end{aligned}$$

14. $\dfrac{x}{a-b}-\dfrac{5a}{a+b}=\dfrac{2bx}{a^2-b^2}.$

$$\begin{aligned}(a+b)x-5a(a-b)&=2bx\\ax+bx-5a^2+5ab&=2bx\\(a-b)x&=5a^2-5ab\\\therefore\ x&=\frac{5a^2-5ab}{a-b}\\&=5a\end{aligned}$$

15. $\dfrac{1}{n}+\dfrac{n}{x+n}=\dfrac{x+n}{nx}.$

$$\begin{aligned}x(x+n)+n^2x&=(x+n)^2\\x^2+nx+n^2x&=x^2+2nx+n^2\\(n^2-n)x&=n^2\\\therefore\ x&=\frac{n^2}{n^2-n}\\&=\frac{n}{n-1}\end{aligned}$$

16. $\dfrac{x}{a}+\dfrac{x}{b-a}=\dfrac{a}{b+a}.$

$$\begin{aligned}x(b^2-a^2)+ax(b+a)&=a^2(b-a)\\x(b^2-a^2+ab+a^2)&=a^2(b-a)\\x(b^2+ab)&=a^2(b-a)\\\therefore\ x&=\frac{a^2(b-a)}{b^2+ab}\end{aligned}$$

17. $\dfrac{x-a}{x-b}=\left(\dfrac{2x-a}{2x-b}\right)^2.$

$$\begin{aligned}(x-a)(2x-b)^2&=(x-b)(2x-a)^2\\4x^3-(4a+4b)x^2+(4ab+b^2)x-ab^2&=4x^3-(4a+4b)x^2+(4ab+a^2)x-a^2b\\(4ab+b^2)x-ab^2&=(4ab+a^2)x-a^2b\\(b^2-a^2)x&=ab^2-a^2b\\\therefore\ x&=\frac{ab^2-a^2b}{b^2-a^2}=\frac{ab}{b+a}\end{aligned}$$

18. $\dfrac{x-a}{2}=\dfrac{(x-b)^2}{2x-a}.$

$$\begin{aligned}(x-a)(2x-a)&=2(x-b)^2\\2x^2-3ax+a^2&=2x^2-4bx+2b^2\\(4b-3a)x&=2b^2-a^2\\\therefore\ x&=\frac{2b^2-a^2}{4b-3a}\end{aligned}$$

19. $\dfrac{\dfrac{m+n}{x}}{\dfrac{1}{m}}=\dfrac{a}{b}.$

$$\frac{m(m+n)}{x}=\frac{a}{b}.$$

$$bm(m+n)=ax$$

$$\therefore\ x=\frac{bm(m+n)}{a}$$

20. $\dfrac{\dfrac{ax-b}{x}}{\dfrac{a}{b}}=\dfrac{\dfrac{ax+b}{x}}{\dfrac{b}{a}}.$

$$\frac{b(ax-b)}{ax}=\frac{a(ax+b)}{bx}$$

$$b^2(ax-b)=a^2(ax+b)$$

$$ab^2x-b^3=a^3x+a^2b$$

$$(ab^2-a^3)x=a^2b+b^3$$

$$\therefore\ x=\frac{a^2b+b^3}{ab^2-a^3}$$

21. $\dfrac{\dfrac{cx+d}{a}}{\dfrac{cx}{d}}=\dfrac{2d}{a}.$

$$\frac{d(cx+d)}{acx}=\frac{2d}{a}$$

$$d(cx+d)=2cdx$$

$$cdx+d^2=2cdx$$

$$-cdx=-d^2$$

$$\therefore\ x=\frac{d}{c}$$

22. $\dfrac{\dfrac{a+x-3}{3}}{\dfrac{a-x+3}{3}}=\dfrac{b}{c}.$

$$\frac{a+x-3}{3}\times\frac{3}{a-x+3}=\frac{b}{c}$$

$$\frac{a+x-3}{a-x+3}=\frac{b}{c}$$

$$c(a+x-3)=b(a-x+3)$$

$$ac+cx-3c=ab-bx+3b$$

$$(b+c)x=ab-ac+3b+3c$$

$$\therefore\ x=\frac{ab-ac+3b+3c}{b+c}$$

23. $\dfrac{a-b}{bx+c}+\dfrac{a+b}{ax-c}=0.$

$$(a-b)(ax-c)+(a+b)(bx+c)=0$$

$$(a^2-ab)x-ac+bc+(ab+b^2)x+ac+bc=0$$

$$(a^2+b^2)x=-2bc$$

$$\therefore\ x=-\frac{2bc}{a^2+b^2}$$

24. $\dfrac{x+a}{x-a}-\dfrac{x-b}{x+b}=\dfrac{2(a+b)}{x}.$

$$x(x+a)(x+b)-x(x-a)(x-b)=2(a+b)(x-a)(x+b)$$

$$x^3+(a+b)x^2+abx-x^3+(a+b)x^2-abx=2(a+b)x^2-2(a^2-b^2)x-2ab(a+b)$$

$$2(a+b)x^2=2(a+b)x^2-2(a^2-b^2)x-2ab(a+b)$$

$$2(a^2-b^2)x=-2ab(a+b)$$

$$2(a-b)x=-2ab$$

$$\therefore\ x=-\frac{ab}{a-b}=\frac{ab}{b-a}$$

25. $\frac{x+a}{b} - x = b - \frac{x-b}{c} + \frac{c-bx}{b}$.

$$c(x+a) - bcx = b^2c - b(x-b) + c(c-bx)$$
$$cx + ac - bcx = b^2c - bx + b^2 + c^2 - bcx$$
$$(b+c)x = b^2c + b^2 + c^2 - ac$$
$$\therefore\ x = \frac{b^2c + b^2 + c^2 - ac}{b+c}$$

26. $\frac{20a - bx}{5a} + \frac{9c - ax}{3c} + \frac{6d - cx}{2d} = 10$.

$$6cd(20a - bx) + 10ad(9c - ax) + 15ac(6d - cx) = 300acd$$
$$120acd - 6bcdx + 90acd - 10a^2dx + 90acd - 15ac^2x = 300acd$$
$$-(6bcd + 10a^2d + 15ac^2)x = 0$$
$$\therefore\ x = 0$$

27. $\frac{ax}{b} - \frac{b-x}{2c} + \frac{a(b-x)}{3d} = a$.

$$6acdx - 3bd(b-x) + 2abc(b-x) = 6abcd$$
$$6acdx - 3b^2d + 3bdx + 2ab^2c - 2abcx = 6abcd$$
$$(6acd + 3bd - 2abc)x = 6abcd + 3b^2d - 2ab^2c$$
$$\therefore\ x = \frac{6abcd + 3b^2d - 2ab^2c}{6acd + 3bd - 2abc}$$
$$= b$$

Exercise 56. Page 150.

1. The sum of the sixth and seventh parts of a number is 13. Find the number.

Let $x =$ the number.

Then, $\frac{x}{6} + \frac{x}{7} =$ the sum of its sixth and seventh parts.

But, $13 =$ this sum.

$$\therefore\ \frac{x}{6} + \frac{x}{7} = 13$$
$$7x + 6x = 546$$
$$13x = 546$$
$$\therefore\ x = 42$$

Therefore the number is 42.

2. The sum of the third, fourth, and sixth parts of a number is 18. Find the number.

Let $x =$ the number.

Then, $\frac{x}{3}+\frac{x}{4}+\frac{x}{6}=$ the sum of its third, fourth, and sixth parts.

But, $18 =$ this sum.

$$\therefore \frac{x}{3}+\frac{x}{4}+\frac{x}{6}=18$$

$$12x+9x+6x=648$$

$$27x=648$$

$$\therefore x=24$$

Therefore the number is 24.

3. The difference between the third and fifth parts of a number is 4. Find the number.

Let $x =$ the number.

Then, $\frac{x}{3}-\frac{x}{5}=$ the difference between its third and fifth parts.

But, $4 =$ this difference.

$$\therefore \frac{x}{3}-\frac{x}{5}=4$$

$$5x-3x=60$$

$$2x=60$$

$$\therefore x=30$$

Therefore the number is 30.

4. The sum of the third, fourth, and fifth parts of a number exceeds the half of the number by 17. Find the number.

Let $x =$ the number.

Then, $\frac{x}{3}+\frac{x}{4}+\frac{x}{5}=$ the sum of the third, fourth, and fifth parts of the number.

$\frac{x}{2}=$ half the number.

$\frac{x}{3}+\frac{x}{4}+\frac{x}{5}-\frac{x}{2}=$ the given excess.

$17 =$ the given excess.

$$\therefore \frac{x}{3}+\frac{x}{4}+\frac{x}{5}-\frac{x}{2}=17$$

$$x\left(\frac{1}{3}+\frac{1}{4}+\frac{1}{5}-\frac{1}{2}\right)=17$$

$$\frac{17x}{60}=17$$

$$\therefore x=60$$

Therefore the number is 60.

5. There are two consecutive numbers, x and $x+1$, such that one-fourth of the smaller exceeds one-ninth of the larger by 11. Find the numbers.

$\frac{x}{4}$ = one-fourth of the smaller number.

$\frac{x+1}{9}$ = one-ninth of the larger number.

$\frac{x}{4}-\frac{x+1}{9}$ = the given excess.

But, 11 = the given excess.

$$\therefore \frac{x}{4}-\frac{x+1}{9}=11$$

$$9x-4x-4=396$$

$$5x=400$$

$$\therefore x=80$$

Therefore the numbers are 80 and 81.

6. Find three consecutive numbers such that if they are divided by 7, 10, and 17, respectively, the sum of the quotients will be 15.

Let x = the smallest number.

Then, $x+1$ = the middle number,

and $x+2$ = the largest number.

$\frac{x}{7}+\frac{x+1}{10}+\frac{x+2}{17}$ = the sum of the respective quotients.

But, 15 = this sum.

$$\therefore \frac{x}{7}+\frac{x+1}{10}+\frac{x+2}{17}=15$$

$$170x+119x+119+70x+140=17850$$

$$359x=17591$$

$$\therefore x=49$$

$$x+1=50$$

$$x+2=51$$

Therefore the numbers are 49, 50, 51.

7. Find a number such that the sum of its sixth and ninth parts shall exceed the difference of its ninth and twelfth parts by 9.

Let x = the number.

Then, $\frac{x}{6}+\frac{x}{9}$ = the sum of its sixth and ninth parts.

$\frac{x}{9}-\frac{x}{12}=$ the difference of its ninth and twelfth parts.

$\frac{x}{6}+\frac{x}{9}-\left(\frac{x}{9}-\frac{x}{12}\right)=$ the given excess.

But, $9=$ the given excess.

$$\frac{x}{6}+\frac{x}{9}-\left(\frac{x}{9}-\frac{x}{12}\right)=9$$

$$\frac{x}{6}+\frac{x}{12}=9$$

$$3x=108$$

$$\therefore x=36$$

Therefore the number is 36.

8. The sum of two numbers is 91, and if the greater is divided by the less the quotient is 2, and the remainder is 1. Find the numbers.

Let $x=$ the smaller number.

Then, $91-x=$ the greater number.

$\frac{91-x}{x}=$ the greater number divided by the less.

$$\therefore \frac{91-x}{x}=2+\frac{1}{x}$$

$$91-x=2x+1$$

$$-3x=-90$$

$$\therefore x=30$$

$$91-x=61$$

Therefore the numbers are 30 and 61.

9. The difference of two numbers is 40, and if the greater is divided by the less the quotient is 4, and the remainder 4. Find the numbers.

Let $x=$ the smaller number.

Then, $40+x=$ the greater number.

$$\therefore \frac{40+x}{x}=4+\frac{4}{x}$$

$$40+x=4x+4$$

$$-3x=-36$$

$$\therefore x=12$$

$$40+x=52$$

Therefore the numbers are 12 and 52.

10. Divide the number 240 into two parts such that the smaller part is contained in the larger part 5 times, with a remainder of 6.

Let $x =$ the smaller part.

Then, $240 - x =$ the larger part.

$$\therefore \frac{240 - x}{x} = 5 + \frac{6}{x}$$

$$240 - x = 5x + 6$$

$$-6x = -234$$

$$\therefore x = 39$$

$$240 - x = 201$$

Therefore the parts are 39 and 201.

11. In a mixture of alcohol and water the alcohol was 24 gallons more than half the mixture, and the water was 4 gallons less than one-fourth the mixture. How many gallons were there of each?

Let $x =$ the entire number of gallons in the mixture.

Then, $\frac{x}{2} + 24 =$ the number of gallons of alcohol,

and $\frac{x}{4} - 4 =$ the number of gallons of water.

$\frac{x}{2} + 24 + \frac{x}{4} - 4 =$ the number of gallons of both alcohol and water.

But, $x =$ the number of gallons of both alcohol and water.

$$\therefore \frac{x}{2} + 24 + \frac{x}{4} - 4 = x$$

$$-\frac{x}{4} = -20$$

$$\therefore x = 80$$

$$\frac{x}{2} + 24 = 64$$

$$\frac{x}{4} - 4 = 16$$

There were 16 gallons of water and 64 gallons of alcohol in the mixture.

12. The width of a room is three-fourths of its length. If the width was 4 feet more and the length 4 feet less, the room would be square. Find its dimensions.

Let $x =$ the length of the room in feet.

Then, $\frac{3x}{4} =$ the width of the room in feet.

$\frac{3x}{4} + 4 =$ the width increased by 4 feet.

$x - 4 =$ the length diminished by 4 feet.

But the last two lengths would be equal.

$$\therefore \frac{3x}{4} + 4 = x - 4$$

$$-\frac{x}{4} = -8$$

$$\therefore x = 32$$

$$\frac{3x}{4} = 24$$

The room is 24 feet wide and 32 feet long.

13. A is 60 years old, and B is three-fourths as old. How many years since B was just one-half as old as A?

Let $x =$ the number of years since B was one-half as old as A.

Then, $60 - x =$ A's age in years at that time.

$45 - x =$ B's age in years at that time.

$$\therefore 45 - x = \frac{60 - x}{2}$$

$$90 - 2x = 60 - x$$

$$\therefore x = 30$$

Therefore it is 30 years since B was half as old as A.

14. A father is 50 years old, and his son is half that age. How many years ago was the father two and one-fourth times as old as his son?

Let $x =$ the number of years since the father's age was $2\frac{1}{4}$ times that of the son.

Then, $50 - x =$ the father's age in years at that time.

$25 - x =$ the son's age in years at that time.

$$\therefore 50 - x = 2\tfrac{1}{4}(25 - x)$$

$$4(50 - x) = 9(25 - x)$$

$$200 - 4x = 225 - 9x$$

$$\therefore x = 5$$

Therefore it is 5 years since the father was two and one-fourth times as old as the son.

15. A father is 40 years old, and his son is one-third that age. In how many years will the son be half as old as his father?

Let $x =$ the number of years before the son is half as old as the father.

Then, $40 + x =$ the father's age in years at that time.

$13\frac{1}{3} + x =$ the son's age in years at that time.

$$\therefore\ 40 + x = 2(13\tfrac{1}{3} + x)$$

$$40 + x = \tfrac{80}{3} + 2x$$

$$\therefore\ x = \tfrac{40}{3} = 13\tfrac{1}{3}$$

Therefore the son will be half as old as the father in $13\frac{1}{3}$ years.

16. A can do a piece of work in 3 days, B in 4 days, and C in 6 days. How long will it take them to do it working together?

Let $x =$ the number of days it will take them all together.

Then, $\frac{1}{x} =$ the part they can do together in one day.

$\frac{1}{3} =$ the part A can do in one day.

$\frac{1}{4} =$ the part B can do in one day.

$\frac{1}{6} =$ the part C can do in one day.

$\frac{1}{3} + \frac{1}{4} + \frac{1}{6} =$ the part they all can do together in one day.

$$\therefore\ \frac{1}{3} + \frac{1}{4} + \frac{1}{6} = \frac{1}{x}$$

$$4x + 3x + 2x = 12$$

$$9x = 12$$

$$\therefore\ x = 1\tfrac{1}{3}$$

They can do the work together in $1\frac{1}{3}$ days.

17. A can do a piece of work in $2\frac{1}{3}$ days, B in $3\frac{1}{2}$ days, and C in $4\frac{2}{3}$ days. How long will it take them to do it working together?

Let $x =$ the number of days it will take them all together.

Then, $\frac{1}{x} =$ the part they can do together in one day.

$\frac{1}{2\frac{1}{3}} = \frac{3}{7} =$ the part A can do in one day.

$\frac{1}{3\frac{1}{2}} = \frac{2}{7} =$ the part B can do in one day.

$\frac{1}{4\frac{2}{3}} = \frac{3}{14} =$ the part C can do in one day.

$\therefore\ \frac{3}{7} + \frac{2}{7} + \frac{3}{14} =$ the part they can all do together in one day.

$$\therefore \frac{3}{7}+\frac{2}{7}+\frac{3}{14}=\frac{1}{x}$$

$$6x+4x+3x=14$$

$$13x=14$$

$$\therefore x=1\tfrac{1}{13}$$

They can do the work together in $1\frac{1}{13}$ days.

18. A and B can separately do a piece of work in 12 days and 20 days, and with the help of C they can do it in 6 days. How long would it take C to do the work?

Let x = the number of days it would take C to do the work.

Then, $\frac{1}{x}$ = the part C can do in one day.

$\frac{1}{12}$ = the part A can do in one day.

$\frac{1}{20}$ = the part B can do in one day.

$\therefore \frac{1}{x}+\frac{1}{12}+\frac{1}{20}$ = the part they can all do together in one day.

But $\frac{1}{6}$ = this part.

$$\therefore \frac{1}{x}+\frac{1}{12}+\frac{1}{20}=\frac{1}{6}$$

$$60+5x+3x=10x$$

$$2x=60$$

$$\therefore x=30$$

Therefore C can do the work in 30 days.

19. A and B together can do a piece of work in 10 days, A and C in 12 days, and A by himself in 18 days. How many days will it take B and C together to do the work? How many days will it take A, B, and C together?

Let x = the number of days it will take B and C together to do the work.

Then, $\frac{1}{x}$ = the part B and C can do together in one day.

But $\frac{1}{18}$ = the part A can do in one day,

and $\frac{1}{10}$ = the part A and B can do together in one day.

Hence $\frac{1}{10}-\frac{1}{18}$ = the part B can do in one day.

Also $\frac{1}{12}$ = the part A and C can do in one day.

Hence $\frac{1}{12} - \frac{1}{18} =$ the part C can do alone in one day.

$\frac{1}{10} - \frac{1}{18} + \frac{1}{12} - \frac{1}{18} =$ the part B and C can do together in one day.

$$\therefore \frac{1}{10} - \frac{1}{18} + \frac{1}{12} - \frac{1}{18} = \frac{1}{x}$$

$$\frac{13}{180} = \frac{1}{x}$$

$$\therefore x = \tfrac{180}{13} = 13\tfrac{11}{13}$$

Therefore B and C can do the work in $13\frac{11}{13}$ days.

Again, since B and C do $\frac{13}{180}$, and A $\frac{1}{18}$, of the work in one day, all together do $\frac{23}{180}$ of the work in one day.

Therefore they can do the work together in $\frac{180}{23}$, or $7\frac{19}{23}$ days.

20. A and B can do a piece of work in 10 days, A and C in 12 days, B and C in 15 days. How long will it take them to do the work if they all work together?

Let $x =$ the number of days it will take them all together to do the work.

Then, $\frac{1}{x} =$ the part they can do all together in one day.

But $\frac{1}{10} =$ the part A and B can do together in one day.

Hence, $\frac{1}{x} - \frac{1}{10} =$ the part C can do in one day.

Similarly, $\frac{1}{x} - \frac{1}{12} =$ the part B can do in one day.

$\frac{1}{x} - \frac{1}{15} =$ the part A can do in one day.

Hence,

$\frac{1}{x} - \frac{1}{10} + \frac{1}{x} - \frac{1}{12} + \frac{1}{x} - \frac{1}{15} =$ the part they can all do together in one day.

$$\therefore \frac{1}{x} - \frac{1}{10} + \frac{1}{x} - \frac{1}{12} + \frac{1}{x} - \frac{1}{15} = \frac{1}{x}$$

$$\frac{2}{x} = \frac{1}{10} + \frac{1}{12} + \frac{1}{15}$$

$$\frac{2}{x} = \frac{1}{4}$$

$$\therefore x = 8$$

Therefore they can do the work together in 8 days.

21. A cistern can be filled by three pipes in 15, 20, and 30 minutes respectively. In what time will it be filled if the pipes are all running together?

Let $x =$ the number of minutes it takes all the pipes to fill the cistern.

Then $\frac{1}{x} =$ the part that all the pipes can fill in one minute.

But $\frac{1}{15} =$ the part the first pipe can fill in one minute.

$\frac{1}{20} =$ the part the second pipe can fill in one minute.

$\frac{1}{30} =$ the part the third pipe can fill in one minute.

$\therefore \frac{1}{15} + \frac{1}{20} + \frac{1}{30} =$ the part all the pipes can fill in one minute.

$$\frac{1}{15}+\frac{1}{20}+\frac{1}{30}=\frac{1}{x}$$

$$\frac{9}{60}=\frac{1}{x}$$

$$\therefore x = 6\tfrac{2}{3}$$

Therefore the three pipes running together can fill the cistern in $6\frac{2}{3}$ minutes.

22. A cistern can be filled by two pipes in 25 minutes and 30 minutes, respectively, and emptied by a third in 20 minutes. In what time will it be filled if all three pipes are running together?

Let $x =$ the required time in minutes.

Then $\frac{1}{x} =$ the part the three pipes can fill in one minute.

But $\frac{1}{25} =$ the part the first pipe can fill in one minute.

$\frac{1}{30} =$ the part the second pipe can fill in one minute.

$\frac{1}{20} =$ the part the third pipe can empty in one minute.

$\therefore \frac{1}{25} + \frac{1}{30} - \frac{1}{20} =$ the part the three pipes can fill in one minute.

$$\therefore \frac{1}{25}+\frac{1}{30}-\frac{1}{20}=\frac{1}{x}$$

$$\frac{7}{300}=\frac{1}{x}$$

$$\therefore x = \tfrac{300}{7} = 42\tfrac{6}{7}$$

Therefore the cistern will be filled in $42\frac{6}{7}$ minutes.

23. A tank can be filled by three pipes in 1 hour and 20 minutes, 2 hours and 20 minutes, and 3 hours and 20 minutes, respectively. In how many minutes can it be filled by all three together?

1 hour 20 minutes $= \frac{4}{3}$ hours.
2 hours 20 minutes $= \frac{7}{3}$ hours.
3 hours 20 minutes $= \frac{10}{3}$ hours.

Let $x =$ the required time in hours.

Then $\frac{1}{x} =$ the part that the three pipes can fill in one hour.

But $\frac{3}{4} =$ the part the first pipe can fill in one hour.

$\frac{3}{7} =$ the part the second pipe can fill in one hour.

$\frac{3}{10} =$ the part the third pipe can fill in one hour.

$\therefore \frac{3}{4} + \frac{3}{7} + \frac{3}{10} =$ the part all the pipes together can fill in one hour.

$$\therefore \frac{3}{4} + \frac{3}{7} + \frac{3}{10} = \frac{1}{x}$$

$$\frac{207}{140} = \frac{1}{x}$$

$$\therefore x = \frac{140}{207}$$

Therefore the tank can be filled by the three pipes together in $\frac{140}{207}$ hours, or $40\frac{40}{69}$ minutes.

24. A sets out and travels at the rate of 9 miles in 2 hours. Seven hours afterwards B sets out from the same place and travels in the same direction at the rate of 5 miles an hour. In how many hours will B overtake A?

Let $x =$ the number of hours in which B will overtake A.

Then $x + 7 =$ the number of hours A will have travelled.

$\frac{9}{2}(x + 7) =$ the number of miles A will have travelled.

$5x =$ the number of miles B will have travelled.

These two distances are equal.

$$\therefore \frac{9}{2}(x + 7) = 5x$$

$$9x + 63 = 10x$$

$$\therefore x = 63$$

Therefore B will overtake A in 63 hours.

25. A man walks to the top of a mountain at the rate of $2\frac{1}{2}$ miles an hour, and down the same way at the rate of 4 miles an hour, and is out 5 hours. How far is it to the top of the mountain?

Let $x =$ the number of miles to the top of the mountain.

Then $\frac{x}{2\frac{1}{2}} =$ the number of hours required for the ascent.

$\frac{x}{4} =$ the number of hours required for the descent.

$\frac{x}{2\frac{1}{2}} + \frac{x}{4} =$ the number of hours required for both.

$$\therefore \frac{x}{2\frac{1}{2}} + \frac{x}{4} = 5$$

$$\frac{2x}{5} + \frac{x}{4} = 5$$

$$8x + 5x = 100$$

$$\therefore x = \tfrac{100}{13} = 7\tfrac{9}{13}$$

Therefore it is $7\frac{9}{13}$ miles to the top of the mountain.

26. In going from Boston to Portland, a passenger train, at 27 miles an hour, occupies 2 hours less time than a freight train at 18 miles an hour. Find the distance from Boston to Portland.

Let $x =$ the number of miles from Boston to Portland.

Then $\frac{x}{27} =$ the number of hours required by the passenger train for the trip.

$\frac{x}{18} =$ the number of hours required by the freight train.

$$\therefore \frac{x}{18} = \frac{x}{27} + 2$$

$$3x = 2x + 108$$

$$x = 108$$

Therefore the distance is 108 miles.

27. A person has 6 hours at his disposal. How far may he ride at 6 miles an hour so as to return in time, walking back at the rate of 3 miles an hour?

Let $x =$ the number of miles he may ride.

Then $\frac{x}{6} =$ the number of hours he may ride.

$\frac{x}{3} =$ the number of hours he must walk.

$\frac{x}{6} + \frac{x}{3} =$ the number of hours required to go and to return.

$$\therefore \frac{x}{6} + \frac{x}{3} = 6$$

$$x + 2x = 36$$

$$\therefore x = 12$$

Therefore he may ride out 12 miles.

28. A boy starts from Exeter and walks towards Andover at the rate of 3 miles an hour, and 2 hours later another boy starts from Andover and walks towards Exeter at the rate of $2\frac{1}{2}$ miles an hour. The distance from Exeter to Andover is 28 miles. How far from Exeter will they meet?

Let $x =$ the number of miles their meeting point is distant from Exeter.

Then $x =$ the number of miles the first boy walks.

$28 - x =$ the number of miles the second boy walks.

$\frac{x}{3} =$ the number of hours the first boy walks.

$\frac{28 - x}{2\frac{1}{2}} =$ the number of hours the second boy walks.

But the first boy walks 2 hours more than the second.

$$\therefore \frac{x}{3} = \frac{28 - x}{2\frac{1}{2}} + 2$$

$$\frac{x}{3} = \frac{2(28 - x)}{5} + 2$$

$$5x = 6(28 - x) + 30$$

$$5x = 168 - 6x + 30$$

$$11x = 198$$

$$\therefore x = 18$$

Therefore the boys will meet at a point 18 miles distant from Exeter.

29. A hare takes 7 leaps while a dog takes 5, and 5 of the dog's leaps are equal to 8 of the hare's. The hare has a start of 50 of her own leaps. How many leaps will the hare take before she is caught?

Let $7x =$ the number of leaps the hare will take.

Then $5x =$ the number of leaps the dog will take.

Also, let $5a =$ the number of feet in one leap of the hare.

Then $8a =$ the number of feet in one leap of the dog.

Hence, $35ax =$ the number of feet the hare will have run.

$40ax =$ the number of feet the dog will have run.

$250a =$ the number of feet start that the hare has.

$$\therefore 40ax = 35ax + 250a$$

$$5ax = 250a$$

$$\therefore x = 50$$

$$7x = 350$$

Therefore the hare will have taken 350 leaps.

30. A dog makes 4 leaps while a hare makes 5, but 3 of the dog's leaps are equal to 4 of the hare's. The hare has a start of 60 of the dog's leaps. How many leaps will each take before the hare is caught?

Let $5x =$ the number of leaps the hare will take.
Then $4x =$ the number of leaps the dog will take.
Also, let $3a =$ the number of feet in one leap of the hare.
Then, $4a =$ the number of feet in one leap of the dog.
Hence, $15ax =$ the number of feet passed over by the hare.
$16ax =$ the number of feet passed over by the dog.
$240a =$ the number of feet start that the hare has.

$$\therefore\ 16ax = 15ax + 240a$$
$$ax = 240a$$
$$\therefore\ x = 240$$
$$5x = 1200$$

Therefore the hare will take 1200 leaps; dog, 960 leaps.

31. A rectangle whose length is $2\frac{1}{2}$ times its breadth would have its area increased by 60 square feet if its length and breadth were each 5 feet more. Find its dimensions.

Let $2x =$ the breadth of the rectangle in feet.
Then $5x =$ its length in feet.
$10x^2 =$ its area in square feet.
$2x + 5 =$ the increased breadth in feet.
$5x + 5 =$ the increased length in feet.
$(2x + 5)(5x + 5) =$ the increased area in square feet.

$$\therefore\ (2x + 5)(5x + 5) = 10x^2 + 60$$
$$10x^2 + 35x + 25 = 10x^2 + 60$$
$$35x = 35$$
$$\therefore\ x = 1$$
$$2x = 2$$
$$5x = 5$$

Therefore the rectangle is 2 feet wide and 5 feet long.

32. A rectangle has its length 4 feet longer and its width 3 feet shorter than the side of the equivalent square. Find its area.

Let $x =$ the length of the equivalent square in feet.
Then $x^2 =$ the area of the square in square feet.
$x + 4 =$ the length of the rectangle in feet.

$x-3=$ the width of the rectangle in feet.

$(x-3)(x+4)=$ the area of the rectangle in square feet.

$$\therefore\ (x-3)(x+4)=x^2$$
$$x^2+x-12=x^2$$
$$\therefore\ x=12$$
$$x^2=144$$

Therefore the area of the rectangle is 144 square feet.

33. The width of a rectangle is an inch more than half its length, and if a strip 5 inches wide is taken off from the four sides, the area of the strip is 510 square inches. Find the dimensions of the rectangle.

Let $x=$ the width of the rectangle in inches.

Then, $2(x-1)=$ its length in inches.

$2x(x-1)=$ the area in square inches.

$x-10=$ the width of the diminished rectangle in inches.

$2(x-1)-10=$ the length of the diminished rectangle in inches.

$[2(x-1)-10](x-10)=$ the area of the diminished rectangle in inches.

$2x(x-1)-[2(x-1)-10](x-10)=$ the area of the strip in inches.

$$\therefore\ 2x(x-1)-[2(x-1)-10](x-10)=510$$
$$2x(x-1)-(2x-12)(x-10)=510$$
$$2x^2-2x-2x^2+32x-120=510$$
$$30x=630$$
$$\therefore\ x=21$$
$$2(x-1)=40$$

Therefore the rectangle is 21 inches wide and 40 inches long.

34. If 1 pound of tin loses $\frac{5}{37}$ of a pound, and 1 pound of lead loses $\frac{2}{23}$ of a pound, when weighed in water, how many pounds of tin and of lead in a mass of 60 pounds that loses 7 pounds when weighed in water?

Let $x=$ the number of pounds of tin in the mass.

Then, $60-x=$ the number of pounds of lead in the mass.

$\frac{5x}{37}=$ the number of pounds the tin loses.

$\frac{2(60-x)}{23}=$ the number of pounds the lead loses.

$\frac{5x}{37}+\frac{2(60-x)}{23}=$ the entire number of pounds lost.

$$\therefore \frac{5x}{37}+\frac{2(60-x)}{23}=7$$

$$115x+74(60-x)=7\times 37\times 23$$

$$115x+4440-74x=5957$$

$$41x=1517$$

$$\therefore x=37$$

$$60-x=23$$

Therefore there are 37 pounds of tin and 23 pounds of lead in the mass.

35. If 19 ounces of gold lose 1 ounce, and 10 ounces of silver lose 1 ounce, when weighed in water, how many ounces of gold and of silver in a mass of gold and silver weighing 530 ounces in air and 495 ounces in water?

Let $x =$ the number of ounces of gold.

Then, $530-x =$ the number of ounces of silver.

$\frac{x}{19} =$ the number of ounces the gold loses.

$\frac{530-x}{10} =$ the number of ounces the silver loses.

$\frac{x}{19}+\frac{530-x}{10} =$ the entire number of ounces lost.

$$\therefore \frac{x}{19}+\frac{530-x}{10}=530-495$$

$$10x+19(530-x)=190\times 35$$

$$10x+10070-19x=6650$$

$$9x=3420$$

$$\therefore x=380$$

$$530-x=150$$

Therefore there are 380 ounces of gold, and 150 ounces of silver.

36. At what time between 2 and 3 o'clock are the hands of a watch at right angles?

When the hands are at right angles to each other the minute hand will be 15 minute spaces ahead of the hour hand.

At 2 o'clock the hour hand is 10 minute spaces ahead of the minute hand.

Hence, when the hands are at right angles to each other the minute hand has gained 25 minute spaces on the hour hand.

Let $x =$ the number of spaces the minute hand moves over.

Then, $x - 25 =$ the number of spaces the hour hand moves over.

$12(x - 25) =$ the number of spaces the minute hand moves over.

$$\therefore 12(x - 25) = x$$
$$12x - 300 = x$$
$$11x = 300$$
$$\therefore x = 27\tfrac{3}{11}$$

Therefore the required time is $27\frac{3}{11}$ minutes past 2 o'clock.

37. At what time between 3 and 4 o'clock are the hands of a watch pointing in opposite directions?

When the hands point in opposite directions the minute hand is 30 minute spaces ahead of the hour hand.

At 3 o'clock the hour hand is 15 minute spaces ahead of the minute hand.

Hence at the instant when the hands point in opposite directions, the minute hand has gained 45 minute spaces on the hour hand.

Let $x =$ the number of minute spaces the minute hand has moved over.

Then $x - 45 =$ the number of minute spaces the hour hand has moved over.

$\therefore 12(x - 45) =$ the number of minute spaces the minute hand has moved over.

$$\therefore 12(x - 45) = x$$
$$12x - 540 = x$$
$$\therefore x = \tfrac{540}{11} = 49\tfrac{1}{11}$$

Therefore the hands will point in opposite directions at $49\frac{1}{11}$ minutes past 3 o'clock.

38. At what time between 7 and 8 o'clock are the hands of a watch together?

At 7 o'clock the hour hand is 35 minute spaces ahead of the minute hand.

Let $x =$ the number of minute spaces the minute hand has moved over.

Then, $x - 35 =$ the number of minute spaces the hour hand has moved over.

$\therefore 12(x - 35) =$ the number of minute spaces the minute hand has moved over.

$$\therefore\ 12(x-35)=x$$
$$12x-420=x$$
$$\therefore\ x=38\tfrac{2}{11}$$

Therefore the hands will be together at $38\frac{2}{11}$ minutes past 7 o'clock.

39. A trader adds yearly to his capital one-fourth of it, but takes from it, at the end of each year, $800 for expenses. At the end of the third year, after deducting the last $800, he has $1\frac{1}{8}$ times his original capital. How much had he at first?

Let $x=$ the number of dollars he has at first.

Then, $\frac{5x}{4}-800=$ the number of dollars he has at the end of the first year.

$\frac{5}{4}\left(\frac{5x}{4}-800\right)-800=$ the number of dollars he has at the end of the second year.

$\frac{5}{4}\left[\frac{5}{4}\left(\frac{5x}{4}-800\right)-800\right]-800=$ the number of dollars he has at the end of the third year.

But, $\frac{15x}{8}=$ the number of dollars he has at the end of the third year.

$$\therefore\ \frac{5}{4}\left[\frac{5}{4}\left(\frac{5x}{4}-800\right)-800\right]-800=\frac{15x}{8}$$
$$\frac{5}{4}\left(\frac{25x}{16}-1000-800\right)-800=\frac{15x}{8}$$
$$\frac{5}{4}\left(\frac{25x}{16}-1800\right)-800=\frac{15x}{8}$$
$$\tfrac{125}{64}x-2250-800=\frac{15x}{8}$$
$$\frac{5x}{64}=3050$$
$$\therefore\ x=39040$$

Therefore he had $39,040 at first.

40. A trader adds yearly to his capital one-fifth of it, but takes from it, at the end of each year, $2500 for expenses. At the end of the third year, after deducting the last $2500, he has $1\frac{7}{10}$ times his original capital. Find his original capital.

Let $x=$ the number of dollars he had at first.

Then, $\frac{6x}{5}-2500=$ the number of dollars he had at the end of the first year.

$\frac{6}{5}\left(\frac{6x}{5}-2500\right)-2500=$ the number of dollars he had at the end of the second year.

$\frac{6}{5}\left[\frac{6}{5}\left(\frac{6x}{5}-2500\right)-2500\right]-2500=$ the number of dollars he had at the end of the third year.

But, $\frac{17x}{10}=$ the number of dollars he had at the end of the third year.

$$\therefore\ \frac{6}{5}\left[\frac{6}{5}\left(\frac{6x}{5}-2500\right)-2500\right]-2500=\frac{17x}{10}$$

$$\frac{6}{5}\left[\frac{36x}{25}-3000-2500\right]-2500=\frac{17x}{10}$$

$$\frac{6}{5}\left(\frac{36x}{25}-5500\right)-2500=\frac{17x}{10}$$

$$\tfrac{216}{125}x-6600-2500=\frac{17x}{10}$$

$$\frac{7x}{250}=9100$$

$$\therefore\ x=325000$$

Therefore his original capital was $325,000.

41. A's age now is two-fifths of B's. Eight years ago A's age was two-ninths of B's. Find their ages.

Let $5x=$ B's age in years.

Then, $2x=$ A's age in years.

$5x-8=$ B's age in years, 8 years ago.

$2x-8=$ A's age in years, 8 years ago.

$$\therefore\ 2x-8=\tfrac{2}{9}(5x-8)$$

$$18x-72=10x-16$$

$$\therefore\ x=7$$

$$5x=35$$

$$2x=14$$

Therefore A's age is 14 years, and B's is 35 years.

42. A had five times as much money as B. He gave B 5 dollars, and then had only twice as much as B. How much had each at first?

Let $x=$ the number of dollars B had at first.

Then, $5x=$ the number of dollars A had at first.

$x+5=$ the number of dollars B had finally.

$5x-5=$ the number of dollars A had finally.

$$\therefore\ 5x - 5 = 2(x + 5)$$
$$5x - 5 = 2x + 10$$
$$\therefore\ x = 5$$
$$5x = 25$$

Therefore A had $25 and B $5 at first.

43. At what time between 12 and 1 o'clock are the hour and minute hands pointing in opposite directions?

At 12 o'clock the hands are together.

When the hands point in opposite directions the minute hand has gained 30 minute spaces on the hour hand.

Let $x =$ the number of minute spaces the minute hand has passed over.

Then, $x - 30 =$ the number of minute spaces the hour hand has passed over.

$$\therefore\ 12(x - 30) = x$$
$$11x = 360$$
$$\therefore\ x = 32\tfrac{8}{11}$$

Therefore the hands will point in opposite directions at $32\frac{8}{11}$ minutes past 12 o'clock.

44. Eleven-sixteenths of a certain principal was at interest at 5 per cent, and the balance at 4 per cent. The entire income was $1500. Find the principal.

Let $16x =$ the number of dollars in the principal.

Then, $11x =$ the number of dollars invested at 5%.

$5x =$ the number of dollars invested at 4%.

$\frac{55x}{100} =$ the number of dollars interest on the first amount.

$\frac{20x}{100} =$ the number of dollars interest on the second amount.

$\frac{55x}{100} + \frac{20x}{100} =$ the total number of dollars interest.

$$\therefore\ \frac{55x}{100} + \frac{20x}{100} = 1500$$
$$\frac{75x}{100} = 1500$$
$$\therefore\ x = 2000$$
$$16x = 32000$$

Therefore the principal was $32,000.

45. A train which travels 36 miles an hour is $\frac{3}{4}$ of an hour in advance of a second train which travels 42 miles an hour. In how long a time will the last train overtake the first?

Let $x =$ the number of hours in which the last train will overtake the first.

Then, $42x =$ the distance the second train will have passed over.

$36x =$ the distance the first train will have passed over.

But the second train was 27 miles behind the first at the start.

$$\therefore\ 42x = 36x + 27$$
$$6x = 27$$
$$x = 4\tfrac{1}{2}$$

Therefore the last train will overtake the first in $4\frac{1}{2}$ hours.

46. An express train which travels 40 miles an hour starts from a certain place 50 minutes after a freight train, and overtakes the freight train in 2 hours and 5 minutes. Find the rate per hour of the freight train.

Let $x =$ the rate of the freight train in miles per hour.

Then, $\frac{175}{60}x =$ the number of miles the freight train travels before it is overtaken.

$40 \times \frac{125}{60} =$ the number of miles the express train has travelled.

But these two distances are equal.

$$\therefore\ \tfrac{175}{60}x = 40 \times \tfrac{125}{60}$$
$$x = \tfrac{200}{7} = 28\tfrac{4}{7}$$

Therefore the rate of the freight train is $28\frac{4}{7}$ miles per hour.

47. A messenger starts to carry a despatch, and 5 hours after a second messenger sets out to overtake the first in 8 hours. In order to do this, he is obliged to travel $2\frac{1}{2}$ miles an hour more than the first. How many miles an hour does the first travel?

Let $x =$ the number of miles the first messenger travels per hour.

Then, $x + 2\frac{1}{2} =$ the number of miles the second messenger travels per hour.

$13x =$ the number of miles the first messenger travels before he is overtaken.

$8(x+2\frac{1}{2})=$ the number of miles the second messenger travels before he overtakes the first.

But these two distances are equal.

$$\therefore\ 13x=8(x+2\tfrac{1}{2})$$
$$x=4$$

Therefore the first messenger travels 4 miles per hour.

48. The fore and hind wheels of a carriage are respectively $9\frac{1}{2}$ feet and $11\frac{2}{5}$ feet in circumference. What distance will the carriage have made when one of the fore wheels has made 160 revolutions more than one of the hind wheels?

Let $x=$ the number of feet the carriage will have travelled.

Then, $\frac{x}{9\frac{1}{2}}=$ the number of times the fore wheel will have revolved.

$\frac{x}{11\frac{2}{5}}=$ the number of times the hind wheel will have revolved.

$$\therefore\ \frac{x}{9\frac{1}{2}}-\frac{x}{11\frac{2}{5}}=160$$
$$\frac{2x}{19}-\frac{5x}{57}=160$$
$$\frac{x}{57}=160$$
$$x=9120$$

Therefore the carriage will have travelled 9120 feet, or $1\frac{8}{11}$ miles.

49. When a certain brigade of troops is formed in a solid square there is found to be 100 men over; but when formed in column with 5 men more in front and 3 men less in depth than before, the column needs 5 men to complete it. Find the number of troops.

Let $x=$ the number of men in a side of the solid square.

Then, $x^2+100=$ the entire number of troops.

$(x+5)(x-3)-5=$ the entire number of troops.

$$\therefore\ (x+5)(x-3)-5=x^2+100$$
$$x^2+2x-15-5=x^2+100$$
$$2x=120$$
$$x=60$$
$$x^2+100=3700$$

Therefore there are 3700 troops.

50. An officer can form his men in a hollow square 14 deep. The whole number of men is 3136. Find the number of men in the front of the hollow square.

Let $x =$ the number of men in the front of the hollow square.
Then, $x - 28 =$ the number of men in an inside face of the hollow square.
$x^2 =$ the number of men required to form a solid square of the same size as the hollow square.
$(x - 28)^2 =$ the number of additional men required to fill out the hollow square.
$\therefore x^2 - (x - 28)^2 =$ the entire number of men.

$$\therefore x^2 - (x - 28)^2 = 3136$$
$$x^2 - x^2 + 56x - 784 = 3136$$
$$56x = 3920$$
$$x = 70$$

Therefore there are 70 men in the front of the hollow square.

51. A trader increases his capital each year by one-fourth of it, and at the end of each year takes out \$2400 for expenses. At the end of 3 years, after deducting the last \$2400, he finds his capital to be \$10,000. Find his original capital.

Let $x =$ the number of dollars he had at first.
Then, $\frac{5x}{4} - 2400 =$ the number of dollars he has at the end of the first year.
$\frac{5}{4}\left(\frac{5x}{4} - 2400\right) - 2400 =$ the number of dollars he has at the end of two years.
$\frac{5}{4}\left[\frac{5}{4}\left(\frac{5x}{4} - 2400\right) - 2400\right] - 2400 =$ the number of dollars he has at the end of three years.

$$\therefore \frac{5}{4}\left[\frac{5}{4}\left(\frac{5x}{4} - 2400\right) - 2400\right] - 2400 = 10000$$
$$\frac{5}{4}\left(\frac{25x}{16} - 3000 - 2400\right) - 2400 = 10000$$
$$\frac{5}{4}\left(\frac{25x}{16} - 5400\right) - 2400 = 10000$$
$$\frac{125x}{64} - 6750 - 2400 = 10000$$
$$\frac{125x}{64} = 19150$$
$$\therefore x = 9804\tfrac{4}{5}$$

Therefore his original capital was \$9804.80.

52. A and B together can do a piece of work in $1\frac{1}{2}$ days, A and C together in $1\frac{3}{4}$ days, and B and C together in $1\frac{7}{8}$ days. How many days will it take each alone to do the work?

Let $x =$ the number of days in which A alone can do the work.

Then, $\frac{1}{x} =$ the part A can do in one day.

But, $\frac{2}{3} =$ the part A and B can do together in one day.

Hence, $\frac{2}{3} - \frac{1}{x} =$ the part B can do alone in one day.

Also, $\frac{4}{7} =$ the part A and C can do together in one day.

Hence, $\frac{4}{7} - \frac{1}{x} =$ the part C can do alone in one day.

$\frac{2}{3} - \frac{1}{x} + \frac{4}{7} - \frac{1}{x} =$ the part B and C can do together in one day.

But, $\frac{8}{15} =$ this part.

$$\therefore \frac{2}{3} - \frac{1}{x} + \frac{4}{7} - \frac{1}{x} = \frac{8}{15}$$

$$-\frac{2}{x} = -\frac{74}{105}$$

$$\therefore x = \tfrac{105}{37}$$

$$\frac{2}{3} - \frac{1}{x} = \frac{33}{105} = \frac{11}{35}$$

$$\frac{4}{7} - \frac{1}{x} = \frac{23}{105}$$

Therefore A can do the work alone in $\frac{105}{37}$ or $2\frac{31}{37}$ days; B, in $\frac{35}{11}$ or $3\frac{2}{11}$ days; and C, in $\frac{105}{23}$ or $4\frac{13}{23}$ days.

53. A fox pursued by a hound has a start of 100 of her leaps. The fox makes 3 leaps while the hound makes 2; but 3 leaps of the hound are equivalent to 5 of the fox. How many leaps will each take before the hound catches the fox?

Let $2x =$ the number of leaps the hound makes.

Then, $3x =$ the number of leaps the fox makes.

Also let $5a =$ the number of feet in a leap of the hound.

Then, $3a =$ the number of feet in a leap of the fox.

$10ax =$ the number of feet passed over by the hound.

$9ax =$ the number of feet passed over by the fox.

$300a =$ the number of feet start the fox has.

$$\begin{aligned} \therefore\ 10\,ax &= 9\,ax + 300\,a \\ ax &= 300\,a \\ \therefore\ x &= 300 \\ 2\,x &= 600 \\ 3\,x &= 900 \end{aligned}$$

Therefore the hound will take 600 leaps and the fox 900 before the fox is overtaken.

Exercise 57. Page 161.

1. The sum of two numbers is 40, and their difference is 10. Find the numbers.

$$\frac{s+d}{2} = \frac{40+10}{2} = 25, \quad \frac{s-d}{2} = \frac{40-10}{2} = 15.$$

Hence the numbers are 25 and 15.

2. The sum of two angles is 100^{C}, and their difference is 21° 30′. Find the angles.

$$\frac{s+d}{2} = \frac{100+21\frac{1}{2}}{2} = 60\tfrac{3}{4}, \quad \frac{s-d}{2} = \frac{100-21\frac{1}{2}}{2} = 39\tfrac{1}{4}.$$

Hence the angles are 60° 45′ and 39° 15′.

3. The sum of two angles is 116° 24′ 30″, and their difference is 56° 21′ 44″. Find the angles.

$$\frac{s+d}{2} = \frac{116^\circ\,24'\,30'' + 56^\circ\,21'\,44''}{2} = 86^\circ\,23'\,7''.$$

$$\frac{s-d}{2} = \frac{116^\circ\,24'\,30'' - 56^\circ\,21'\,44''}{2} = 30^\circ\,1'\,23''.$$

Hence the angles are 86° 23′ 7″ and 30° 1′ 23″.

4. A can do a piece of work in 6 days, and B in 5 days. How long will it take both together to do it?

$$a = 6, \quad b = 5. \qquad \therefore\ \frac{ab}{a+b} = \frac{30}{11}.$$

Hence they can both do it together in $2\frac{8}{11}$ days.

5. Find the interest of $2750 for 3 years at $4\frac{1}{2}$ per cent.

$$i = prt$$

$$p = 2750,\ r = \tfrac{9}{200},\ t = 3$$

$$\therefore\ i = 2750 \times 3 \times \tfrac{9}{200} = 371\tfrac{1}{4}$$

Hence the interest is $371.25.

6. Find the interest of $950 for 2 years 6 months at 5 per cent.

$$i = prt$$
$$p = 950,\ r = \tfrac{5}{100},\ t = 2\tfrac{1}{2}$$
$$\therefore\ i = 950 \times \tfrac{5}{100} \times \tfrac{5}{2}$$
$$= 118\tfrac{3}{4}$$

Hence the interest is $118.75.

7. Find the amount of $2000 for 7 years 4 months at 6 per cent.

$$a = p + prt$$
$$p = 2000,\ r = \tfrac{6}{100},\ t = 7\tfrac{1}{3}$$
$$\therefore\ a = 2000 + 2000 \times \tfrac{6}{100} \times \tfrac{22}{3}$$
$$= 2880$$

Hence the amount is $2880.

8. Find the rate if the interest on $680 for 7 months is $35.70.

$$r = \frac{a-p}{pt}$$
$$a - p = 35\tfrac{70}{100},\ p = 680,\ t = \tfrac{7}{12}$$
$$\therefore\ r = \frac{35.70}{680 \times \tfrac{7}{12}}$$
$$= \tfrac{9}{100}$$

Hence the rate is 9%.

9. Find the rate if the amount of $750 for 4 years is $900.

$$r = \frac{a-p}{pt}$$
$$a = 900,\ p = 750,\ t = 4$$
$$\therefore\ r = \frac{900 - 750}{750 \times 4}$$
$$= \tfrac{5}{100}$$

Hence the rate is 5%.

10. Find the rate if a sum of money doubles in 16 years and 8 months.

$$r = \frac{a-p}{pt}$$
$$a = 2p,\ t = 16\tfrac{2}{3}$$
$$\therefore\ r = \frac{2p - p}{16\tfrac{2}{3}p}$$
$$= \frac{1}{16\tfrac{2}{3}}$$
$$= \tfrac{6}{100}$$

Hence the rate is 6%.

11. Find the time required for the interest on $2130 to be $436.65 at 6 per cent.

$$i = prt$$
$$p = 2130,\ r = \tfrac{6}{100},\ i = 436.65$$
$$\therefore\ 436.65 = 2130 \times \tfrac{6}{100}t$$
$$\therefore\ t = \frac{100 \times 436.65}{6 \times 2130}$$
$$= \tfrac{41}{12} = 3\tfrac{5}{12}$$

Hence the time is 3 years and 5 months.

12. Find the time required for the interest on a sum of money to be equal to the principal at 5 per cent.

$$i = prt$$
$$i = p,\ r = \tfrac{5}{100}$$
$$\therefore\ p = p\tfrac{5}{100}t$$
$$t = 20$$

Hence the time is 20 years.

13. Find the principal that will produce $161.25 interest in 3 years 9 months at 8 per cent.

$$i = prt$$
$$i = 161.25,\ r = \tfrac{8}{100},\ t = 3\tfrac{3}{4}$$
$$\therefore\ 161.25 = p\tfrac{8}{100} \times \tfrac{15}{4}$$
$$p = \frac{4 \times 16125}{8 \times 15}$$
$$= 537.50$$

Hence the principal is $537.50.

14. Find the principal that will amount to \$1500 in 3 years 4 months at 6 per cent.

$$p=\frac{a}{1+rt}$$

$$a=1500,\ r=\tfrac{6}{100},\ t=3\tfrac{1}{3}$$

$$\therefore\ p=\frac{1500}{1+\frac{6}{100}\times\frac{10}{3}}$$

$$=1250$$

Hence the principal is \$1250.

15. How much money is required to yield \$2000 interest annually if the money is invested at 5 per cent?

$$i=prt$$

$$i=2000,\ r=\tfrac{5}{100},\ t=1$$

$$\therefore\ 2000=\tfrac{5}{100}p$$

$$p=40000$$

Hence the principal is \$40,000.

16. Find the time in which \$640 will amount to \$1000 at 6 per cent.

$$t=\frac{a-p}{pr}$$

$$a=1000,\ p=640,\ r=\tfrac{6}{100}$$

$$\therefore\ t=\frac{1000-640}{640\times\frac{6}{100}}$$

$$=\tfrac{75}{8}=9\tfrac{3}{8}$$

Hence the time is 9 years $4\frac{1}{2}$ months.

17. Find the principal that will produce \$100 per month, at 6 per cent.

$$i=prt$$

$$i=100,\ r=\tfrac{6}{100},\ t=\tfrac{1}{12}$$

$$\therefore\ 100=\tfrac{6}{100}\times\tfrac{1}{12}p$$

$$p=20000$$

Hence the principal is \$20,000.

18. Find the rate if the interest on \$700 for 10 months is \$25.

$$i=prt$$

$$i=25,\ p=700,\ t=\tfrac{5}{6}$$

$$\therefore\ 25=700\times\tfrac{5}{6}r$$

$$r=\tfrac{3}{70}=\frac{\frac{30}{7}}{100}$$

Hence the rate is $4\frac{2}{7}\%$.

Exercise 58. Page 165.

Solve by addition or subtraction:

1.
$$\begin{cases}5x+2y=39 & (1)\\ 2x-\ \ y=\ 3 & (2)\end{cases}$$

Multiply (2) by 2, and add to (1).

$$\begin{aligned}5x+2y&=39 \qquad (1)\\ 4x-2y&=\ 6 \qquad (3)\\ \hline 9x\qquad&=45\end{aligned}$$

$$\therefore\ x=5$$

Substitute the value of x in (2).

$$10-y=3$$

$$\therefore\ y=7$$

2.
$$\begin{cases}x+3y=22 & (1)\\ 2x-4y=\ 4 & (2)\end{cases}$$

Multiply (1) by 2, and subtract (2) from the result.

$$\begin{aligned}2x+6y&=44 \qquad (3)\\ 2x-4y&=\ 4 \qquad (2)\\ \hline 10y&=40\end{aligned}$$

$$\therefore\ y=\ 4$$

Substitute the value of y in (1).

$$x+12=22$$

$$\therefore\ x=10$$

3.
$$\left\{\begin{array}{l} 7x - 2y = 11 \\ x + 5y = 28 \end{array}\right. \qquad \begin{array}{l}(1)\\(2)\end{array}$$

Multiply (2) by 7, and subtract from (1).

$$\begin{array}{rl} 7x - 2y = 11 & (1) \\ \underline{7x + 35y = 196} & (3) \\ -37y = -185 & \end{array}$$

$$\therefore y = 5$$

Substitute the value of y in (2).

$$x + 25 = 28$$
$$\therefore x = 3$$

4.
$$\left\{\begin{array}{l} 4x - 5y = 26 \\ 3x - 6y = 15 \end{array}\right. \qquad \begin{array}{l}(1)\\(2)\end{array}$$

Divide (2) by 3, multiply by 4, and subtract from (1).

$$\begin{array}{rl} 4x - 5y = 26 & (1) \\ \underline{4x - 8y = 20} & (3) \\ 3y = 6 & \end{array}$$

$$\therefore y = 2$$

Substitute the value of y in (1).

$$4x - 10 = 26$$
$$\therefore x = 9$$

5.
$$\left\{\begin{array}{l} x + 2y = 35 \\ 3x - 2y = 17 \end{array}\right. \qquad \begin{array}{l}(1)\\(2)\end{array}$$

Add (1) and (2).

$$4x = 52$$
$$\therefore x = 13$$

Substitute the value of x in (1).

$$13 + 2y = 35$$
$$\therefore y = 11$$

6.
$$\left\{\begin{array}{l} x + 4y = 35 \\ 2x - 3y = 26 \end{array}\right. \qquad \begin{array}{l}(1)\\(2)\end{array}$$

Multiply (1) by 2, and subtract (2) from the result.

$$\begin{array}{rl} 2x + 8y = 70 & (3) \\ \underline{2x - 3y = 26} & (2) \\ 11y = 44 & \end{array}$$

$$\therefore y = 4$$

Substitute the value of y in (1).

$$x + 16 = 35$$
$$\therefore x = 19$$

7.
$$\left\{\begin{array}{l} 3x + 5y = 50 \\ x - 7y = 8 \end{array}\right. \qquad \begin{array}{l}(1)\\(2)\end{array}$$

Multiply (2) by 3, and subtract from (1).

$$\begin{array}{rl} 3x + 5y = 50 & (1) \\ \underline{3x - 21y = 24} & (3) \\ 26y = 26 & \end{array}$$

$$\therefore y = 1$$

Substitute the value of y in (2).

$$x - 7 = 8$$
$$\therefore x = 15$$

8.
$$\left\{\begin{array}{l} 5x + 2y = 36 \\ 2x + 3y = 43 \end{array}\right. \qquad \begin{array}{l}(1)\\(2)\end{array}$$

Multiply (1) by 3, and (2) by 2, and subtract.

$$\begin{array}{rl} 15x + 6y = 108 & (3) \\ \underline{4x + 6y = 86} & (4) \\ 11x \qquad = 22 & \end{array}$$

$$\therefore x = 2$$

Substitute the value of x in (1).

$$10 + 2y = 36$$
$$\therefore y = 13$$

9.
$$\left\{\begin{array}{l} 3x + 7y = 50 \\ 5x - 2y = 15 \end{array}\right. \qquad \begin{array}{l}(1)\\(2)\end{array}$$

Multiply (1) by 5, and (2) by 3, and subtract.

$$\begin{array}{rl} 15x + 35y = 250 & (3) \\ \underline{15x - 6y = 45} & (4) \\ 41y = 205 & \end{array}$$

$$\therefore y = 5$$

Substitute the value of y in (1).

$$3x + 35 = 50$$
$$\therefore x = 5$$

10. $$\left\{\begin{array}{l} 2x+y=3 \\ 7x+5y=21 \end{array}\right. \qquad \begin{array}{c} (1) \\ (2) \end{array}$$

Multiply (1) by 5, and subtract (2) from the result.

$$\begin{array}{rl} 10x+5y= & 15 \qquad (3) \\ 7x+5y= & 21 \qquad (2) \\ \hline 3x \qquad = & -6 \end{array}$$

$$\therefore x=-2$$

Substitute the value of x in (1).

$$-4+y=3$$
$$\therefore y=7$$

11. $$\left\{\begin{array}{l} x+2y=9 \\ 3x-3y=90 \end{array}\right. \qquad \begin{array}{c} (1) \\ (2) \end{array}$$

Divide (2) by 3, and subtract from (1).

$$\begin{array}{rl} x+2y= & 9 \qquad (3) \\ x-y= & 30 \qquad (2) \\ \hline 3y= & -21 \end{array}$$

$$\therefore y=-7$$

Substitute the value of y in (1).

$$x-14=9$$
$$\therefore x=23$$

12. $$\left\{\begin{array}{l} 4x-3y=39 \\ 3x-4y=17 \end{array}\right. \qquad \begin{array}{c} (1) \\ (2) \end{array}$$

Add, $\quad 7x-7y=56$

$$x-y=8 \qquad (3)$$

Multiply (3) by 4, and subtract from (1).

$$\begin{array}{rl} 4x-3y= & 39 \qquad (1) \\ 4x-4y= & 32 \qquad (4) \\ \hline y= & 7 \end{array}$$

Substitute the value of y in (2).

$$3x-28=17$$
$$\therefore x=15$$

13. $$\left\{\begin{array}{l} 7x-2y=69 \\ x-10y=39 \end{array}\right. \qquad \begin{array}{c} (1) \\ (2) \end{array}$$

Multiply (1) by 5, and subtract (2) from the result.

$$\begin{array}{rl} 35x-10y= & 345 \\ x-10y= & 39 \\ \hline 34x \qquad = & 306 \end{array}$$

$$\therefore x=9$$

Substitute the value of x in (2).

$$9-10y=39$$
$$\therefore y=-3$$

14. $$\left\{\begin{array}{l} 3x+7y=16 \\ 2x+5y=13 \end{array}\right. \qquad \begin{array}{c} (1) \\ (2) \end{array}$$

Multiply (1) by 2, and (2) by 3, and subtract.

$$\begin{array}{rl} 6x+14y= & 32 \\ 6x+15y= & 39 \\ \hline -\qquad y= & -7 \end{array}$$

$$\therefore y=7$$

Substitute the value of y in (2).

$$2x+35=13$$
$$\therefore x=-11$$

Exercise 59. Page 167.

Solve by substitution:

1. $$\left\{\begin{array}{l} 2x-7y=0 \\ 3x-5y=11 \end{array}\right. \qquad \begin{array}{c} (1) \\ (2) \end{array}$$

Transpose $-7y$ in (1).

$$2x=7y$$
$$x=\tfrac{7}{2}y \qquad (3)$$

Substitute the value of x in (2).

$$\tfrac{21}{2}y-5y=11$$
$$21y-10y=22$$
$$\therefore y=2$$

Substitute the value of y in (3).

$$\therefore x=7$$

2. $$\begin{cases} 4x - 5y = 4 & (1) \\ 3x - 2y = 10 & (2) \end{cases}$$

Transpose $-5y$ in (1).

$$4x = 4 + 5y$$

$$x = \frac{4 + 5y}{4} \quad (3)$$

Substitute the value of x in (2).

$$3\left(\frac{4 + 5y}{4}\right) - 2y = 10$$

$$12 + 15y - 8y = 40$$

$$\therefore\ y = 4$$

Substitute the value of y in (3).

$$\therefore\ x = 6$$

3. $$\begin{cases} 2x - 3y = 1 & (1) \\ 3x - 2y = 29 & (2) \end{cases}$$

Transpose $-3y$ in (1).

$$2x = 1 + 3y$$

$$x = \frac{1 + 3y}{2} \quad (3)$$

Substitute the value of x in (2).

$$3\left(\frac{1 + 3y}{2}\right) - 2y = 29$$

$$3 + 9y - 4y = 58$$

$$\therefore\ y = 11$$

Substitute the value of y in (3).

$$\therefore\ x = 17$$

4. $$\begin{cases} x + y = 19 & (1) \\ 2x + 7y = 88 & (2) \end{cases}$$

Transpose y in (1).

$$x = 19 - y \quad (3)$$

Substitute the value of x in (2).

$$2(19 - y) + 7y = 88$$

$$38 - 2y + 7y = 88$$

$$\therefore\ y = 10$$

Substitute the value of y in (3).

$$\therefore\ x = 9$$

5. $$\begin{cases} 2x - y = 5 & (1) \\ x + 2y = 25 & (2) \end{cases}$$

Transpose $2y$ in (2).

$$x = 25 - 2y \quad (3)$$

Substitute the value of x in (1).

$$2(25 - 2y) - y = 5$$

$$50 - 4y - y = 5$$

$$\therefore\ y = 9$$

Substitute the value of y in (3).

$$\therefore\ x = 7$$

6. $$\begin{cases} 19x - 15y = 23 & (1) \\ 13x - 5y = 21 & (2) \end{cases}$$

Transpose $13x$ in (2)

$$-5y = 21 - 13x$$

$$y = \frac{13x - 21}{5} \quad (3)$$

Substitute the value of y in (1).

$$19x - 15\left(\frac{13x - 21}{5}\right) = 23$$

$$19x - 39x + 63 = 23$$

$$\therefore\ x = 2$$

Substitute the value of x in (3).

$$\therefore\ y = 1$$

7. $$\begin{cases} x + 10y = 73 & (1) \\ 7x - 2y = 7 & (2) \end{cases}$$

Transpose $10y$ in (1).

$$x = 73 - 10y \quad (3)$$

Substitute the value of x in (2).

$$7(73 - 10y) - 2y = 7$$

$$511 - 70y - 2y = 7$$

$$\therefore\ y = 7$$

Substitute the value of y in (3).

$$\therefore\ x = 3$$

8. $$\begin{cases} 3x - 2y = 28 & (1) \\ 2x + 5y = 63 & (2) \end{cases}$$

Transpose $-2y$ in (1).

$$3x = 28 + 2y$$

$$x = \frac{28 + 2y}{3} \quad (3)$$

Substitute the value of x in (2).

$$2\left(\frac{28+2y}{3}\right)+5y=63$$

$$56+4y+15y=189$$

$$\therefore\ y=7$$

Substitute the value of y in (3)

$$\therefore\ x=14$$

9. $$\begin{cases}2x-3y=23 & (1)\\ 5x+2y=29 & (2)\end{cases}$$

Transpose $-3y$ in (1)

$$2x=23+3y$$

$$x=\frac{23+3y}{2}\quad(3)$$

Substitute the value of x in (2).

$$5\left(\frac{23+3y}{2}\right)+2y=29$$

$$115+15y+4y=58$$

$$\therefore\ y=-3$$

Substitute the value of y in (3).

$$\therefore\ x=7$$

10. $$\begin{cases}6x-7y=11 & (1)\\ 5x-6y=\ 8 & (2)\end{cases}$$

Transpose $-7y$ in (1).

$$6x=11+7y$$

$$x=\frac{11+7y}{6}\quad(3)$$

Substitute the value of y in (2).

$$5\left(\frac{11+7y}{6}\right)-6y=8$$

$$55+35y-36y=48$$

$$\therefore\ y=7$$

Substitute the value of y in (3).

$$\therefore\ x=10$$

11. $$\begin{cases}7x+6y=20 & (1)\\ 2x+5y=32 & (2)\end{cases}$$

Transpose $5y$ in (2).

$$2x=32-5y$$

$$x=\frac{32-5y}{2}\quad(3)$$

Substitute the value of x in (1).

$$7\left(\frac{32-5y}{2}\right)+6y=20$$

$$224-35y+12y=40$$

$$\therefore\ y=8$$

Substitute the value of y in (3).

$$\therefore\ x=-4$$

12. $$\begin{cases}x+5y=37 & (1)\\ 3x+2y=46 & (2)\end{cases}$$

Transpose $5y$ in (1),

$$x=37-5y\quad(3)$$

Substitute the value of x in (2).

$$3(37-5y)+2y=46$$

$$111-15y+2y=46$$

$$\therefore\ y=5$$

Substitute the value of y in (3).

$$\therefore\ x=12$$

13. $$\begin{cases}3x-7y=40 & (1)\\ 4x-3y=\ 9 & (2)\end{cases}$$

Transpose $-7y$ in (1).

$$3x=40+7y$$

$$x=\frac{40+7y}{3}\quad(3)$$

Substitute the value of x in (2).

$$4\left(\frac{40+7y}{3}\right)-3y=9$$

$$160+28y-9y=27$$

$$\therefore\ y=-7$$

Substitute the value of y in (3).

$$\therefore\ x=-3$$

14. $$\begin{cases} 5x + 9y = -17 & (1) \\ 3x + 11y = 1 & (2) \end{cases}$$

Transpose $9y$ in (1).

$$5x = -17 - 9y$$

$$x = -\frac{17 + 9y}{5} \qquad (3)$$

Substitute the value of x in (2).

$$-3\left(\frac{17 + 9y}{5}\right) + 11y = 1$$

$$-51 - 27y + 55y = 5$$

$$\therefore y = 2$$

Substitute the value of y in (3).

$$\therefore x = -7$$

Exercise 60. Page 168.

Solve by comparison:

1. $$\begin{cases} x + y = 30 & (1) \\ 3x - 2y = 25 & (2) \end{cases}$$

Transpose y in (1), and $-2y$ in (2).

$$x = 30 - y \qquad (3)$$

$$3x = 25 + 2y \qquad (4)$$

Divide (4) by 3.

$$x = 30 - y \qquad (3)$$

$$x = \frac{25 + 2y}{3} \qquad (5)$$

Equate the values of x.

$$30 - y = \frac{25 + 2y}{3} \qquad (6)$$

$$90 - 3y = 25 + 2y$$

$$\therefore y = 13$$

Substitute the value of y in (3).

$$\therefore x = 17$$

2. $$\begin{cases} 7x + 3y = 70 & (1) \\ 5x - 4y = 7 & (2) \end{cases}$$

Transpose $3y$ in (1), and $-4y$ in (2).

$$7x = 70 - 3y \qquad (3)$$

$$5x = 7 + 4y \qquad (4)$$

Divide (3) by 7, and (4) by 5.

$$x = \frac{70 - 3y}{7} \qquad (5)$$

$$x = \frac{7 + 4y}{5} \qquad (6)$$

Equate the values of x.

$$\frac{70 - 3y}{7} = \frac{7 + 4y}{5}$$

$$350 - 15y = 49 + 28y$$

$$\therefore y = 7$$

Substitute the value of y in (5).

$$\therefore x = 7$$

3. $$\begin{cases} 9x + 4y = 54 & (1) \\ 4x + 9y = 89 & (2) \end{cases}$$

Transpose $4y$ in (1), and $9y$ in (2).

$$9x = 54 - 4y \qquad (3)$$

$$4x = 89 - 9y \qquad (4)$$

Divide (3) by 9, and (4) by 4.

$$x = \frac{54 - 4y}{9} \qquad (5)$$

$$x = \frac{89 - 9y}{4} \qquad (6)$$

Equate the values of x.

$$\frac{54 - 4y}{9} = \frac{89 - 9y}{4}$$

$$216 - 16y = 801 - 81y$$

$$\therefore y = 9$$

Substitute the value of y in (5).

$$\therefore x = 2$$

4. $$\begin{cases} 7x + 2y = 63 & (1) \\ 8x - y = 3 & (2) \end{cases}$$

Transpose $2y$ in (1), and $-y$ in (2).

$$7x = 63 - 2y \qquad (3)$$
$$8x = 3 + y \qquad (4)$$

Divide (3) by 7, and (4) by 8.

$$x = \frac{63 - 2y}{7} \qquad (5)$$
$$x = \frac{3 + y}{8} \qquad (6)$$

Equate the values of x.

$$\frac{63 - 2y}{7} = \frac{3 + y}{8}$$
$$504 - 16y = 21 + 7y$$
$$\therefore\ y = 21$$

Substitute the value of y in (5).

$$\therefore\ x = 3$$

5. $$\begin{cases} 2x - 33y = 29 & (1) \\ 3x - 47y = 46 & (2) \end{cases}$$

Transpose $-33y$ in (1) and $-47y$ in (2).

$$2x = 29 + 33y \qquad (3)$$
$$3x = 46 + 47y \qquad (4)$$

Divide (3) by 2, and (4) by 3.

$$x = \frac{29 + 33y}{2} \qquad (5)$$
$$x = \frac{46 + 47y}{3} \qquad (6)$$

Equate the values of x.

$$\frac{29 + 33y}{2} = \frac{46 + 47y}{3}$$
$$87 + 99y = 92 + 94y$$
$$\therefore\ y = 1$$

Substitute the value of y in (5).

$$\therefore\ x = 31$$

6. $$\begin{cases} 2x - y = 9 & (1) \\ 5x - 3y = 14 & (2) \end{cases}$$

Transpose $-y$ in (1) and $-3y$ in (2).

$$2x = 9 + y \qquad (3)$$
$$5x = 14 + 3y \qquad (4)$$

Divide (3) by 2, and (4) by 5.

$$x = \frac{9 + y}{2} \qquad (5)$$
$$x = \frac{14 + 3y}{5} \qquad (6)$$

Equate the values of x.

$$\frac{9 + y}{2} = \frac{14 + 3y}{5}$$
$$45 + 5y = 28 + 6y$$
$$\therefore\ y = 17$$

Substitute the value of y in (5).

$$\therefore\ x = 13$$

7. $$\begin{cases} 11x - 7y = 6 & (1) \\ 9x - 5y = 10 & (2) \end{cases}$$

Transpose $-7y$ in (1) and $-5y$ in (2).

$$11x = 6 + 7y \qquad (3)$$
$$9x = 10 + 5y \qquad (4)$$

Divide (3) by 11, and (4) by 9.

$$x = \frac{6 + 7y}{11} \qquad (5)$$
$$x = \frac{10 + 5y}{9} \qquad (6)$$

Equate the values of x.

$$\frac{6 + 7y}{11} = \frac{10 + 5y}{9}$$
$$54 + 63y = 110 + 55y$$
$$\therefore\ y = 7$$

Substitute the value of y in (5).

$$\therefore\ x = 5$$

8. $$\begin{cases} 5x+9y=188 & (1)\\ 13x-2y=\ \ 57 & (2)\end{cases}$$

Transpose $9y$ in (1), and $-2y$ in (2).

$$5x=188-9y \qquad (3)$$
$$13x=57+2y \qquad (4)$$

Divide (3) by 5, and (4) by 13.

$$x=\frac{188-9y}{5} \qquad (5)$$
$$x=\frac{57+2y}{13} \qquad (6)$$

Equate the values of x.

$$\frac{188-9y}{5}=\frac{57+2y}{13}$$
$$2444-117y=285+10y$$
$$\therefore\ y=17$$

Substitute the value of y in (5).

$$\therefore\ x=7$$

9. $$\begin{cases} 2x-3y=\ \ 1 & (1)\\ 5x+2y=126 & (2)\end{cases}$$

Transpose $-3y$ in (1), and $2y$ in (2).

$$2x=1+3y \qquad (3)$$
$$5x=126-2y \qquad (4)$$

Divide (3) by 2, and (4) by 5.

$$x=\frac{1+3y}{2} \qquad (5)$$
$$x=\frac{126-2y}{5} \qquad (6)$$

Equate the values of x.

$$\frac{1+3y}{2}=\frac{126-2y}{5}$$
$$5+15y=252-4y$$
$$\therefore\ y=13$$

Substitute the value of y in (5).

$$\therefore\ x=20$$

10. $$\begin{cases} 50x-9y=1 & (1)\\ 7x-\ \ y=3 & (2)\end{cases}$$

Transpose $-9y$ in (1), and $-y$ in (2).

$$50x=1+9y \qquad (3)$$
$$7x=3+y \qquad (4)$$

Divide (3) by 50, and (4) by 7.

$$x=\frac{1+9y}{50} \qquad (5)$$
$$x=\frac{3+y}{7} \qquad (6)$$

Equate the values of x.

$$\frac{1+9y}{50}=\frac{3+y}{7}$$
$$7+63y=150+50y$$
$$\therefore\ y=11$$

Substitute the value of y in (5).

$$\therefore\ x=2$$

11. $$\begin{cases} x+21y\ \ =\ \ 2 & (1)\\ 27y+2x=19 & (2)\end{cases}$$

Transpose $21y$ in (1), and $27y$ in (2).

$$x=2-21y \qquad (3)$$
$$2x=19-27y \qquad (4)$$

Divide (4) by 2.

$$x=2-21y \qquad (5)$$
$$x=\frac{19-27y}{2} \qquad (6)$$

Equate the values of x.

$$2-21y=\frac{19-27y}{2}$$
$$4-42y=19-27y$$
$$\therefore\ y=-1$$

Substitute the value of y in (5).

$$\therefore\ x=23$$

12. $$\begin{cases} 10x + 3y = 174 & (1) \\ 3x + 10y = 125 & (2) \end{cases}$$

Transpose $3y$ in (1), and $10y$ in (2).

$$10x = 174 - 3y \qquad (3)$$
$$3x = 125 - 10y \qquad (4)$$

Divide (3) by 10, and (4) by 3.

$$x = \frac{174 - 3y}{10} \qquad (5)$$
$$x = \frac{125 - 10y}{3} \qquad (6)$$

Equate the values of x.

$$\frac{174 - 3y}{10} = \frac{125 - 10y}{3}$$
$$522 - 9y = 1250 - 100y$$
$$\therefore\ y = 8$$

Substitute the value of y in (5).

$$\therefore\ x = 15$$

13. $$\begin{cases} 6x - 13y = 2 & (1) \\ 5x - 12y = 4 & (2) \end{cases}$$

Transpose $-13y$ in (1), and $-12y$ in (2).

$$6x = 2 + 13y \qquad (3)$$
$$5x = 4 + 12y \qquad (4)$$

Divide (3) by 6, and (4) by 5.

$$x = \frac{2 + 13y}{6} \qquad (5)$$
$$x = \frac{4 + 12y}{5} \qquad (6)$$

Equate the values of x.

$$\frac{2 + 13y}{6} = \frac{4 + 12y}{5}$$
$$10 + 65y = 24 + 72y$$
$$\therefore\ y = -2$$

Substitute the value of y in (5).

$$\therefore\ x = -4$$

14. $$\begin{cases} 2x + y = 108 & (1) \\ 10x + 2y = 60 & (2) \end{cases}$$

Transpose y in (1), and $2y$ in (2).

$$2x = 108 - y \qquad (3)$$
$$10x = 60 - 2y \qquad (3)$$

Divide (3) by 2, and (4) by 10.

$$x = \frac{108 - y}{2} \qquad (5)$$
$$x = \frac{30 - y}{5} \qquad (6)$$

Equate the values of x.

$$\frac{108 - y}{2} = \frac{30 - y}{5}$$
$$540 - 5y = 60 - 2y$$
$$\therefore\ y = 160.$$

Substitute the value of y in (5).

$$\therefore\ x = -26$$

15. $$\begin{cases} 3x - 5y = 5 & (1) \\ 7x + y = 265 & (2) \end{cases}$$

Transpose $-5y$ in (1), and y in (2).

$$3x = 5 + 5y \qquad (3)$$
$$7x = 265 - y \qquad (4)$$

Divide (3) by 3, and (4) by 7.

$$x = \frac{5 + 5y}{3} \qquad (5)$$
$$x = \frac{265 - y}{7} \qquad (6)$$

Equate the values of x.

$$\frac{5 + 5y}{3} = \frac{265 - y}{7}$$
$$35 + 35y = 795 - 3y$$
$$\therefore\ y = 20$$

Substitute the value of y in (5).

$$\therefore\ x = 35$$

16. $\begin{cases} 12x + 7y = 176 & (1) \\ 3y - 19x = 3 & (2) \end{cases}$

Transpose $7y$ in (1), and $3y$ in (2).

$$12x = 176 - 7y \quad (3)$$

$$-19x = 3 - 3y \quad (4)$$

Divide (3) by 12, and (4) by -19.

$$x = \frac{176 - 7y}{12} \quad (5)$$

$$x = \frac{3y - 3}{19} \quad (6)$$

Equate the values of x.

$$\frac{176 - 7y}{12} = \frac{3y - 3}{19}$$

$$3344 - 133y = 36y - 36$$

$$\therefore y = 20$$

Substitute the value of y in (6).

$$\therefore x = 3$$

Exercise 61. Page 169.

Solve:

1. $\begin{cases} \dfrac{x}{3} + \dfrac{y}{2} = \dfrac{4}{3} & (1) \\ \dfrac{x}{2} + \dfrac{y}{3} = \dfrac{7}{6} & (2) \end{cases}$

Multiply (1) and (2) by 6.

$$2x + 3y = 8 \quad (3)$$

$$3x + 2y = 7 \quad (4)$$

Add, $5x + 5y = 15$

$$x + y = 3 \quad (5)$$

Multiply (5) by 2, and subtract from (3).

$$\therefore y = 2$$

Substitute the value of y in (5).

$$\therefore x = 1$$

2. $\begin{cases} \dfrac{x + y}{3} + \dfrac{y - x}{2} = 9 & (1) \\ \dfrac{x}{2} + \dfrac{x + y}{9} = 5 & (2) \end{cases}$

Multiply (1) by 6, and (2) by 18.

$$2(x + y) + 3(y - x) = 54 \quad (3)$$

$$9x + 2(x + y) = 90 \quad (4)$$

From (3), $5y - x = 54$ (5)

From (4), $11x + 2y = 90$ (6)

From (5), $x = 5y - 54$

Substitute the value of x in (6).

$$11(5y - 54) + 2y = 90$$

$$55y - 594 + 2y = 90$$

$$\therefore y = 12$$

Substitute the value of y in (5).

$$\therefore x = 6$$

3. $\begin{cases} \dfrac{4x + 5y}{40} = x - y & (1) \\ \dfrac{2x - y}{3} + 2y = \dfrac{1}{2} & (2) \end{cases}$

Multiply (1) by 40, and (2) by 6.

$$4x + 5y = 40x - 40y \quad (3)$$

$$4x - 2y + 12y = 3 \quad (4)$$

From (3), $36x - 45y = 0$ (5)

From (4), $4x + 10y = 3$ (6)

Divide (5) by 9.

$$4x - 5y = 0$$

$$4x = 5y \quad (7)$$

Substitute the value of $4x$ in (6).

$$5y + 10y = 3$$

$$\therefore y = \tfrac{1}{5}$$

Substitute the value of y in (7).

$$\therefore x = \tfrac{1}{4}$$

$$4.\quad \begin{cases} \dfrac{x-1}{8}+\dfrac{y-2}{5}=2 & (1)\\[2ex] 2x+\dfrac{2y-5}{3}=21 & (2)\end{cases}$$

Multiply (1) by 40, and (2) by 3.

$$5x-5+8y-16=80 \qquad (3)$$

$$6x+2y-5=63 \qquad (4)$$

From (3), $5x+8y=101$ (5)

From (4), $6x+2y=68$ (6)

Multiply (6) by 4, and subtract from (5).

$$\begin{array}{rcrl} 5x+8y & = & 101 & (5)\\ 24x+8y & = & 272 & (6)\\ \hline -19x & = & -171 & \end{array}$$

$$\therefore\ x=9$$

Substitute the value of x in (5).

$$\therefore\ y=7$$

$$5.\quad \begin{cases} \dfrac{3x-5y}{2}+3=\dfrac{2x+y}{5} & (1)\\[2ex] 8-\dfrac{x-2y}{4}=\dfrac{x}{2}+\dfrac{y}{3} & (2)\end{cases}$$

Multiply (1) by 10, and (2) by 12.

$$15x-25y+30=4x+2y \qquad (3)$$

$$96-3x+6y=6x+4y \qquad (4)$$

From (3),

$$11x-27y=-30 \qquad (5)$$

From (4),

$$9x-2y=96 \qquad (6)$$

Multiply (5) by 9, and (6) by 11, and subtract.

$$\begin{array}{rcr} 99x-243y & = & -\ 270\\ 99x-\ \ 22y & = & 1056\\ \hline -221y & = & -1326 \end{array}$$

$$\therefore\ y=6$$

Substitute the value of y in (6).

$$\therefore\ x=12$$

$$6.\quad \begin{cases} \dfrac{x+3y}{x-y}=8 & (1)\\[2ex] \dfrac{7x-13}{3y-5}=4 & (2)\end{cases}$$

From (1), $x+3y=8x-8y$

$$7x=11y \qquad (3)$$

From (2), $7x-13=12y-20$

$$7x-12y=-7 \qquad (4)$$

Substitute the value of $7x$ from (3) in (4).

$$11y-12y=-7$$

$$\therefore\ y=7$$

Substitute the value of y in (3).

$$\therefore\ x=11$$

$$7.\quad \begin{cases} \dfrac{x+1}{3}-\dfrac{y+2}{4}=\dfrac{2(x-y)}{5} & (1)\\[2ex] \dfrac{x+6}{4}-\dfrac{y+1}{3}=\dfrac{2x-y}{2} & (2)\end{cases}$$

Multiply (1) by 60, and (2) by 12.

$$20x+20-15y-30=24x-24y \qquad (3)$$

$$3x + 18 - 4y - 4 = 12x - 6y \qquad (4)$$

From (3), $4x - 9y = -10$ (5)

From (4), $9x - 2y = 14$ (6)

Multiply (5) by 9, and (6) by 4, and subtract.

$$\begin{array}{r} 36x - 81y = -\ \ 90 \\ \underline{36x - \ \ 8y = \quad 56} \\ -73y = -146 \end{array}$$

$$\therefore\ y = 2$$

Substitute the value of y in (5).

$$\therefore\ x = 2$$

8. $$\begin{cases} \dfrac{x + 2y + 1}{2x - y + 1} = 2 & (1) \\ \dfrac{3x - y + 1}{x - y + 3} = 5 & (2) \end{cases}$$

From (1), $x + 2y + 1 = 4x - 2y + 2$

$$3x - 4y = -1 \qquad (3)$$

From (2), $3x - y + 1 = 5x - 5y + 15$

$$2x - 4y = -14 \qquad (4)$$

Subtract (4) from (3).

$$\begin{array}{rl} 3x - 4y = -\ \ 1 & (3) \\ \underline{2x - 4y = -14} & (4) \\ \therefore x \qquad = \ \ 13 & \end{array}$$

Substitute the value of x in (3).

$$\therefore\ y = 10$$

9. $$\begin{cases} \dfrac{x - 2}{5} - \dfrac{10 - x}{3} = \dfrac{y - 10}{4} & (1) \\ \dfrac{2y + 4}{3} = \dfrac{4x + y + 26}{8} & (2) \end{cases}$$

Multiply (1) by 60, and (2) by 24.

$$12x - 24 - 200 + 20x = 15y - 150 \qquad (3)$$

$$16y + 32 = 12x + 3y + 78 \qquad (4)$$

From (3), $32x - 15y = 74$ (5)

From (4), $12x - 13y = -46$ (6)

Multiply (5) by 3, and (6) by 8, and subtract.

$$\begin{array}{r} 96x - \ \ 45y = \quad 222 \\ \underline{96x - 104y = -368} \\ 59y = \quad 590 \end{array}$$

$$\therefore\ y = 10$$

Substitute the value of y in (6).

$$\therefore\ x = 7$$

10. $$\begin{cases} \dfrac{2x-y+3}{3}=\dfrac{x-2y+19}{4} & (1) \\[2ex] \dfrac{3x-4y+3}{4}=\dfrac{2y-4x+21}{3} & (2) \end{cases}$$

Multiply (1) and (2) by 12.

$$8x-4y+12=3x-6y+57 \tag{3}$$
$$9x-12y+9=8y-16x+84 \tag{4}$$

From (3), $$5x+2y=45 \tag{5}$$

From (4), $$25x-20y=75 \tag{6}$$

Divide (6) by 5, and subtract from (5).

$$\begin{array}{rl} 5x+2y=45 & (5) \\ 5x-4y=15 & (7) \\ \hline 6y=30 & \end{array}$$

$$\therefore\ y=5$$

Substitute the value of y in (5).

$$\therefore\ x=7$$

11. $$\begin{cases} \dfrac{7+x}{5}-\dfrac{2x-y}{4}=3y-5 & (1) \\[2ex] \dfrac{5y-7}{2}+\dfrac{4x-3}{6}=18-5x & (2) \end{cases}$$

Multiply (1) by 20, and (2) by 6.

$$28+4x-10x+5y=60y-100 \tag{3}$$
$$15y-21+4x-3=108-30x \tag{4}$$

From (3), $$6x+55y=128 \tag{5}$$

From (4), $$34x+15y=132 \tag{6}$$

Multiply (5) by 3, and (6) by 11, and subtract.

$$\begin{array}{rl} 18x+165y= & 384 \\ 374x+165y= & 1452 \\ \hline 356x\qquad\quad = & 1068 \end{array}$$

$$\therefore\ x=3$$

Substitute the value of x in (6).

$$\therefore\ y=2$$

12. $$\begin{cases} \dfrac{26+5x-6y}{13}=4y-3x & (1) \\[2ex] 12+\dfrac{5x-6y}{6}=\dfrac{3(x+2y)}{4} & (2) \end{cases}$$

Multiply (1) by 13, and (2) by 12.

$$26 + 5x - 6y = 52y - 39x \qquad (3)$$

$$144 + 10x - 12y = 9x + 18y \qquad (4)$$

From (3), $$44x - 58y = -26 \qquad (5)$$

From (4), $$x - 30y = -144 \qquad (6)$$

Divide (5) by 2, and multiply (6) by 22, and subtract.

$$\begin{array}{r}
22x - 29y = -13 \\
\underline{22x - 660y = -3168} \\
631y = 3155
\end{array}$$

$$\therefore\ y = 5$$

Substitute the value of y in (6).

$$\therefore\ x = 6$$

13. $$\begin{cases} \dfrac{x+8}{2} = 2 - \dfrac{3y-x}{6} & (1) \\[2ex] \dfrac{2x+y}{2} - \dfrac{9x-7}{8} = \dfrac{7y-4x+36}{16} & (2) \end{cases}$$

Multiply (1) by 6, and (2) by 16.

$$3x + 24 = 12 - 3y + x \qquad (3)$$

$$16x + 8y - 18x + 14 = 7y - 4x + 36 \qquad (4)$$

From (3), $$2x + 3y = -12 \qquad (5)$$

From (4), $$2x + y = 22 \qquad (6)$$

Subtract, $$2y = -34$$

$$\therefore\ y = -17$$

Substitute the value of y in (6).

$$\therefore\ x = 19\tfrac{1}{2}.$$

14. $$\begin{cases} \dfrac{x+2y+3}{13} = \dfrac{5y-4x-6}{3} & (1) \\[2ex] \dfrac{6x-5y+4}{3} = \dfrac{3x+2y+1}{19} & (2) \end{cases}$$

Multiply (1) by 39, and (2) by 57.

$$3x + 6y + 9 = 65y - 52x - 78 \qquad (3)$$

$$114x - 95y + 76 = 9x + 6y + 3 \qquad (4)$$

From (3), $$55x - 59y = -87 \qquad (5)$$

From (4), $$105x - 101y = -73 \qquad (6)$$

Add, $$160x - 160y = -160$$

$$x - y = -1 \qquad (7)$$

Multiply (7) by 55, and subtract from (5).

$$\begin{aligned}55x-59y&=-87\\ 55x-55y&=-55\\ \hline -4y&=-32\\ \therefore y&=8\end{aligned}$$

Substitute the value of y in (7).

$$\therefore x=7$$

15. $$\begin{cases}\dfrac{x+y}{y-x}=\dfrac{5}{3} & (1)\\ 3x-\dfrac{3y+44}{7}=13 & (2)\end{cases}$$

Multiply (1) by $3(y-x)$, and (2) by 7.

$$3x+3y=5y-5x \qquad (3)$$

$$21x-3y-44=91 \qquad (4)$$

From (3), $$8x-2y=0 \qquad (5)$$

From (4), $$21x-3y=135 \qquad (6)$$

Divide (5) by 2, and (6) by 3, and subtract.

$$\begin{aligned}4x-y&=\ \ 0\\ 7x-y&=45\\ \hline 3x\qquad&=45\\ \therefore x&=15\end{aligned}$$

Substitute the value of x in (5).

$$\therefore y=60$$

16. $$\begin{cases}\dfrac{y-4}{4}=\dfrac{x+1}{10} & (1)\\ \dfrac{x}{6}+\dfrac{y+2}{5}=3\frac{1}{2} & (2)\end{cases}$$

Multiply (1) by 20, and (2) by 30.

$$5y-20=2x+2 \qquad (3)$$

$$5x+6y+12=105 \qquad (4)$$

From (3), $$2x-5y=-22 \qquad (5)$$

From (4), $$5x+6y=93 \qquad (6)$$

Multiply (5) by 5, and (6) by 2, and subtract.

$$\begin{aligned}10x-25y&=-110\\ 10x+12y&=\ \ \ 186\\ \hline 37y&=\ \ \ 296\\ \therefore y&=8\end{aligned}$$

Substitute the value of y in (5).

$$\therefore x=9$$

17.
$$\begin{cases} \dfrac{5x-6y}{11}+2x=3(y-1) & (1)\\[1ex] \dfrac{5x+6y}{10}-\dfrac{4x-3y}{3}=y-2 & (2)\end{cases}$$

Multiply (1) by 11, and (2) by 30.

$$5x-6y+22x=33y-33 \qquad (3)$$
$$15x+18y-40x+30y=30y-60 \qquad (4)$$

From (3), $27x-39y=-33$ (5)

From (4), $25x-18y=60$ (6)

Multiply (5) by 6, and (6) by 13, and subtract.

$$\begin{array}{rcr} 162x-234y & = & -198\\ 325x-234y & = & 780\\ \hline 163x & = & 978 \end{array}$$

$$\therefore\ x=6$$

Substitute the value of x in (6).

$$\therefore\ y=5$$

18.
$$\begin{cases} \dfrac{4x-3y-7}{5}=\dfrac{9x-4y-25}{30} & (1)\\[1ex] \dfrac{y-1}{3}+\dfrac{10x-3y-20}{20}=\dfrac{3x+2y+3}{30} & (2)\end{cases}$$

Multiply (1) by 30, and (2) by 60.

$$24x-18y-42=9x-4y-25 \qquad (3)$$
$$20y-20+30x-9y-60=6x+4y+6 \qquad (4)$$

From (3), $15x-14y=17$ (5)

From (4), $24x+7y=86$ (6)

Multiply (6) by 2, and add to (5).

$$\begin{array}{rcr} 15x-14y & = & 17\\ 48x+14y & = & 172\\ \hline 63x & = & 189 \end{array}$$

$$\therefore\ x=3$$

Substitute the value of x in (6).

$$\therefore\ y=2$$

19.
$$\begin{cases} \dfrac{6y+5}{8}-\dfrac{4x-5y+3}{4x-2y}=\dfrac{9y-4}{12} & (1)\\[1ex] \dfrac{8x+3}{4}+\dfrac{x-3y}{7-x}=\dfrac{6x-1}{3} & (2)\end{cases}$$

Multiply (1) by 24, and (2) by 12.

$$18y+15-24\left(\frac{4x-5y+3}{4x-2y}\right)=18y-8 \tag{3}$$

$$24x+9+12\left(\frac{x-3y}{7-x}\right)=24x-4 \tag{4}$$

From (3), $$24\left(\frac{4x-5y+3}{4x-2y}\right)=23 \tag{5}$$

From (4), $$12\left(\frac{x-3y}{7-x}\right)=-13 \tag{6}$$

Multiply (5) by $2x-y$, and (6) by $7-x$.

$$48x-60y+36=46x-23y \tag{7}$$

$$12x-36y=-91+13x \tag{8}$$

From (7), $$2x-37y=-36 \tag{9}$$

From (8), $$-x-36y=-91 \tag{10}$$

Multiply (10) by 2, and add to (9).

$$\begin{array}{r} 2x-\ \ 37y=-\ \ 36 \\ -2x-\ \ 72y=-182 \\ \hline -109y=-218 \end{array}$$

$$\therefore\ y=2$$

Substitute the value of y in (10).

$$\therefore\ x=19$$

20. $$\begin{cases} x-\dfrac{2y-x}{23-x}=x-9\frac{1}{2} & (1) \\ y+\dfrac{y-3}{x-18}=y-5\frac{2}{3} & (2) \end{cases}$$

Cancel x in (1), and y in (2).

$$-\frac{2y-x}{23-x}=-9\frac{1}{2} \tag{3}$$

$$\frac{y-3}{x-18}=-5\frac{2}{3} \tag{4}$$

Multiply (3) by $2(23-x)$, and (4) by $3(x-18)$.

$$4y-2x=437-19x \tag{5}$$

$$3y-9=-17x+306 \tag{6}$$

From (5), $$17x+4y=437 \tag{7}$$

From (6), $$17x+3y=315 \tag{8}$$

Subtract, $$y=122$$

Substitute the value of y in (8).

$$\therefore\ x=-3$$

21. $$\begin{cases} \dfrac{4x+7}{3}+\dfrac{5x-4y}{2x+8}=\dfrac{17+8x}{6} & (1) \\[2ex] \dfrac{5x-12}{4}-\dfrac{4x-6y-13}{2x-3y}=\dfrac{10x-53}{8} & (2) \end{cases}$$

Multiply (1) by 6, and (2) by 8.

$$8x+14+6\left(\frac{5x-4y}{2x+8}\right)=17+8x \qquad (3)$$

$$10x-24-8\left(\frac{4x-6y-13}{2x-3y}\right)=10x-53 \qquad (4)$$

From (3), $$6\left(\frac{5x-4y}{2x+8}\right)=3 \qquad (5)$$

From (4), $$8\left(\frac{4x-6y-13}{2x-3y}\right)=29 \qquad (6)$$

Multiply (5) by $2x+8$, and (6) by $2x-3y$.

$$30x-24y=6x+24 \qquad (7)$$

$$32x-48y-104=58x-87y \qquad (8)$$

From (7), $$24x-24y=24 \qquad (9)$$

From (8), $$26x-39y=-104 \qquad (10)$$

Divide (9) by 24, and (10) by 13.

$$x-y=1 \qquad (11)$$

$$2x-3y=-8 \qquad (12)$$

Multiply (11) by 2, and subtract from (12).

$$\begin{array}{r} 2x-3y=-8 \\ 2x-2y=2 \\ \hline \therefore y=10 \end{array}$$

Substitute the value of y in (11).

$$\therefore x=11$$

22. $$\begin{cases} \dfrac{7+8x}{10}-\dfrac{3(x-2y)}{2(x-4)}=\dfrac{11+4x}{5} & (1) \\[2ex] \dfrac{3(2y+3)}{4}=\dfrac{6y+21}{4}-\dfrac{3y+5x}{2(2y-3)} & (2) \end{cases}$$

Multiply (1) by 10, and (2) by 4.

$$7+8x-15\left(\frac{x-2y}{x-4}\right)=22+8x \qquad (3)$$

$$6y+9=6y+21-2\left(\frac{3y+5x}{2y-3}\right) \qquad (4)$$

From (3),
$$\frac{x-2y}{x-4}=-1 \tag{5}$$

From (4),
$$\frac{3y+5x}{2y-3}=6 \tag{6}$$

Multiply (5) by $x-4$, and (6) by $2y-3$.

$$x-2y=-x+4 \tag{7}$$
$$3y+5x=12y-18 \tag{8}$$

From (7),
$$2x-2y=4 \tag{9}$$

From (8),
$$5x-9y=-18 \tag{10}$$

Multiply (9) by $\frac{9}{2}$, and subtract from (10).

$$\begin{array}{rl} 5x-9y= & -18 \\ 9x-9y= & 18 \\ \hline 4x \quad = & 36 \end{array}$$

$$\therefore x=9$$

Substitute the value of x in (9).

$$\therefore y=7$$

Exercise 62. Page 172.

Solve:

1.
$$\begin{cases} x+y=s & (1) \\ x-y=d & (2) \end{cases}$$

Add,
$$2x=s+d$$
$$\therefore x=\frac{s+d}{2}$$

Again,
$$x+y=s \tag{1}$$
$$x-y=d \tag{2}$$

Subtract,
$$2y=s-d$$
$$\therefore y=\frac{s-d}{2}$$

2.
$$\begin{cases} mx+ny=r & (1) \\ m'x+n'y=r' & (2) \end{cases}$$

Multiply (1) by m', and (2) by m, and subtract.

$$mm'x+m'ny=m'r \tag{3}$$
$$mm'x+mn'y=mr' \tag{4}$$
$$(m'n-mn')y=m'r-mr'$$
$$\therefore y=\frac{m'r-mr'}{m'n-mn'}$$

Again, multiply (1) by n', and (2) by n, and subtract.

$$mn'x+nn'y=n'r \tag{5}$$
$$m'nx+nn'y=nr' \tag{6}$$
$$(mn'-m'n)x=n'r-nr'$$
$$\therefore x=\frac{n'r-nr'}{mn'-m'n}$$

3.
$$\begin{cases} ax-by=c & (1) \\ a'x+b'y=c' & (2) \end{cases}$$

Multiply (1) by a', and (2) by a, and subtract.

$$aa'x-a'by=a'c \tag{3}$$
$$aa'x+ab'y=ac' \tag{4}$$
$$(ab'+a'b)y=ac'-a'c$$
$$\therefore y=\frac{ac'-a'c}{ab'+a'b}$$

Again, multiply (1) by b', and (2) by b, and add.

$$\begin{array}{rr} ab'x - bb'y = b'c & (5) \\ a'bx + bb'y = bc' & (6) \\ \hline (ab' + a'b)x = b'c + bc' & \end{array}$$

$$\therefore\ x = \frac{b'c + bc'}{ab' + a'b}$$

4. $\begin{cases} x - y = mn & (1) \\ cx + aby = ms & (2) \end{cases}$

Multiply (1) by c, and subtract from (2).

$$\begin{array}{rr} cx + aby = ms & (2) \\ cx - \ \ cy = cmn & (3) \\ \hline (ab + c)y = ms - cmn & \end{array}$$

$$\therefore\ y = \frac{m(s - cn)}{ab + c}$$

Again, multiply (1) by ab, and add to (2).

$$\begin{array}{rr} abx - aby = abmn & (4) \\ cx + aby = ms & (2) \\ \hline (ab + c)x = abmn + ms & \end{array}$$

$$\therefore\ x = \frac{m(abn + s)}{ab + c}$$

5. $\begin{cases} bx + ay = abc & (1) \\ x = dy & (2) \end{cases}$

Substitute the value of x from (2) in (1).

$$bdy + ay = abc \qquad (3)$$

$$\therefore\ y = \frac{abc}{bd + a}$$

Substitute the value of y in (2).

$$\therefore\ x = \frac{abcd}{bd + a}$$

6. $\begin{cases} bx + ay = 1 & (1) \\ b'x - a'y = 1 & (2) \end{cases}$

Multiply (1) by b', and (2) by b, and subtract.

$$\begin{array}{rr} bb'x + ab'y = b' & (3) \\ bb'x - a'by = b & (4) \\ \hline (ab' + a'b)y = b' - b & \end{array}$$

$$\therefore\ y = \frac{b' - b}{ab' + a'b}$$

Again, multiply (1) by a', and (2) by a, and add.

$$\begin{array}{rr} a'bx + aa'y = a' & (5) \\ ab'x - aa'y = a & (6) \\ \hline (a'b + ab')x = a' + a & \end{array}$$

$$\therefore\ x = \frac{a' + a}{a'b + ab'}$$

7. $\begin{cases} 3bx + 2ay = 3ab & (1) \\ 4bx - 3ay = \frac{7}{6}ab & (2) \end{cases}$

Multiply (1) by 4, and (2) by 3, and subtract.

$$\begin{array}{rr} 12bx + 8ay = 12ab & (3) \\ 12bx - 9ay = \frac{7}{2}ab & (4) \\ \hline 17ay = \frac{17}{2}ab & \end{array}$$

$$\therefore\ y = \frac{b}{2}$$

Substitute the value of y in (1).

$$3bx + ab = 3ab$$

$$\therefore\ x = \frac{2a}{3}$$

8. $\begin{cases} 2x - 3y = a - b & (1) \\ 3x - 2y = a + b & (2) \end{cases}$

Add, $\quad 5x - 5y = 2a$

$$x - y = \frac{2a}{5} \qquad (3)$$

Again, subtract (1) from (2).

$$x + y = 2b \qquad (4)$$

Add (3) and (4).

$$2x = \frac{2a}{5} + 2b$$

$$\therefore\ x = \frac{a}{5} + b$$

Substitute the value of x in (3).

$$\therefore\ y = b - \frac{a}{5}$$

9. $$\begin{cases} \dfrac{bx}{a^2-b^2} + \dfrac{cy}{b^2-a^2} = \dfrac{1}{a+b} & (1) \\ bx + cy = a + b & (2) \end{cases}$$

From (1), $$\frac{bx-cy}{a^2-b^2} = \frac{1}{a+b}$$

$$bx - cy = a - b$$

(2) is $$bx + cy = a + b$$

Add, $$2bx = 2a$$

$$\therefore\ x = \frac{a}{b}$$

Substitute the value of x in (2).

$$a + cy = a + b$$

$$\therefore\ y = \frac{b}{c}$$

10. $$\begin{cases} \dfrac{x+m}{y-n} = \dfrac{a}{b} & (1) \\ bx + ay = c & (2) \end{cases}$$

Multiply (1) by $b(y-n)$.

$$bx + bm = ay - an$$

$$bx - ay = -an - bm \qquad (3)$$

(2) is $$bx + ay = c \qquad (2)$$

Add, $$2bx = c - an - bm$$

$$\therefore\ x = \frac{c - an - bm}{2b}$$

Subtract (3) from (2).

$$2ay = c + an + bm$$

$$\therefore\ y = \frac{c + an + bm}{2a}$$

11. $$\begin{cases} \dfrac{x}{a} + \dfrac{2y}{b} = 1 & (1) \\ \dfrac{2x}{a} - \dfrac{y}{b} = \dfrac{1}{3} & (2) \end{cases}$$

Multiply (2) by 2, and add to (1).

$$\frac{x}{a} + \frac{2y}{b} = 1 \qquad (1)$$

$$\frac{4x}{a} - \frac{2y}{b} = \frac{2}{3} \qquad (3)$$

$$\frac{5x}{a} = \frac{5}{3}$$

$$\therefore\ x = \frac{a}{3}$$

Substitute the value of x in (1).

$$\frac{1}{3} + \frac{2y}{b} = 1$$

$$\therefore\ y = \frac{b}{3}$$

12. $$\begin{cases} \dfrac{x}{a+b} + \dfrac{y}{a-b} = 2 & (1) \\ x + y = 2a & (2) \end{cases}$$

Multiply (1) by $a^2 - b^2$.

$$(a-b)x + (a+b)y = 2(a^2-b^2) \qquad (3)$$

Multiply (2) by $a - b$, and subtract from (3).

$$(a-b)x + (a+b)y = 2(a^2-b^2) \qquad (3)$$

$$(a-b)x + (a-b)y = 2a(a-b) \qquad (4)$$

$$2by = 2ab - 2b^2$$

$$\therefore\ y = a - b$$

Substitute the value of y in (2).

$$\therefore\ x = a + b$$

13. $$\begin{cases} \dfrac{2x}{3a} + \dfrac{4y}{3b} = \dfrac{3y}{b} - \dfrac{x}{a} & (1) \\ x - y = a - b & (2) \end{cases}$$

Multiply (1) by $3ab$.

$$2bx + 4ay = 9ay - 3bx \qquad (3)$$

$$\therefore\ 5bx - 5ay = 0$$

$$bx - ay = 0$$

$$\therefore\ x = \frac{ay}{b}$$

Substitute the value of x in (2).

$$\frac{ay}{b} - y = a - b$$

$$\frac{a-b}{b}y = a - b$$

$$\therefore y = b$$

Substitute the value of y in (2).

$$\therefore x = a$$

14.
$$\begin{cases} ax + by = c & (1) \\ bx + ay = c & (2) \end{cases}$$

Subtract, $(a-b)x + (b-a)y = 0$

Divide by $a - b$.

$$x - y = 0$$

$$\therefore x = y$$

Substitute the value of x in (1).

$$ay + by = c$$

$$\therefore y = \frac{c}{a+b}$$

$$\therefore x = \frac{c}{a+b}$$

15.
$$\begin{cases} 3a^2 + ax = b^2 + by & (1) \\ ax + 2by = d & (2) \end{cases}$$

From (1), $ax - by = b^2 - 3a^2$

(2) is, $ax + 2by = d$

Subtract, $3by = d - b^2 + 3a^2$

$$\therefore y = \frac{d - b^2 + 3a^2}{3b}$$

Substitute the value of y in (2).

$$ax + \tfrac{2}{3}(d - b^2 + 3a^2) = d$$

$$\therefore ax = \frac{d + 2b^2 - 6a^2}{3}$$

$$\therefore x = \frac{d + 2b^2 - 6a^2}{3a}$$

16.
$$\begin{cases} \dfrac{x}{a+b} - \dfrac{y}{a-b} = \dfrac{1}{a+b} & (1) \\[2ex] \dfrac{x}{a+b} + \dfrac{y}{a-b} = \dfrac{1}{a-b} & (2) \end{cases}$$

Add,
$$\frac{2x}{a+b} = \frac{1}{a+b} + \frac{1}{a-b}$$

$$\therefore \frac{2x}{a+b} = \frac{2a}{a^2 - b^2}$$

$$\therefore x = \frac{a}{a-b}$$

Again subtract (1) from (2).

$$\frac{2y}{a-b} = \frac{1}{a-b} - \frac{1}{a+b}$$

$$\therefore \frac{2y}{a-b} = \frac{2b}{a^2 - b^2}$$

$$\therefore y = \frac{b}{a+b}$$

17.
$$\begin{cases} \dfrac{x}{m-a} + \dfrac{y}{m-b} = 1 & (1) \\[2ex] \dfrac{x}{n-a} + \dfrac{y}{n-b} = 1 & (2) \end{cases}$$

Multiply (1) by $(m-a)(m-b)$ and (2) by $(n-a)(n-b)$.

$$(m-b)x + (m-a)y = m^2 - am - bm + ab \qquad (3)$$

$$(n-b)x + (n-a)y = n^2 - an - bn + ab \qquad (4)$$

Subtract, $(m-n)x + (m-n)y = m^2 - n^2 - a(m-n) - b(m-n)$ (5)

Divide by $m - n$, $x + y = m + n - a - b$ (6)

Again, subtract (2) from (1).

$$x\left(\frac{1}{m-a} - \frac{1}{n-a}\right) + y\left(\frac{1}{m-b} - \frac{1}{n-b}\right) = 0$$

$$x\frac{n-m}{(m-n)(n-a)}+y\frac{n-m}{(m-b)(n-b)}=0$$

Divide by $n-m$.

$$\frac{x}{(m-a)(n-a)}+\frac{y}{(m-b)(n-b)}=0$$

$$\therefore\ x=-\frac{(m-a)(n-a)}{(m-b)(n-b)}y \qquad (7)$$

Substitute the value of x in (6).

$$\left(-\frac{(m-a)(n-a)}{(m-b)(n-b)}+1\right)y=m+n-a-b$$

Simplify the coefficient of y.

$$\frac{(m-b)(n-b)-(m-a)(n-a)}{(m-b)(n-b)}y=m+n-a-b$$

$$\frac{(a-b)(m+n)+b^2-a^2}{(m-b)(n-b)}y=m+n-a-b$$

$$\therefore\ y=\frac{(m+n-a-b)(m-b)(n-b)}{(a-b)(m+n)+b^2-a^2}$$

$$=\frac{(m+n-a-b)(m-b)(n-b)}{(a-b)(m+n-a-b)}$$

$$=\frac{(m-b)(n-b)}{a-b}$$

Substitute the value of y in (7).

$$\therefore\ x=-\frac{(m-a)(n-a)}{a-b}$$

$$=\frac{(m-a)(n-a)}{b-a}$$

Exercise 63. Page 175.

Solve:

1. $$\begin{cases}\dfrac{5}{x}+\dfrac{6}{y}=2 & (1)\\[2ex] \dfrac{15}{x}-\dfrac{3}{y}=2\tfrac{1}{2} & (2)\end{cases}$$

Multiply (2) by 2, and add to (1).

$$\frac{5}{x}+\frac{6}{y}=2 \qquad (1)$$

$$\frac{30}{x}-\frac{6}{y}=5 \qquad (3)$$

$$\frac{35}{x}\qquad =7$$

$$7x=35$$

$$\therefore\ x=5$$

Substitute the value of x in (1)

$$1+\frac{6}{y}=2$$

$$\therefore\ \frac{6}{y}=1$$

$$\therefore\ y=6$$

2. $$\begin{cases} \dfrac{5}{x}+\dfrac{13}{y}=49 & (1) \\ \dfrac{7}{x}+\dfrac{3}{y}=23 & (2) \end{cases}$$

Multiply (1) by 7, and (2) by 5, and subtract.

$$\begin{aligned} \frac{35}{x}+\frac{91}{y}&=343 && (3) \\ \frac{35}{x}+\frac{15}{y}&=115 && (4) \\ \hline \frac{76}{y}&=228 \\ 76&=228y \\ \therefore y&=\tfrac{1}{3} \end{aligned}$$

Substitute the value of y in (1).

$$\begin{aligned} \frac{5}{x}+39&=49 \\ \frac{5}{x}&=10 \\ \therefore x&=\tfrac{1}{2} \end{aligned}$$

3. $$\begin{cases} \dfrac{3}{x}+\dfrac{8}{y}=3 & (1) \\ \dfrac{15}{x}-\dfrac{4}{y}=4 & (2) \end{cases}$$

Multiply (2) by 2 and add to (1).

$$\begin{aligned} \frac{3}{x}+\frac{8}{y}&=3 && (1) \\ \frac{30}{x}-\frac{8}{y}&=8 && (3) \\ \hline \frac{33}{x}&=11 \\ 33&=11x \\ \therefore x&=3 \end{aligned}$$

Substitute the value of x in 1.

$$\begin{aligned} 1+\frac{8}{y}&=3 \\ \therefore y&=4 \end{aligned}$$

4. $$\begin{cases} \dfrac{2}{x}+\dfrac{5}{y}=19 & (1) \\ \dfrac{8}{x}-\dfrac{3}{y}=7 & (2) \end{cases}$$

Multiply (1) by 3, and (2) by 5, and add.

$$\begin{aligned} \frac{6}{x}+\frac{15}{y}&=57 && (3) \\ \frac{40}{x}-\frac{15}{y}&=35 && (4) \\ \hline \frac{46}{x}&=92 \\ 46&=92x \\ \therefore x&=\tfrac{1}{2} \end{aligned}$$

Substitute the value of x in (1).

$$\begin{aligned} 4+\frac{5}{y}&=19 \\ \therefore y&=\tfrac{1}{3} \end{aligned}$$

5. $$\begin{cases} \dfrac{4}{5x}+\dfrac{5}{6y}=5\tfrac{11}{15} & (1) \\ \dfrac{5}{4x}-\dfrac{4}{5y}=\dfrac{11}{20} & (2) \end{cases}$$

Multiply (1) by 30, and (2) by 20.

$$\begin{aligned} \frac{24}{x}+\frac{25}{y}&=172 && (3) \\ \frac{25}{x}-\frac{16}{y}&=11 && (4) \end{aligned}$$

Multiply (1) by 25, and (2) by 24, and subtract.

$$\begin{aligned} \frac{600}{x}+\frac{625}{y}&=4300 && (5) \\ \frac{600}{x}-\frac{384}{y}&=264 && (6) \\ \hline \frac{1009}{y}&=4036 \\ \therefore y&=\tfrac{1}{4} \end{aligned}$$

Substitute the value of y in (2).

$$\frac{5}{4x}-\frac{16}{5}=\frac{11}{20}$$

$$\frac{5}{4x}=\frac{75}{20}$$

$$\therefore\ x=\tfrac{1}{3}$$

6. $$\begin{cases} \dfrac{1}{2x}+\dfrac{2}{3y}=3 & (1) \\ \dfrac{3}{4x}+\dfrac{4}{5y}=3.9 & (2) \end{cases}$$

Multiply (1) by $\frac{3}{2}$, and subtract from (2).

$$\begin{array}{ll} \dfrac{3}{4x}+\dfrac{4}{5y}=3.9 & (2) \\ \dfrac{3}{4x}+\dfrac{1}{y}\ =\dfrac{9}{2} & (3) \\ \hline \end{array}$$

$$\frac{1}{5y}=\frac{6}{10}$$

$$10=30y$$

$$\therefore\ y=\tfrac{1}{3}$$

Substitute the value of y in (1).

$$\frac{1}{2x}+2=3$$

$$\frac{1}{2x}=1$$

$$\therefore\ x=\tfrac{1}{2}$$

7. $$\begin{cases} \dfrac{1}{x}+\dfrac{1}{y}=m & (1) \\ \dfrac{1}{x}-\dfrac{1}{y}=n & (2) \end{cases}$$

Add, $$\frac{2}{x}=m+n$$

$$2=(m+n)x$$

$$\therefore\ x=\frac{2}{m+n}$$

Again, subtract (2) from (1).

$$\frac{2}{y}=m-n$$

$$2=(m-n)y$$

$$\therefore\ y=\frac{2}{m-n}$$

8. $$\begin{cases} \dfrac{m}{x}+\dfrac{1}{y}=b & (1) \\ \dfrac{n}{x}+\dfrac{1}{y}=c & (2) \end{cases}$$

Subtract, $$\frac{m-n}{x}=b-c$$

$$m-n=(b-c)x$$

$$\therefore\ x=\frac{m-n}{b-c}$$

Again, multiply (1) by n, and 2 by m, and subtract.

$$\begin{array}{ll} \dfrac{mn}{x}+\dfrac{n}{y}=bn & (3) \\ \dfrac{mn}{x}+\dfrac{m}{y}=cm & (4) \\ \hline \end{array}$$

$$\frac{n-m}{y}=bn-cm$$

$$(n-m)=(bn-cm)y$$

$$\therefore\ y=\frac{n-m}{bn-cm}$$

9. $$\begin{cases} \dfrac{m}{x}+\dfrac{n}{y}=a & (1) \\ \dfrac{r}{x}+\dfrac{s}{y}=b & (2) \end{cases}$$

Multiply (1) by r, and 2 by m, and subtract.

$$\frac{mr}{x}+\frac{nr}{y}=ar \qquad (3)$$

$$\frac{mr}{x}+\frac{ms}{y}=bm \qquad (4)$$

$$\frac{nr-ms}{y}=ar-bm$$

$$nr-ms=(ar-bm)y$$

$$\therefore\ y=\frac{nr-ms}{ar-bm}$$

Again, multiply (1) by s, and (2) by n, and subtract.

$$\frac{ms}{x}+\frac{ns}{y}=as \qquad (5)$$

$$\frac{nr}{x}+\frac{ns}{y}=bn \qquad (6)$$

$$\frac{ms-nr}{x}=as-bn$$

$$ms-nr=(as-bn)x$$

$$\therefore\ x=\frac{ms-nr}{as-bn}$$

10.
$$\begin{cases}\dfrac{a}{x}-\dfrac{b}{y}=\dfrac{ac}{b} & (1)\\[2ex] \dfrac{b}{x}-\dfrac{a}{y}=\dfrac{bc}{a} & (2)\end{cases}$$

Multiply (1) by b, and (2) by a, and subtract.

$$\frac{ab}{x}-\frac{b^2}{y}=ac \qquad (3)$$

$$\frac{ab}{x}-\frac{a^2}{y}=bc \qquad (4)$$

$$\frac{a^2-b^2}{y}=ac-bc$$

$$(a^2-b^2)=(ac-bc)y$$

$$\therefore\ y=\frac{a+b}{c}$$

Substitute the value of y in (1).

$$\frac{a}{x}-\frac{bc}{a+b}=\frac{ac}{b}$$

$$\frac{a}{x}=\frac{ac(a+b)+b^2c}{b(a+b)}$$

$$=\frac{c(a^2+ab+b^2)}{b(a+b)}$$

$$\frac{x}{a}=\frac{b(a+b)}{c(a^2+ab+b^2)}$$

$$\therefore\ x=\frac{ab(a+b)}{c(a^2+ab+b^2)}$$

11.
$$\begin{cases}\dfrac{3}{ax}-\dfrac{2}{by}=5 & (1)\\[2ex] \dfrac{2}{ax}-\dfrac{3}{by}=2 & (2)\end{cases}$$

Multiply (1) by 2, and (2) by 3, and subtract.

$$\frac{6}{ax}-\frac{4}{by}=10 \qquad (3)$$

$$\frac{6}{ax}-\frac{9}{by}=\ 6 \qquad (4)$$

$$\frac{5}{by}=\ 4$$

$$5=4by$$

$$\therefore\ y=\frac{5}{4b}$$

Substitute the value of y in (1).

$$\frac{3}{ax}-\frac{8}{5}=5$$

$$\frac{3}{ax}=\frac{33}{5}$$

$$\therefore\ x=\frac{5}{11a}$$

12.
$$\begin{cases}\dfrac{a}{bx}+\dfrac{b}{ay}=a+b & (1)\\[2ex] \dfrac{b}{x}+\dfrac{a}{y}=a^2+b^2 & (2)\end{cases}$$

Multiply (1) by $\frac{b^2}{a}$, and subtract from (2).

$$\begin{aligned} \frac{b}{x}+\frac{a}{y} &= a^2+b^2 && (2)\\ \frac{b}{x}+\frac{b^3}{a^2y} &= b^2+\frac{b^3}{a} && (3)\\ \hline \left(a-\frac{b^3}{a^2}\right)\frac{1}{y} &= a^2-\frac{b^3}{a} &&\\ \frac{a^3-b^3}{a^2y} &= \frac{a^3-b^3}{a} &&\\ a &= a^2y && \end{aligned}$$

$$\therefore y=\frac{1}{a}$$

Substitute the value of y in (1).

$$\begin{aligned} \frac{a}{bx}+b &= a+b\\ \frac{a}{bx} &= a\\ \frac{1}{bx} &= 1\\ bx &= 1\\ \therefore x &= \frac{1}{b} \end{aligned}$$

Exercise 64. Page 177.

1. $$\begin{cases} x+y-8=0 & (1)\\ y+z-28=0 & (2)\\ x+z-14=0 & (3) \end{cases}$$

Add, $2x+2y+2z-50=0$

$$x+y+z=25 \quad (4)$$

(1) is $x+y \quad = 8$

Subtract, $z=17$

Substitute the value of z in (2) and (3).

$$\begin{aligned} y+17-28 &= 0\\ \therefore y &= 11\\ x+17-14 &= 0\\ \therefore x &= -3 \end{aligned}$$

2. $$\begin{cases} 4x+3y+2z=25 & (1)\\ 3x-2y+5z=20 & (2)\\ 10x-5y+3z=17 & (3) \end{cases}$$

Subtract (2) from (1).

$$x+5y-3z=5 \quad (4)$$

Add (4) to (3).

$$\begin{aligned} 11x &= 22\\ \therefore x &= 2 \end{aligned}$$

Substitute the value of x in (1) and (2).

$$\begin{aligned} 3y+2z &= 17 && (5)\\ -2y+5z &= 14 && (6) \end{aligned}$$

Multiply (5) by 2, and (6) by 3, and add.

$$\begin{aligned} 6y+4z &= 34\\ -6y+15z &= 42\\ \hline 19z &= 76\\ \therefore z &= 4 \end{aligned}$$

Substitute the value of z in (5).

$$\therefore y=3$$

3. $$\begin{cases} 5x-2y-2z=12 & (1)\\ x+y+z=8 & (2)\\ 7x+3y+4z=42 & (3) \end{cases}$$

(1) is $5x-2y-2z=12$

2×(2) is $2x+2y+2z=16$

Add, $7x \quad =28$

$$\therefore x=4$$

Substitute the value of x in (2) and (3).

$$\begin{aligned} y+z &= 4 && (4)\\ 3y+4z &= 14 && (5) \end{aligned}$$

(5) is $3y + 4z = 14$

$3 \times (4)$ is $3y + 3z = 12$

Subtract, $z = 2$

Substitute the value of z in (4).

$\therefore y = 2$

$$4. \quad \begin{cases} x - y + z = 11 & (1) \\ 3x + 3y - 2z = 60 & (2) \\ 10x - 5y - 3z = 0 & (3) \end{cases}$$

$2 \times (1)$ is $2x - 2y + 2z = 22$

(2) is $3x + 3y - 2z = 60$

Add, $5x + y = 82$ (4)

$3 \times (1)$ is $3x - 3y + 3z = 33$

(3) is $10x - 5y - 3z = 0$

Add, $13x - 8y = 33$ (5)

$8 \times (4)$ is $40x + 8y = 656$

(5) is $13x - 8y = 33$

Add, $53x = 689$

$\therefore x = 13$

Substitute the value of x in (4).

$\therefore y = 17$

Substitute the values of x and y in (1).

$\therefore z = 15$

$$5. \quad \begin{cases} 10x - y + 3z = 42 & (1) \\ 7x + 2y + z = 51 & (2) \\ 3x + 3y - z = 24 & (3) \end{cases}$$

(1) is $10x - y + 3z = 42$

$3 \times (3)$ is $9x + 9y - 3z = 72$

Add, $19x + 8y = 114$ (4)

(2) is $7x + 2y + z = 51$

(3) is $3x + 3y - z = 24$

Add, $10x + 5y = 75$

Divide by 5, $2x + y = 15$ (5)

(4) is $19x + 8y = 114$

$8 \times (5)$ is $16x + 8y = 120$

Subtract, $3x = -6$

$\therefore x = -2$

Substitute the value of x in (5).

$\therefore y = 19$

Substitute the values of x and y in (2).

$\therefore z = 27$

$$6. \quad \begin{cases} 5x + 2y - 3z = 160 & (1) \\ 3x + 9y + 8z = 115 & (2) \\ 2x - 3y - 5z = 45 & (3) \end{cases}$$

$8 \times (1)$ is $40x + 16y - 24z = 1280$

$3 \times (2)$ is $9x + 27y + 24z = 345$

Add, $49x + 43y = 1625$ (4)

$5 \times (1)$ is $25x + 10y - 15z = 800$

$3 \times (3)$ is $6x - 9y - 15z = 135$

Subtract, $19x + 19y = 665$

Divide by 19, $x + y = 35$ (5)

(4) is $49x + 43y = 1625$

$49 \times (5)$ is $49x + 49y = 1715$

Subtract, $6y = 90$

$\therefore y = 15$

Substitute the value of y in (5).

$\therefore x = 20$

Substitute the values of x and y in (1).

$\therefore z = -10$

$$7. \quad \begin{cases} 6x - 2y + 5z = 53 & (1) \\ 5x + 3y + 7z = 33 & (2) \\ x + y + z = 5 & (3) \end{cases}$$

(1) is $6x - 2y + 5z = 53$

$5 \times (3)$ is $5x + 5y + 5z = 25$

Subtract, $x - 7y = 28$ (4)

(2) is $5x + 3y + 7z = 33$

7×(3) is $7x + 7y + 7z = 35$

Subtract, $2x + 4y \quad = 2$

Divide by 2,

$x + 2y = 1$ (5)

(4) is $x - 7y = 28$

Subtract, $9y = -27$

$\therefore y = -3$

Substitute the value of y in (5).

$\therefore x = 7$

Substitute the values of x and y in (3).

$\therefore z = 1$

8. $$\begin{cases} 3x - 3y + 4z = 20 & (1) \\ 6x + 2y - 7z = 5 & (2) \\ 2x - y + 8z = 45 & (3) \end{cases}$$

(1) is $3x - 3y + 4z = 20$

3 × (3) is $6x - 3y + 24z = 135$

Subtract, $3x \quad + 20z = 115$ (4)

(2) is $6x + 2y - 7z = 5$

2 × (3) is $4x - 2y + 16z = 90$

Add, $10x \quad + 9z = 95$ (5)

10 × (4) is $30x + 200z = 1150$

3 × (5) is $30x + 27z = 285$

Subtract, $173z = 865$

$\therefore z = 5$

Substitute the value of z in 4.

$\therefore x = 5$

Substitute the values of x and z in (1).

$\therefore y = 5$

9. $$\begin{cases} 2x + 7y + 10z = 25 & (1) \\ x + y - z = 9 & (2) \\ 7x - 7y - 11z = 73 & (3) \end{cases}$$

(1) is $2x + 7y + 10z = 25$

2 × (2) is $2x + 2y - 2z = 18$

Subtract, $5y + 12z = 7$ (4)

(3) is $7x - 7y - 11z = 73$

7 × (2) is $7x + 7y - 7z = 63$

Subtract, $14y + 4z = -10$

Divide by 2,

$7y + 2z = -5$ (5)

(4) is $5y + 12z = 7$

6 × (5) is $42y + 12z = -30$

Subtract, $37y \quad = -37$

$\therefore y = -1$

Substitute the value of y in (4).

$\therefore z = 1$

Substitute the values of y and z in (2).

$\therefore x = 11$

10. $$\begin{cases} 5x + 2y - 20z = 20 & (1) \\ 3x - 6y + 7z = 51 & (2) \\ 4x + 8y - 9z = 53 & (3) \end{cases}$$

3 × (1) is $15x + 6y - 60z = 60$

(2) is $3x - 6y + 7z = 51$

Add, $18x \quad - 53z = 111$ (4)

4 × (1) is $20x + 8y - 80z = 80$

(3) is $4x + 8y - 9z = 53$

Subtract, $16x \quad - 71z = 27$ (5)

8 × (4) is $144x - 424z = 888$

9 × (5) is $144x - 639z = 243$

Subtract, $215z = 645$

$\therefore z = 3$

Substitute the value of z in (4).

$\therefore x = 15$

Substitute the values of x and z in (1).

$\therefore y = \frac{5}{2}$

11. $$\begin{cases} x+2y+10z=44 & (1)\\ 3x+3y+7z=384 & (2)\\ 2x+y+z=256 & (3)\end{cases}$$

$$\begin{aligned}
&3\times(1)\text{ is} && 3x+6y+30z=132\\
&(2)\text{ is} && 3x+3y+7z=384\\ \hline
&\text{Subtract,} && 3y+23z=-252 \quad (4)\\
&2\times(1)\text{ is} && 2x+4y+20z=88\\
&(3)\text{ is} && 2x+y+z=256\\ \hline
&\text{Subtract,} && 3y+19z=-168 \quad (5)\\
&(4)\text{ is} && 3y+23z=-252\\
&(5)\text{ is} && 3y+19z=-168\\ \hline
&\text{Subtract,} && 4z=-84\\
& && \therefore z=-21
\end{aligned}$$

Substitute the value of z in (4).

$$\therefore y=77$$

Substitute the values of y and z in (1).

$$\therefore x=100$$

12. $$\begin{cases} 10x=y+4z+56 & (1)\\ 3y=2x+3z-98 & (2)\\ 2z=x-3y-18 & (3)\end{cases}$$

Rearrange,

$$\begin{aligned}
10x-y-4z&=56 \quad (4)\\
2x-3y+3z&=98 \quad (5)\\
x-3y-2z&=18 \quad (6)
\end{aligned}$$

$$\begin{aligned}
&3\times(4)\text{ is} && 30x-3y-12z=168\\
&(5)\text{ is} && 2x-3y+3z=98\\ \hline
&\text{Subtract,} && 28x-15z=70 \quad (7)\\
&(5)\text{ is} && 2x-3y+3z=98\\
&(6)\text{ is} && x-3y-2z=18\\ \hline
&\text{Subtract,} && x+5z=80 \quad (8)\\
&(7)\text{ is} && 28x-15z=70\\
&28\times(8)\text{ is} && 28x+140z=2240\\ \hline
&\text{Subtract,} && 155z=2170\\
& && \therefore z=14
\end{aligned}$$

Substitute the value of z in (8).

$$\therefore x=10$$

Substitute the values of x and z in (1).

$$\therefore y=-12$$

13. $$\begin{cases} 3x-5y-2z=14 & (1)\\ 5x-8y-z=12 & (2)\\ x-3y-3z=1 & (3)\end{cases}$$

$$\begin{aligned}
&(1)\text{ is} && 3x-5y-2z=14\\
&2\times(2)\text{ is} && 10x-16y-2z=24\\ \hline
&\text{Subtract,} && 7x-11y=10 \quad (4)\\
&(3)\text{ is} && x-3y-3z=1\\
&3\times(2)\text{ is} && 15x-24y-3z=36\\ \hline
&\text{Subtract,} && 14x-21y=35\\
&\text{Divide by 7,} && 2x-3y=5 \quad (5)\\
&2\times(4)\text{ is} && 14x-22y=20\\
&7\times(5)\text{ is} && 14x-21y=35\\ \hline
&\text{Subtract,} && y=15
\end{aligned}$$

Substitute the value of y in (5).

$$\therefore x=25$$

Substitute the values of x and y in (1).

$$\therefore z=-7$$

14. $$\begin{cases} 2x+3y+z=31 & (1)\\ x-y+3z=13 & (2)\\ 10y+5x-2z=48 & (3)\end{cases}$$

$$\begin{aligned}
&(2)\text{ is} && x-y+3z=13\\
&3\times(1)\text{ is} && 6x+9y+3z=93\\ \hline
&\text{Subtract,} && 5x+10y=80\\
&\text{Divide by 5,} && x+2y=16 \quad (4)\\
&(3)\text{ is} && 5x+10y-2z=48\\
&2\times(1)\text{ is} && 4x+6y+2z=62\\ \hline
&\text{Add,} && 9x+16y=110 \quad (5)
\end{aligned}$$

$$\begin{array}{ll} 9 \times (4) \text{ is} & 9x + 18y = 144 \\ (5) \text{ is} & \underline{9x + 16y = 110} \\ \text{Subtract,} & 2y = 34 \end{array}$$

$$\therefore\ y = 17$$

Substitute the value of y in (4).

$$\therefore\ x = -18$$

Substitute the values of x and y in (1).

$$\therefore\ z = 16$$

15. $\left\{\begin{array}{rl} 2x + 3y - 4z = 1 & (1) \\ 10x - 6y + 12z = 6 & (2) \\ x + 12y + 2z = 5 & (3) \end{array}\right.$

$$\begin{array}{lll} 2 \times (1) \text{ is} & 4x + 6y - 8z = 2 & \\ (2) \text{ is} & \underline{10x - 6y + 12z = 6} & \\ \text{Add,} & 14x + 4z = 8 & \\ \text{Divide by 2,} & 7x + 2z = 4 & (4) \\ 4 \times (1) \text{ is} & 8x + 12y - 16z = 4 & \\ (3) \text{ is} & \underline{x + 12y + 2z = 5} & \\ \text{Subtract,} & 7x - 18z = -1 & (5) \\ (4) \text{ is} & 7x + 2z = 4 & \\ (5) \text{ is} & \underline{7x - 18z = -1} & \\ \text{Subtract,} & 20z = 5 & \end{array}$$

$$\therefore\ z = \tfrac{1}{4}$$

Substitute the value of z in (4).

$$\therefore\ x = \tfrac{1}{2}$$

Substitute the values of x and z in (1).

$$\therefore\ y = \tfrac{1}{3}$$

16. $\left\{\begin{array}{rl} 3x + 6y + 2z = 3 & (1) \\ 12y + 4z - 6x = 2 & (2) \\ 9x + 18y - 4z = 4 & (3) \end{array}\right.$

$$\begin{array}{ll} (1) \text{ is} & 3x + 6y + 2z = 3 \\ \tfrac{1}{2} \times (2) \text{ is} & \underline{-3x + 6y + 2z = 1} \\ \text{Subtract,} & 6x = 2 \end{array}$$

$$\therefore\ x = \tfrac{1}{3}$$

$$\begin{array}{ll} 3 \times (1) \text{ is} & 9x + 18y + 6z = 9 \\ (3) \text{ is} & \underline{9x + 18y - 4z = 4} \\ \text{Subtract,} & 10z = 5 \end{array}$$

$$\therefore\ z = \tfrac{1}{2}$$

Substitute the values of x and z in (1).

$$\therefore\ y = \tfrac{1}{6}$$

17. $\left\{\begin{array}{rl} 2x + y + 2z = 3 & (1) \\ 5y - 4x - 4z = 1 & (2) \\ 3x + 9y + z = 9 & (3) \end{array}\right.$

$$\begin{array}{ll} 2 \times (1) \text{ is} & 4x + 2y + 4z = 6 \\ (2) \text{ is} & \underline{-4x + 5y - 4z = 1} \\ \text{Add,} & 7y = 7 \end{array}$$

$$\therefore\ y = 1$$

Substitute the value of y in (2) and (3).

$$\begin{array}{lll} & 4x + 4z = 4 & (4) \\ & 3x + z = 0 & (5) \\ \tfrac{1}{4} \times (4) \text{ is} & x + z = 1 & \\ (5) \text{ is} & \underline{3x + z = 0} & \\ \text{Subtract,} & 2x = -1 & \end{array}$$

$$\therefore\ x = -\tfrac{1}{2}$$

Substitute the value of x in (5).

$$\therefore\ z = \tfrac{3}{2}$$

18. $\left\{\begin{array}{rl} 3x + 2y + z = 20\tfrac{1}{2} & (1) \\ 2x - y + 3z = 26\tfrac{1}{2} & (2) \\ x + y + 10z = 55 & (3) \end{array}\right.$

$$\begin{array}{lll} (1) \text{ is} & 3x + 2y + z = 20\tfrac{1}{2} & \\ 2 \times (2) \text{ is} & \underline{4x - 2y + 6z = 53} & \\ \text{Add,} & 7x + 7z = 73\tfrac{1}{2} & \\ \text{Divide by 7,} & x + z = 10\tfrac{1}{2} & (4) \\ (2) \text{ is} & 2x - y + 3z = 26\tfrac{1}{2} & \\ (3) \text{ is} & \underline{x + y + 10z = 55} & \\ \text{Add,} & 3x + 13z = 81\tfrac{1}{2} & (5) \end{array}$$

$$\begin{array}{ll} 3\times(4) \text{ is} & 3x + 3z = 31\tfrac{1}{2} \\ (5) \text{ is} & 3x + 13z = 81\tfrac{1}{2} \\ \hline \text{Subtract,} & \qquad 10z = 50 \end{array}$$

$$\therefore z = 5$$

Substitute the value of z in (4).

$$\therefore x = 5\tfrac{1}{2}$$

Substitute the values of x and z in (2).

$$\therefore y = -\tfrac{1}{2}$$

19. $$\begin{cases} \dfrac{1}{x} - \dfrac{2}{y} + 4 = 0 & (1) \\ \dfrac{1}{y} - \dfrac{1}{z} + 1 = 0 & (2) \\ \dfrac{2}{z} + \dfrac{3}{x} - 14 = 0 & (3) \end{cases}$$

$$\begin{array}{ll} (1) \text{ is} & \dfrac{1}{x} - \dfrac{2}{y} + 4 = 0 \\ 2\times(2) \text{ is} & \dfrac{2}{y} - \dfrac{2}{z} + 2 = 0 \\ \hline \text{Add,} & \dfrac{1}{x} - \dfrac{2}{z} + 6 = 0 \quad (4) \end{array}$$

$$\begin{array}{ll} (3) \text{ is} & \dfrac{3}{x} + \dfrac{2}{z} - 14 = 0 \\ (4) \text{ is} & \dfrac{1}{x} - \dfrac{2}{z} + 6 = 0 \\ \hline \text{Add,} & \dfrac{4}{x} \qquad - 8 = 0 \end{array}$$

$$\therefore x = \tfrac{1}{2}$$

Substitute the value of x in (1) and (3).

$$\therefore y = \tfrac{1}{3}$$

$$z = \tfrac{1}{4}$$

20. $$\begin{cases} \dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} = 36 & (1, \\ \dfrac{1}{x} + \dfrac{3}{y} - \dfrac{1}{z} = 28 & (2) \\ \dfrac{1}{x} + \dfrac{1}{3y} + \dfrac{1}{2z} = 20 & (3) \end{cases}$$

$$\begin{array}{ll} (1) \text{ is} & \dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} = 36 \\ (2) \text{ is} & \dfrac{1}{x} + \dfrac{3}{y} - \dfrac{1}{z} = 28 \\ \hline \text{Add,} & \dfrac{2}{x} + \dfrac{4}{y} \qquad = 64 \end{array}$$

$$\text{Divide by 2,} \quad \dfrac{1}{x} + \dfrac{2}{y} = 32 \quad (4)$$

$$\begin{array}{ll} (1) \text{ is} & \dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} = 36 \\ 2\times(3) \text{ is} & \dfrac{2}{x} + \dfrac{2}{3y} + \dfrac{1}{z} = 40 \\ \hline \text{Subtract,} & \dfrac{1}{x} - \dfrac{1}{3y} \qquad = 4 \quad (5) \end{array}$$

$$\begin{array}{ll} (4) \text{ is} & \dfrac{1}{x} + \dfrac{2}{y} = 32 \\ (5) \text{ is} & \dfrac{1}{x} - \dfrac{1}{3y} = 4 \\ \hline \text{Subtract,} & \qquad \dfrac{7}{3y} = 28 \end{array}$$

$$\therefore y = \tfrac{1}{12}$$

Substitute the value of y in (4).

$$\therefore x = \tfrac{1}{8}$$

Substitute the values of x and y in (1).

$$\therefore z = \tfrac{1}{16}$$

21.
$$\begin{cases} \dfrac{1}{x}+\dfrac{2}{y}-\dfrac{3}{z}=1 & (1) \\[2ex] \dfrac{5}{x}+\dfrac{4}{y}+\dfrac{6}{z}=24 & (2) \\[2ex] \dfrac{7}{x}-\dfrac{8}{y}+\dfrac{9}{z}=14 & (3) \end{cases}$$

2 × (1) is $\dfrac{2}{x}+\dfrac{4}{y}-\dfrac{6}{z}=2$

(2) is $\dfrac{5}{x}+\dfrac{4}{y}+\dfrac{6}{z}=24$

Add, $\dfrac{7}{x}+\dfrac{8}{y}=26$ (4)

3 × (1) is $\dfrac{3}{x}+\dfrac{6}{y}-\dfrac{9}{z}=3$

(3) is $\dfrac{7}{x}-\dfrac{8}{y}+\dfrac{9}{z}=14$

Add, $\dfrac{10}{x}-\dfrac{2}{y}=17$ (5)

(4) is $\dfrac{7}{x}+\dfrac{8}{y}=26$

4 × (5) is $\dfrac{40}{x}-\dfrac{8}{y}=68$

Add, $\dfrac{47}{x}=94$

$\therefore x=\frac{1}{2}$

Substitute the value of x in (4).

$\therefore y=\frac{2}{3}$

Substitute the values of x and y in (1).

$\therefore z=\frac{3}{4}$

22.
$$\begin{cases} \dfrac{4}{x}-\dfrac{3}{y}=\dfrac{1}{20} & (1) \\[2ex] \dfrac{2}{z}-\dfrac{3}{x}=\dfrac{1}{15} & (2) \\[2ex] \dfrac{4}{z}-\dfrac{5}{y}=\dfrac{1}{12} & (3) \end{cases}$$

2 × (2) is $\dfrac{4}{z}-\dfrac{6}{x}=\dfrac{2}{15}$

(3) is $\dfrac{4}{z}-\dfrac{5}{y}=\dfrac{1}{12}$

Subtract, $\dfrac{6}{x}-\dfrac{5}{y}=-\dfrac{1}{20}$ (4)

2 × (4) is $\dfrac{12}{x}-\dfrac{10}{y}=-\dfrac{1}{10}$

3 × (1) is $\dfrac{12}{x}-\dfrac{9}{y}=\dfrac{3}{20}$

Subtract, $\dfrac{1}{y}=\dfrac{1}{4}$

$\therefore y=4$

Substitute the value of y in (1) and (3).

$\therefore x=5$

$z=3$

23.
$$\begin{cases} \dfrac{1}{x}+\dfrac{1}{y}-\dfrac{1}{a}=0 & (1) \\[2ex] \dfrac{1}{x}+\dfrac{1}{z}-\dfrac{1}{b}=0 & (2) \\[2ex] \dfrac{1}{y}+\dfrac{1}{z}-\dfrac{1}{c}=0 & (3) \end{cases}$$

Add, $\dfrac{2}{x}+\dfrac{2}{y}+\dfrac{2}{z}-\dfrac{1}{a}-\dfrac{1}{b}-\dfrac{1}{c}=0$ (4)

(4) is $\frac{2}{x}+\frac{2}{y}+\frac{2}{z}-\frac{1}{a}-\frac{1}{b}-\frac{1}{c}=0$

2 × (1) is $\frac{2}{x}+\frac{2}{y}\quad-\frac{2}{a}\quad=0$

Subtract, $\frac{2}{z}+\frac{1}{a}-\frac{1}{b}-\frac{1}{c}=0$

$$\therefore\ \frac{2}{z}=\frac{1}{b}+\frac{1}{c}-\frac{1}{a}=\frac{ac+ab-bc}{abc}$$

$$\therefore\ z=\frac{2abc}{ac+ab-bc}$$

(4) is $\frac{2}{x}+\frac{2}{y}+\frac{2}{z}-\frac{1}{a}-\frac{1}{b}-\frac{1}{c}=0$

2 × (2) is $\frac{2}{x}\quad+\frac{2}{z}\quad-\frac{2}{b}\quad=0$

Subtract, $\frac{2}{y}\quad-\frac{1}{a}+\frac{1}{b}-\frac{1}{c}=0$

$$\therefore\ \frac{2}{y}=\frac{1}{a}+\frac{1}{c}-\frac{1}{b}=\frac{bc+ab-ac}{abc}$$

$$\therefore\ y=\frac{2abc}{ab+bc-ac}$$

(4) is $\frac{2}{x}+\frac{2}{y}+\frac{2}{z}-\frac{1}{a}-\frac{1}{b}-\frac{1}{c}=0$

2 × (3) is $\frac{2}{y}+\frac{2}{z}\quad-\frac{2}{c}=0$

Subtract, $\frac{2}{x}\quad-\frac{1}{a}-\frac{1}{b}+\frac{1}{c}=0$

$$\therefore\ \frac{2}{x}=\frac{1}{a}+\frac{1}{b}-\frac{1}{c}=\frac{bc+ac-ab}{abc}$$

$$\therefore\ x=\frac{2abc}{bc+ac-ab}$$

24.
$$\begin{cases}\frac{x}{2}+\frac{y}{3}+\frac{z}{4}=62 & (1)\\[1ex] \frac{x}{3}+\frac{y}{4}+\frac{z}{5}=47 & (2)\\[1ex] \frac{x}{4}+\frac{y}{5}+\frac{z}{6}=38 & (3)\end{cases}$$

Multiply (1) by 12, (2) by 60, and (3) by 60.

$$\begin{array}{lrr}
 & 6x+4y+3z=744 & (4)\\
 & 20x+15y+12z=2820 & (5)\\
 & 15x+12y+10z=2280 & (6)\\
\text{(5) is} & 20x+15y+12z=2820 & \\
4\times(4)\text{ is} & 24x+16y+12z=2976 & \\
\hline
\text{Subtract,} & 4x+y=156 & (7)\\
10\times(4)\text{ is} & 60x+40y+30z=7440 & \\
3\times(6)\text{ is} & 45x+36y+30z=6840 & \\
\hline
\text{Subtract,} & 15x+4y=600 & (8)\\
4\times(7)\text{ is} & 16x+4y=624 & \\
\text{(8) is} & 15x+4y=600 & \\
\hline
\text{Subtract,} & x=24 &
\end{array}$$

Substitute the value of x in (7).

$$\therefore\ y=60$$

Substitute the values of x and y in (4).

$$\therefore\ z=120$$

25. $$\begin{cases}\dfrac{2}{x}+\dfrac{1}{y}-\dfrac{3}{z}=0 & (1)\\[2ex] \dfrac{3}{z}-\dfrac{2}{y}-2=0 & (2)\\[2ex] \dfrac{1}{x}+\dfrac{1}{z}-\dfrac{4}{3}=0 & (3)\end{cases}$$

$$\begin{array}{lrr}
\text{(1) is} & \dfrac{2}{x}+\dfrac{1}{y}-\dfrac{3}{z}=0 & \\[2ex]
\text{(2) is} & -\dfrac{2}{y}+\dfrac{3}{z}=2 & \\[2ex]
\hline
\text{Add,} & \dfrac{2}{x}-\dfrac{1}{y}=2 & (4)\\[2ex]
\text{(1) is} & \dfrac{2}{x}+\dfrac{1}{y}-\dfrac{3}{z}=0 & \\[2ex]
3\times(3)\text{ is} & \dfrac{3}{x}+\dfrac{3}{z}=4 & \\[2ex]
\hline
\text{Add,} & \dfrac{5}{x}+\dfrac{1}{y}=4 & (5)
\end{array}$$

$$\begin{array}{lr}
\text{(4) is} & \dfrac{2}{x}-\dfrac{1}{y}=2\\[2ex]
\text{(5) is} & \dfrac{5}{x}+\dfrac{1}{y}=4\\[2ex]
\hline
\text{Add,} & \dfrac{7}{x}=6
\end{array}$$

$$\therefore\ x=\tfrac{7}{6}$$

Substitute the value of x in (4).

$$\therefore\ y=-\tfrac{7}{2}$$

Substitute the values of x and y in (1).

$$\therefore\ z=\tfrac{21}{10}$$

26. $$\begin{cases} \dfrac{15}{x}-\dfrac{4}{y}+\dfrac{5}{z}=38 & (1) \\[2ex] \dfrac{2}{x}+\dfrac{3}{y}+\dfrac{12}{z}=61 & (2) \\[2ex] \dfrac{8}{x}-\dfrac{5}{y}+\dfrac{40}{z}=161 & (3) \end{cases}$$

2×(1) is $\dfrac{30}{x}-\dfrac{8}{y}+\dfrac{10}{z}=76$

15×(2) is $\dfrac{30}{x}+\dfrac{45}{y}+\dfrac{180}{z}=915$

Subtract, $\dfrac{53}{y}+\dfrac{170}{z}=839$ (4)

(3) is $\dfrac{8}{x}-\dfrac{5}{y}+\dfrac{40}{z}=161$

4×(2) is $\dfrac{8}{x}+\dfrac{12}{y}+\dfrac{48}{z}=244$

Subtract, $\dfrac{17}{y}+\dfrac{8}{z}=83$ (5)

17×(4) is $\dfrac{901}{y}+\dfrac{2890}{z}=14263$

53×(5) is $\dfrac{901}{y}+\dfrac{424}{z}=4399$

Subtract, $\dfrac{2466}{z}=9864$

$\therefore z=\frac{1}{4}$

Substitute the value of z in (5).

$\therefore y=\frac{1}{3}$

Substitute the values of y and z in (1).

$\therefore x=\frac{1}{2}$

Exercise 65. Page 180.

1. If A gives B $100, A will then have half as much money as B; but if B gives A $100, B will have one-third as much as A. How much has each?

Let $x=$ the number of dollars A has,
and $y=$ the number of dollars B has.

Then, after A gives B $100,

$x-100=$ the number of dollars A has.
$y+100=$ the number of dollars B has.

$$\therefore y+100=2(x-100). \quad (1)$$

Again, if B gives A $100,

$x+100=$ the number of dollars A has.
$y-100=$ the number of dollars B has.

$$\therefore x+100=3(y-100) \quad (2)$$

From (1), $2x-y=300$ (3)
From (2), $x-3y=-400$ (4)
(3) is $2x-y=300$
2 × (4) is $2x-6y=-800$
Subtract, $5y=1100$

$\therefore y=220$
$x=260$

Therefore A has $260 and B $220.

2. If the greater of two numbers is divided by the smaller, the quotient is 4 and the remainder 0.37; but if the smaller is divided by the greater, the quotient is 0.23 and the remainder 0.0149. Find the numbers.

Let $x =$ the greater number,
and $y =$ the smaller number.

Then,
$$\frac{x}{y} = 4 + \frac{0.37}{y} \qquad (1)$$
$$\frac{y}{x} = 0.23 + \frac{0.0149}{x} \qquad (2)$$

From (1), $x = 4y + 0.37 \qquad (3)$
From (2), $y = 0.23x + 0.0149 \qquad (4)$

Substitute the value of y from (4) in (3).
$$\therefore\ x = 0.92x + 0.0596 + 0.37$$
$$0.08x = 0.4296$$
$$\therefore\ x = 5.37$$
$$y = 1.25$$

Therefore the numbers are 5.37 and 1.25.

3. A certain number of persons paid a bill. If there had been 10 persons more, each would have paid \$2 less; but if there had been 5 persons less, each would have paid \$2.50 more. Find the number of persons and the amount of the bill.

Let $x =$ the number of persons,
and $y =$ the number of dollars each paid.
Then, $xy =$ the number of dollars they all paid.

If there had been 10 persons more, and each had paid \$2 less,
then, $x + 10 =$ the number of persons,
$y - 2 =$ the number of dollars each paid,
and $(x + 10)(y - 2) =$ the number of dollars they all paid.

But the same number of dollars is to be paid in each case.
$$\therefore\ (x + 10)(y - 2) = xy \qquad (1)$$

Again, if there had been 5 persons less, and each had paid \$2.50 more,
then, $x - 5 =$ the number of persons,
$y + 2\frac{1}{2} =$ the number of dollars each paid,
and $(x - 5)(y + 2\frac{1}{2}) =$ the number of dollars paid by all.
$$(x - 5)(y + 2\tfrac{1}{2}) = xy \qquad (2)$$

From (1),
$$xy + 10y - 2x - 20 = xy$$
$$2x - 10y = -20 \qquad (3)$$

From (2),

$$xy - 5y + 2\tfrac{1}{2}x - 12\tfrac{1}{2} = xy$$

$$5x - 10y = 25 \qquad (4)$$

Subtract (3) from (4), $3x = 45$

$$\therefore\ x = 15$$

$$y = 5$$

$$xy = 75$$

Therefore there were 15 persons, and the amount of the bill was $75.

4. A train proceeded a certain distance at a uniform rate. If the speed had been 6 miles an hour more, the time occupied would have been 5 hours less; but if the speed had been 6 miles an hour less, the time occupied would have been $7\frac{1}{2}$ hours more. Find the distance.

Let x = the number of hours the train travels,
and y = the number of miles it travels per hour.
Then, xy = the total number of miles travelled.

If the speed had been 6 miles an hour more, and the time 5 hours less,
then, $x - 5$ = the number of hours.
$y + 6$ = the number of miles per hour.
$(x - 5)(y + 6)$ = the total number of miles travelled.

$$\therefore\ (x - 5)(y + 6) = xy \qquad (1)$$

Again, if the speed had been 6 miles an hour less, and the time $7\frac{1}{2}$ hours more,
then, $x + 7\frac{1}{2}$ = the number of hours,
$y - 6$ = the number of miles per hour,
and $(x + 7\frac{1}{2})(y - 6)$ = the total number of miles travelled.

$$\therefore\ (x + 7\tfrac{1}{2})(y - 6) = xy \qquad (2)$$

From (1),

$$xy + 6x - 5y - 30 = xy$$

$$6x - 5y = 30 \qquad (3)$$

From (2),

$$xy - 6x + 7\tfrac{1}{2}y - 45 = xy$$

$$6x - 7\tfrac{1}{2}y = -45 \qquad (4)$$

Subtract (4) from (3),

$$2\tfrac{1}{2}y = 75$$

$$\therefore\ y = 30$$

$$x = 30$$

$$xy = 900$$

Therefore the distance is 900 miles.

5. A man bought 10 cows and 50 sheep for $750. He sold the cows at a profit of 10 per cent, and the sheep at a profit of 30 per cent, and received in all $875. Find the average cost of a cow and of a sheep.

Let $x =$ the number of dollars in the average cost of a cow,

and $y =$ the number of dollars in the average cost of a sheep.

Then, $10x =$ the number of dollars the 10 cows cost,

and $50y =$ the number of dollars the 50 sheep cost.

$\therefore\ 10x + 50y = 750$ (1)

Also, $\frac{110}{100} \times 10x =$ the number of dollars he received for the cows.

$\frac{130}{100} \times 50y =$ the number of dollars he received for the sheep.

$\therefore\ \frac{110}{100} \times 10x + \frac{130}{100} \times 50y = 875$ (2)

From (1), $x + 5y = 75$ (3)

From (2), $11x + 65y = 875$ (4)

$11 \times (3)$ is $11x + 55y = 825$

(4) is $11x + 65y = 875$

Subtract, $10y = 50$

$\therefore\ y = 5$

$x = 50$

Therefore the average cost of the cows was $50, and of the sheep $5.

6. It is 40 miles from Dover to Portland. A sets out from Dover, and B from Portland, at 7 o'clock A.M., to meet each other. A walks at the rate of 3½ miles an hour, but stops 1 hour on the way; B walks at the rate of 2½ miles an hour. At what time of day and how far from Portland will they meet?

Let $x =$ the number of miles from Portland to the meeting point,

and $y =$ the number of hours that A walks.

Then, $40 - x =$ the number of miles from Dover to the meeting place.

$y + 1 =$ the number of hours B walks.

$3\frac{1}{2}y =$ the number of miles A walks.

$2\frac{1}{2}(y + 1) =$ the number of miles B walks.

$\therefore\ 3\frac{1}{2}y = 40 - x$ (1)

$2\frac{1}{2}(y + 1) = x$ (2)

From (1), $2x + 7y = 80$

From (2), $2x - 5y = 5$

Subtract, $12y = 75$

$$\therefore y = 6\tfrac{1}{4}$$

$$y + 1 = 7\tfrac{1}{4}$$

$$x = 18\tfrac{1}{8}$$

Therefore they meet at a point $18\frac{1}{8}$ miles from Portland $7\frac{1}{4}$ hours after they start, that is, at quarter past 2 o'clock, P.M.

7. The sum of two numbers is 35, and their difference exceeds one-fifth of the smaller number by 2. Find the numbers.

Let $x =$ the larger number,
and $y =$ the smaller number.

Then, $x + y = 35$ (1)

$x - y = \frac{y}{5} + 2$ (2)

Subtract, $2y = 33 - \frac{y}{5}$

$$10y = 165 - y$$

$$\therefore y = 15$$

$$x = 20$$

Therefore the numbers are 20 and 15.

8. If the greater of two numbers is divided by the smaller, the quotient is 7 and the remainder 4; but if three times the greater number is divided by twice the smaller, the quotient is 11 and the remainder 4. Find the numbers.

Let $x =$ the greater number,
and $y =$ the smaller number.

Then, $\frac{x}{y} = 7 + \frac{4}{y}$ (1)

$\frac{3x}{2y} = 11 + \frac{4}{2y}$ (2)

From (1), $x = 7y + 4$ (3)

From (2), $3x = 22y + 4$ (4)

3 × (3) is $3x - 21y = 12$

(4) is $3x - 22y = 4$

Subtract, $y = 8$

$$\therefore x = 60$$

Therefore the numbers are 60 and 8.

9. If 3 yards of velvet and 12 yards of silk cost \$60, and 4 yards of velvet and 5 yards of silk cost \$58, what is the price of a yard of velvet and of a yard of silk?

Let $x =$ the number of dollars one yard of velvet costs,
and $y =$ the number of dollars one yard of silk costs.

$$
\begin{array}{lrcll}
\text{Then,} & 3x + 12y &=& 60 & (1) \\
 & 4x + 5y &=& 58 & (2) \\
4 \times (1) \text{ is} & 12x + 48y &=& 240 & \\
3 \times (2) \text{ is} & \underline{12x + 15y} &=& \underline{174} & \\
\text{Subtract,} & 33y &=& 66 & \\
 & \therefore\ y &=& 2 & \\
 & x &=& 12 &
\end{array}
$$

Therefore the velvet costs \$12 a yard, and the silk \$2 a yard.

10. If 5 bushels of wheat, 4 of rye, and 3 of oats are sold for \$9; 3 bushels of wheat, 5 of rye, and 6 of oats for \$8.75; and 2 bushels of wheat, 3 of rye, and 9 of oats for \$7.25; what is the price per bushel of each kind of grain?

Let $x =$ the number of dollars one bushel of wheat costs,
$y =$ the number of dollars one bushel of rye costs,
and $z =$ the number of dollars one bushel of oats costs.

$$
\begin{array}{lrcll}
\text{Then,} & 5x + 4y + 3z &=& 9 & (1) \\
 & 3x + 5y + 6z &=& 8\frac{3}{4} & (2) \\
 & 2x + 3y + 9z &=& 7\frac{1}{4} & (3) \\
2 \times (1) \text{ is} & 10x + 8y + 6z &=& 18 & \\
(2) \text{ is} & \underline{3x + 5y + 6z} &=& \underline{8\frac{3}{4}} & \\
\text{Subtract,} & 7x + 3y &=& 9\frac{1}{4} & (4) \\
3 \times (1) \text{ is} & 15x + 12y + 9z &=& 27 & \\
(3) \text{ is} & \underline{2x + 3y + 9z} &=& \underline{7\frac{1}{4}} & \\
\text{Subtract,} & 13x + 9y &=& 19\frac{3}{4} & (5) \\
3 \times (4) \text{ is} & 21x + 9y &=& 27\frac{3}{4} & \\
(5) \text{ is} & \underline{13x + 9y} &=& \underline{19\frac{3}{4}} & \\
\text{Subtract,} & 8x &=& 8 & \\
 & \therefore\ x &=& 1 & \\
 & y &=& \frac{3}{4} & \\
 & z &=& \frac{1}{3} &
\end{array}
$$

Therefore a bushel of wheat costs \$1.00, a bushel of rye \$0.75, and a bushel of oats \$0.33⅓.

11. A certain fraction becomes equal to $\frac{1}{2}$ if 3 is added to its numerator and 1 to its denominator, and equal to $\frac{1}{4}$ if 3 is subtracted from its numerator and from its denominator. Find the fraction.

Let $\frac{x}{y} =$ the required fraction.

Then, $$\frac{x+3}{y+1} = \frac{1}{2} \qquad (1)$$

and $$\frac{x-3}{y-3} = \frac{1}{4} \qquad (2)$$

From (1), $$2x + 6 = y + 1$$

$$2x - y = -5 \qquad (3)$$

From (2), $$4x - 12 = y - 3$$

$$4x - y = 9 \qquad (4)$$

Subtract (3) from (4), $$2x = 14$$

$$\therefore x = 7$$

$$y = 19$$

Therefore the required fraction is $\frac{7}{19}$.

12. A certain fraction becomes equal to $\frac{9}{11}$ if 1 is added to double its numerator, and equal to $\frac{1}{3}$ if 3 is subtracted from its numerator and from its denominator. Find the fraction.

Let $\frac{x}{y} =$ the required fraction.

Then, $$\frac{2x+1}{y} = \frac{9}{11} \qquad (1)$$

and $$\frac{x-3}{y-3} = \frac{1}{3} \qquad (2)$$

From (1), $$22x + 11 = 9y$$

$$22x - 9y = -11 \qquad (3)$$

From (2), $$3x - 9 = y - 3$$

$$3x - y = 6 \qquad (4)$$

$$\begin{array}{ll} \text{(3) is} & 22x - 9y = -11 \\ 9 \times \text{(4) is} & \underline{27x - 9y = 54} \\ \text{Subtract,} & 5x = 65 \end{array}$$

$$\therefore x = 13$$

$$y = 33$$

Therefore the required fraction is $\frac{13}{33}$.

13. Find two fractions with numerators 11 and 5 respectively, such that their sum is $1\frac{4}{9}$, and if their denominators are interchanged their sum is $2\frac{1}{9}$.

Let $\frac{11}{x}$ = the first fraction,

and $\frac{5}{y}$ = the second fraction.

Then, $$\frac{11}{x}+\frac{5}{y}=\frac{13}{9} \tag{1}$$

and $$\frac{11}{y}+\frac{5}{x}=\frac{19}{9} \tag{2}$$

$5 \times (1)$ is $$\frac{55}{x}+\frac{25}{y}=\frac{65}{9}$$

$11 \times (2)$ is $$\frac{55}{x}+\frac{121}{y}=\frac{209}{9}$$

Subtract, $$\frac{96}{y}=\frac{144}{9}$$

$$\therefore y=6$$
$$x=18$$

Therefore the required fractions are $\frac{11}{18}$ and $\frac{5}{6}$.

14. There are two fractions with denominators 20 and 16 respectively. The fraction formed by taking for a numerator the sum of the numerators, and for a denominator the sum of the denominators, of the given fractions, is equal to $\frac{1}{3}$; and the fraction formed by taking for a numerator the difference of the numerators, and for a denominator the difference of the denominators, of the given fractions, is equal to $\frac{1}{2}$. Find the fractions.

Let $\frac{x}{20}$ = the first fraction,

and $\frac{y}{16}$ = the second fraction.

Then, $$\frac{x+y}{36}=\frac{1}{3} \tag{1}$$

and $$\frac{x-y}{4}=\frac{1}{2} \tag{2}$$

From (1), $x+y=12$
From (2), $x-y=\ \ 2$
Add, $2x \quad =14$

$$\therefore x=7$$
$$y=5$$

Therefore the required fractions are $\frac{7}{20}$ and $\frac{5}{16}$.

15. The sum of two digits of a number is 9, and if 27 is subtracted from the number, the digits will be reversed. Find the number.

Let $x =$ the tens' digit,
and $y =$ the units' digit.
Then, $10x + y =$ the number.

$$x + y = 9 \quad (1)$$
$$10x + y - 27 = 10y + x \quad (2)$$

From (2), $9x - 9y = 27$

$$x - y = 3$$

(1) is $x + y = 9$

Add, $2x = 12$

$$\therefore x = 6$$
$$y = 3$$

Therefore the number is 63.

16. The sum of the two digits of a number is 9, and if the number is divided by the sum of the digits, the quotient is 5. Find the number.

Let $x =$ the tens' digit,
and $y =$ the units' digit.
Then, $10x + y =$ the number.

$$x + y = 9 \quad (1)$$
$$\frac{10x + y}{x + y} = 5 \quad (2)$$

From (2), $10x + y = 45$ (3)

Subtract (1) from (3), $9x = 36$

$$\therefore x = 4$$
$$y = 5$$

Therefore the number is 45.

17. A certain number is expressed by two digits. The sum of the digits is 11. If the digits are reversed, the new number exceeds the given number by 27. Find the number.

Let $x =$ the tens' digit,
and $y =$ the units' digit.
Then, $x + y = 11$ (1)

$$10y + x = 10x + y + 27 \quad (2)$$

From (2), $9y - 9x = 27$

$$\therefore x - y = -3 \quad (3)$$

Add (1) and (3),
$$2x = 8$$
$$\therefore\ x = 4$$
$$y = 7$$

Therefore the number is 47.

18. A certain number is expressed by three digits. The sum of the digits is 21. The sum of the first and last digits is twice the middle digit. If the hundreds' and tens' digits are interchanged, the number is diminished by 90. Find the number.

Let $x =$ the hundreds' digit,
$y =$ the tens' digit,
and $z =$ the units' digit.

Then,
$$x + y + z = 21 \tag{1}$$
$$x + z = 2y \tag{2}$$
$$100x + 10y + z = 100y + 10x + z + 90 \tag{3}$$

From (3),
$$90x - 90y = 90$$
$$x - y = 1 \tag{4}$$

Substitute the value of $x + z$ from (4) in (1).
$$3y = 21$$
$$\therefore\ y = 7$$
$$x = 8$$
$$z = 6$$

Therefore the number is 876.

19. A certain number is expressed by three digits, the units' digit being zero. If the hundreds' and tens' digits are interchanged, the number is diminished by 180. If the hundreds' digit is halved, and the tens' and units' digits are interchanged, the number is diminished by 336. Find the number.

Let $x =$ the hundreds' digit,
and $y =$ the tens' digit.

Then, $100x + 10y =$ the number.
$$\therefore\ 100x + 10y = 100y + 10x + 180 \tag{1}$$
$$100x + 10y = 100\frac{x}{2} + y + 336 \tag{2}$$

From (1),
$$90x - 90y = 180$$
$$x - y = 2 \tag{3}$$

From (2),
$$50x + 9y = 336 \tag{4}$$

$$\begin{array}{ll} 9\times(3) \text{ is} & 9x-9y=\ \ 18 \\ (4) \text{ is} & \underline{50x+9y=336} \\ \text{Add,} & 59x \qquad\ =354 \end{array}$$

$$\therefore\ x=6$$

$$y=4$$

Therefore the number is 640.

20. A number is expressed by three digits. If the digits are reversed, the new number exceeds the given number by 99. If the number is divided by nine times the sum of its digits, the quotient is 3. The sum of the hundreds' and units' digits exceeds the tens' digit by 1. Find the number.

Let $x=$ the hundreds' digit,

$y=$ the tens' digit,

and $z=$ the units' digit.

Then, $$100z+10y+x=100x+10y+z+99 \quad (1)$$

$$\frac{100x+10y+z}{9(x+y+z)}=3 \quad (2)$$

$$x+z=y+1 \quad (3)$$

From (1), $$99x-99z=-99$$

$$x-z=-1 \quad (4)$$

From (2), $$100x+10y+z=27x+27y+27z$$

$$73x-17y-26z=0 \quad (5)$$

From (3), $$x-y+z=1 \quad (6)$$

From (4), $$x=z-1$$

Substitute this value of x in (5) and (6).

$$73z-73-17y-26z=0$$

$$47z-17y=73 \quad (7)$$

$$z-1-y+z=1$$

$$2z-y=2 \quad (8)$$

$$\begin{array}{ll} 17\times(8) \text{ is} & 34z-17y=34 \\ (7) \text{ is} & \underline{47z-17y=73} \\ \text{Subtract,} & 13z \qquad\ =39 \end{array}$$

$$\therefore\ z=3$$

$$y=4$$

$$x=2$$

Therefore the number is 243.

21. A boatman rows 20 miles down a river and back in 8 hours. He finds that he can row 5 miles down the river in the same time that he rows 3 miles up the river. Find the time he was rowing down and up respectively.

Let $x =$ the boatman's rate in still water,
and $y =$ the rate of the current.

Then, $\dfrac{1}{x+y} =$ the number of hours it takes him to row 1 mile down stream.

$\dfrac{1}{x-y} =$ the number of hours it takes him to row 1 mile up stream.

$$\therefore \frac{20}{x+y} + \frac{20}{x-y} = 8 \tag{1}$$

$$\frac{5}{x+y} = \frac{3}{x-y} \tag{2}$$

From (2),

$$5x - 5y = 3x + 3y$$

$$2x = 8y$$

$$x = 4y \tag{3}$$

Substitute the value of x in (1),

$$\frac{20}{5y} + \frac{20}{3y} = 8$$

$$\frac{160}{15y} = 8$$

$$\therefore y = \tfrac{4}{3}$$

$$x = \tfrac{16}{3}$$

$$\therefore \frac{20}{x+y} = 3$$

$$\frac{20}{x-y} = 5$$

Therefore it takes him 3 hours to row down, and 5 hours to row back.

22. A boat's crew which can pull down a river at the rate of 10 miles an hour finds that it takes twice as long to row a mile up the river as to row a mile down. Find the rate of their rowing in still water and the rate of the stream.

Let $x =$ the rate of the crew in still water,
and $y =$ the rate of the current.

Then, $\dfrac{1}{x+y} =$ the number of hours it takes the crew to row a mile down stream, and $\frac{1}{x-y}$ = the number of hours it takes the crew to row a mile up stream.

$$x+y=10 \qquad (1)$$

$$\therefore \frac{1}{x-y}=\frac{2}{x+y} \qquad (2)$$

From (2), $x+y=2x-2y$

$$x=3y$$

Substitute the value of x in (1).

$$4y=10$$

$$\therefore y=\tfrac{5}{2}$$

$$x=\tfrac{15}{2}$$

Therefore the rate of the crew in still water is $7\frac{1}{2}$ miles per hour, and the rate of the current is $2\frac{1}{2}$ miles per hour.

23. A boatman rows down a stream, which runs at the rate of $2\frac{1}{2}$ miles an hour, for a certain distance in 1 hour and 30 minutes; it takes him 4 hours and 30 minutes to return. Find the distance he pulled down the stream and his rate of rowing in still water.

Let x = his rate in still water,

and y = the number of miles he rowed down stream.

Then, $\frac{3}{2}(x+2\frac{1}{2})$ = the number of miles he rowed down stream in $1\frac{1}{2}$ hours.

$\frac{9}{2}(x-2\frac{1}{2})$ = the number of miles he rowed up stream in $4\frac{1}{2}$ hours.

$$\therefore \tfrac{3}{2}(x+2\tfrac{1}{2})=y \qquad (1)$$

$$\tfrac{9}{2}(x-2\tfrac{1}{2})=y \qquad (2)$$

Subtract, $3x-15=0$

$$\therefore x=5$$

$$y=\tfrac{45}{4}=11\tfrac{1}{4}$$

Therefore he rowed $11\frac{1}{4}$ miles down stream, and his rate in still water is 5 miles per hour.

24. A and B can do a piece of work together in 3 days, A and C in 4 days, B and C in $4\frac{1}{2}$ days. How long will it take each alone to do the work?

Let x = the number of days it takes A to do the work,

y = the number of days it takes B to do the work,

and z = the number of days it takes C to do the work.

Then, $\frac{1}{x}+\frac{1}{y}=$ the part A and B can do in 1 day.

$\frac{1}{x}+\frac{1}{z}=$ the part A and C can do in 1 day.

$\frac{1}{y}+\frac{1}{z}=$ the part B and C can do in 1 day.

$$\therefore \frac{1}{x}+\frac{1}{y}=\frac{1}{3} \qquad (1)$$

$$\frac{1}{x}+\frac{1}{z}=\frac{1}{4} \qquad (2)$$

$$\frac{1}{y}+\frac{1}{z}=\frac{2}{9} \qquad (3)$$

Add, $$\frac{2}{x}+\frac{2}{y}+\frac{2}{z}=\frac{29}{36} \qquad (4)$$

Subtract 2 × (1), $$\frac{2}{x}+\frac{2}{y}=\frac{2}{3}$$

$$\frac{2}{z}=\frac{5}{36}$$

$$\therefore z=\tfrac{72}{5}$$

Substitute the value of z in (2) and (3).

$$\therefore x=\tfrac{72}{13}$$

$$y=\tfrac{72}{11}$$

Therefore A can do the work in $5\frac{7}{13}$ days, B in $6\frac{6}{11}$ days, and C in $14\frac{2}{5}$ days.

25. A and B can do a piece of work in $2\frac{1}{2}$ days, A and C in $3\frac{1}{3}$ days, B and C in 4 days. How long will it take each alone to do the work?

Let $x=$ the number of days it takes A to do the work,

$y=$ the number of days it takes B to do the work,

and $z=$ the number of days it takes C to do the work.

Then, $\frac{1}{x}+\frac{1}{y}=$ the part A and B can do in 1 day.

$\frac{1}{x}+\frac{1}{z}=$ the part A and C can do in 1 day.

$\frac{1}{y}+\frac{1}{z}=$ the part B and C can do in 1 day.

$$\therefore \frac{1}{x}+\frac{1}{y}=\frac{2}{5} \qquad (1)$$

$$\frac{1}{x}+\frac{1}{z}=\frac{3}{10} \qquad (2)$$

$$\frac{1}{y}+\frac{1}{z}=\frac{1}{4} \qquad (3)$$

Add, $$\frac{2}{x}+\frac{2}{y}+\frac{2}{z}=\frac{19}{20}$$

Subtract 2 × (1), $$\frac{2}{x}+\frac{2}{y}=\frac{4}{5}$$

$$\frac{2}{z}=\frac{3}{20}$$

$$\therefore z=\frac{40}{3}$$

Substitute the value of z in (2) and (3).

$$\therefore x=\frac{40}{9}$$

$$y=\frac{40}{7}$$

Therefore A can do the work in $4\frac{4}{9}$ days, B in $5\frac{5}{7}$ days, and C in $13\frac{1}{3}$ days.

26. A and B can do a piece of work in a days, A and C in b days, B and C in c days. How long will it take each alone to do the work?

Let $x=$ the number of days it takes A to do the work,
$y=$ the number of days it takes B to do the work,
and $z=$ the number of days it takes C to do the work.

Then, $\frac{1}{x}+\frac{1}{y}=$ the part A and B can do in 1 day.
$\frac{1}{x}+\frac{1}{z}=$ the part A and C can do in 1 day.
$\frac{1}{y}+\frac{1}{z}=$ the part B and C can do in 1 day.

$$\therefore \frac{1}{x}+\frac{1}{y}=\frac{1}{a} \qquad (1)$$

$$\frac{1}{x}+\frac{1}{z}=\frac{1}{b} \qquad (2)$$

$$\frac{1}{y}+\frac{1}{z}=\frac{1}{c} \qquad (3)$$

Add, $$\frac{2}{x}+\frac{2}{y}+\frac{2}{z}=\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \qquad (4)$$

$$\frac{2}{x}+\frac{2}{y}+\frac{2}{z}=\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \qquad (4)$$

Subtract 2 × (1),
$$\frac{2}{x}+\frac{2}{y}=\frac{2}{a}$$

$$\frac{2}{z}=\frac{1}{b}+\frac{1}{c}-\frac{1}{a}$$

$$\therefore z=\frac{2abc}{ac+ab-bc}$$

(4) is
$$\frac{2}{x}+\frac{2}{y}+\frac{2}{z}=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$$

Subtract 2 × (2),
$$\frac{2}{x}+\frac{2}{z}=\frac{2}{b}$$

$$\frac{2}{y}=\frac{1}{a}-\frac{1}{b}+\frac{1}{c}$$

$$\therefore y=\frac{2abc}{bc-ac+ab}$$

(4) is
$$\frac{2}{x}+\frac{2}{y}+\frac{2}{z}=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$$

Subtract 2 × (3),
$$\frac{2}{y}+\frac{2}{z}=\frac{2}{c}$$

$$\frac{2}{x}=\frac{1}{a}+\frac{1}{b}-\frac{1}{c}$$

$$\therefore x=\frac{2abc}{bc+ac-ab}$$

Therefore A can do the work in $\frac{2abc}{bc+ac-ab}$ days, B in $\frac{2abc}{bc-ac+ab}$ days, and C in $\frac{2abc}{ac+ab-bc}$ days.

27. If the length of a rectangular field were increased by 5 yards and its breadth increased by 10 yards, its area would be increased by 450 square yards; but if its length were increased by 5 yards and its breadth diminished by 10 yards, its area would be diminished by 350 square yards. Find its dimensions.

Let $x =$ the number of yards in the length of the field,
and $y =$ the number of yards in the breadth of the field.

Then, xy = the number of square yards in the area of the field.

$(x+5)(y+10)$ = the number of square yards in the area of the increased field.

$(x+5)(y-10)$ = the number of square yards in the area of the diminished field.

$$\therefore\ (x+5)(y+10)-xy=450 \qquad (1)$$

$$xy-(x+5)(y-10)=350 \qquad (2)$$

From (1), $10x+5y+50=450$

$$2x+y=80 \qquad (3)$$

From (2), $10x-5y+50=350$

$$2x-y=60 \qquad (4)$$

Add (3) and (4), $4x=140$

$$\therefore\ x=35$$

$$y=10$$

Therefore the field is 35 yards long and 10 yards wide.

28. If the floor of a certain hall had been 2 feet longer and 4 feet wider, it would have contained 528 square feet more; but if the length and width were each 2 feet less, it would contain 316 square feet less. Find its dimensions.

Let x = the number of feet in the length of the hall,
and y = the number of feet in the width of the hall.

Then, xy = the number of square feet in the area of the hall.

$(x+2)(y+4)$ = the number of square feet in the area of the enlarged hall.

$(x-2)(y-2)$ = the number of square feet in the area of the diminished hall.

$$\therefore\ (x+2)(y+4)-xy=528 \qquad (1)$$

$$xy-(x-2)(y-2)=316 \qquad (2)$$

From (1), $4x+2y+8=528$

$$2x+y=260 \qquad (3)$$

From (2), $2x+2y-4=316$

$$x+y=160 \qquad (4)$$

Subtract (4) from (3), $x=100$

$$y=60$$

Therefore the floor is 100 feet long and 60 wide.

29. If the length of a rectangle was 4 feet less and the width 3 feet more, the figure would be a square of the same area as the given rectangle. Find the dimensions of the rectangle.

Let $x =$ the number of feet in the length of the rectangle,
and $y =$ the number of feet in the breadth of the rectangle.
Then, $xy =$ the number of square feet in the area of the rectangle.
$(x-4)(y+3) =$ the number of square feet in the area of the rectangle as changed.

But since the new rectangle is a square its sides are of equal length.

$$\therefore\ x-4=y+3 \qquad (1)$$

$$(x-4)(y+3)=xy \qquad (2)$$

From (1), $x-y=7$ (3)

From (2), $3x-4y=12$ (4)

$$\begin{array}{ll} 3\times(3) \text{ is} & 3x-3y=21 \\ (4) \text{ is} & \underline{3x-4y=12} \\ \text{Subtract,} & \qquad y=\ \ 9 \\ & \therefore\ x=16 \end{array}$$

Therefore the rectangle is 9 feet wide and 16 feet long.

30. A man has $10,000 invested. For a part of this sum he receives 5 per cent interest, and for the rest 6 per cent; the income from his 5 per cent investment is $60 more than from his 6 per cent. How much has he in each investment?

Let $x =$ the number of dollars in the one part,
and $y =$ the number of dollars in the other part.
Then, $\frac{5x}{100} =$ the number of dollars interest on the first part.
$\frac{6y}{100} =$ the number of dollars interest on the second part.

$$\therefore\ x+y=10000 \qquad (1)$$

$$\frac{5x}{100}=\frac{6y}{100}+60 \qquad (2)$$

$$\begin{array}{ll} \text{From (2),} & 5x-6y=\ \ 6000 \\ 5\times(1) \text{ is} & \underline{5x+5y=50000} \\ \text{Subtract,} & \qquad 11y=44000 \\ & \therefore\ y=4000 \\ & \qquad x=6000 \end{array}$$

Therefore $6000 is invested at 5%, and $4000 at 6%.

31. A sum of money, at simple interest, amounted in 4 years to $29,000, and in 5 years to $30,000. Find the sum and the rate of interest.

Let $x =$ the number of dollars in the sum,
and $y =$ the rate of interest in per cent.

Then,
$$x + \frac{4xy}{100} = 29000 \qquad (1)$$
$$x + \frac{5xy}{100} = 30000 \qquad (2)$$

Subtract,
$$\frac{xy}{100} = 1000$$

Substitute the value of $\frac{xy}{100}$ in (1).
$$x + 4000 = 29000$$
$$\therefore\ x = 25000$$
$$y = 4$$

Therefore the sum was $25,000, and the rate of interest 4 %.

32. A sum of money, at simple interest, amounted in 10 months to $2100, and in 18 months to $2180. Find the sum and the rate of interest.

Let $x =$ the number of dollars in the sum,
and $y =$ the rate of interest in per cent.

Then,
$$x + \frac{5}{6}\left(\frac{xy}{100}\right) = 2100 \qquad (1)$$
$$x + \frac{3}{2}\left(\frac{xy}{100}\right) = 2180 \qquad (2)$$

Subtract,
$$\frac{2}{3}\left(\frac{xy}{100}\right) = 80$$
$$\therefore\ \frac{xy}{100} = 120$$

Substitute the value of $\frac{xy}{100}$ in (1).
$$x + 100 = 2100$$
$$\therefore\ x = 2000$$
$$y = 6$$

Therefore the sum was $2000, and the rate of interest 6 %.

33. A person has a certain capital invested at a certain rate per cent. Another person has \$2000 more capital, and his capital invested at one per cent better than the first and he receives an income of \$150 greater. A third person has \$3000 more capital, and his capital invested at two per cent better than the first, and he receives an income of \$280 greater. Find the capital of each and the rate at which it is invested.

Let $x =$ the number of dollars in the first person's capital,
and $y =$ the rate of interest on this capital in per cent.
Then, $x + 2000 =$ the number of dollars in the second person's capital.
$y + 1 =$ the rate of interest on this capital in per cent.
$x + 3000 =$ the number of dollars in the third person's capital.
$y + 2 =$ the rate of interest on this capital in per cent.

$$(x + 2000)\left(\frac{y+1}{100}\right) = \frac{xy}{100} + 150 \qquad (1)$$

$$(x + 3000)\left(\frac{y+2}{100}\right) = \frac{xy}{100} + 280 \qquad (2)$$

From (1), $(x + 2000)(y + 1) = xy + 15000$

$$xy + 2000y + x + 2000 = xy + 15000$$

$$x + 2000y = 13000 \qquad (3)$$

From (2), $(x + 3000)(y + 2) = xy + 28000$

$$xy + 3000y + 2x + 6000 = xy + 28000$$

$$2x + 3000y = 22000$$

$$x + 1500y = 11000 \qquad (4)$$

Subtract (4) from (3), $500y = 2000$

$$\therefore\ y = 4$$

$$x = 5000$$

Therefore the first person has \$5000 invested at 4%, the second person \$7000 at 5%, and the third person \$8000 at 6%.

34. A sum of money, at simple interest, amounted in m years to c dollars, and in n years to d dollars. Find the sum and the rate of interest.

Let $x =$ the number of dollars in the sum,
and $y =$ the rate of interest in per cent.

Then,
$$x + \frac{mxy}{100} = c \qquad (1)$$
$$x + \frac{nxy}{100} = d \qquad (2)$$

Subtract,
$$\frac{(m-n)xy}{100} = c - d$$
$$\therefore \frac{xy}{100} = \frac{c-d}{m-n}$$

Substitute the value of $\frac{xy}{100}$ in (1).
$$x + \frac{m(c-d)}{m-n} = c$$
$$\therefore x = \frac{c(m-n) - m(c-d)}{m-n} = \frac{dm - cn}{m-n}$$
$$\therefore y = \frac{100(c-d)}{x(m-n)} = \frac{100(c-d)}{dm-cn}$$

Therefore the sum was $\frac{dm-cn}{m-n}$ dollars, and the rate of interest $\frac{100(c-d)}{dm-cn}$ per cent.

35. A sum of money at simple interest, amounted in m months to a dollars, and in n months to b dollars. Find the sum and the rate of interest.

Let $x =$ the number of dollars in the sum,
and $y =$ the rate of interest in per cent.

Then,
$$x + \frac{m}{12} \times \frac{xy}{100} = a \qquad (1)$$
$$x + \frac{n}{12} \times \frac{xy}{100} = b \qquad (2)$$

Subtract,
$$\frac{(m-n)xy}{1200} = a - b$$
$$\therefore \frac{xy}{1200} = \frac{a-b}{m-n}$$

Substitute the value of $\frac{xy}{1200}$ in (1).
$$x + \frac{m(a-b)}{m-n} = a$$

$$\therefore\ x = \frac{mb - na}{m - n}$$

$$y = \frac{(1200)(a - b)}{x(m - n)}$$

$$= 1200\frac{a - b}{mb - na}$$

Therefore the sum was $\frac{mb - na}{m - n}$ dollars and the rate of interest $1200\frac{a - b}{mb - na}$.

36. A person has $18,375 to invest. He can buy 3 per cent bonds at 75, and 5 per cent bonds at 120. How much of his money must he invest in each kind of bonds in order to have the same income from each investment?

Let x = the number of dollars he must invest in 3% bonds.

and y = the number of dollars he must invest in 5% bonds.

Then, $x + y = 18375$ (1)

$$\frac{4x}{100} = \frac{25y}{600} \quad (2)$$

From (2), $24x = 25y$

$$x = \tfrac{25}{24}y$$

Substitute the value of x in (1).

$$\tfrac{49}{24}y = 18375$$

$$\therefore\ y = 9000$$

$$x = 9375$$

Therefore he must invest $9000 in 5 % bonds and $9375 in 3% bonds.

37. A man makes an investment at 4 per cent, and a second investment at $4\frac{1}{2}$ per cent. His income from the two investments is $715. If the first investment had been at $4\frac{1}{2}$ per cent and the second at 4 per cent, his income would have been $730. Find the amount of each investment.

Let x = the number of dollars in the first investment,

and y = the number of dollars in the second investment.

Then, $\frac{4}{100}x + \frac{9}{200}y = 715$ (1)

$\frac{9}{200}x + \frac{4}{100}y = 730$ (2)

9 × (1) is $\frac{36}{100}x + \frac{81}{200}y = 6435$

8 × (2) is $\frac{36}{100}x + \frac{32}{100}y = 5840$

Subtract, $\frac{17}{200}y = 595$

$$\therefore\ y = 7000$$

$$x = 10000$$

Therefore the two investments were $10,000 and $7000, respectively.

38. Two men, A and B, run a mile, and A wins by 2 seconds. In the second trial B has a start of $18\frac{1}{3}$ yards, and wins by 1 second. Find the number of yards each runs a second, and the number of miles each would run in an hour.

Let x = the number of yards A runs in 1 second,

and y = the number of yards B runs in 1 second.

Then, $\frac{1760}{x}$ = the number of seconds it takes A to run a mile.

$\frac{1760}{y}$ = the number of seconds it takes B to run a mile.

$\frac{1760 - 18\frac{1}{3}}{y}$ = the number of seconds it takes B to run a mile, less $18\frac{1}{3}$ yards.

$$\therefore\ \frac{1760}{y} - \frac{1760}{x} = 2 \qquad (1)$$

$$\frac{1760}{x} - \frac{1760 - 18\frac{1}{3}}{y} = 1 \qquad (2)$$

Add, $$\frac{18\frac{1}{3}}{y} = 3$$

$$\therefore\ y = \frac{55}{9}$$

$$\frac{1760 \times 9}{55} - \frac{1760}{x} = 2$$

$$\therefore\ x = \frac{80}{13}$$

1 hour = 3600 seconds.

$$3600 \times \frac{55}{9} = 22000$$

$$3600 \times \frac{80}{13} = \frac{288000}{13}$$

Therefore A runs $6\frac{2}{13}$ yards per second, and B $6\frac{1}{9}$ yards per second.

In one hour A will run $\frac{288000}{13}$ yards, or $12\frac{84}{143}$ miles, and B 22,000 yards, or $12\frac{1}{2}$ miles.

39. In a mile race A gives B a start of 3 seconds, and is beaten by $12\frac{4}{7}$ yards. In the second trial A gives B a start of 10 yards, and the

race is a tie. Find the number of yards each runs a second. At this rate, how many miles could each run in an hour?

Let $x =$ the number of yards A runs in 1 second,
and $y =$ the number of yards B runs in 1 second.

Then, $\dfrac{1760 - 12\frac{4}{7}}{x} =$ the number of seconds A runs in the first trial.

$\dfrac{1760}{y} =$ the number of seconds B runs in the first trial.

But B runs 3 seconds longer than A.

$$\therefore \frac{1760 - 12\frac{4}{7}}{x} = \frac{1760}{y} - 3 \qquad (1)$$

In the second trial A runs 1760 yards while B runs 1750 yards.

$$\therefore \frac{1760}{x} = \frac{1750}{y} \qquad (2)$$

Subtract (1) from (2), $\dfrac{12\frac{4}{7}}{x} = 3 - \dfrac{10}{y}$

$$\frac{88}{7x} = \frac{3y - 10}{y}$$

$$\therefore x = \frac{88y}{7(3y - 10)}$$

But from (2), $x = \frac{176}{175}y$

$$\therefore \tfrac{176}{175}y = \frac{88y}{7(3y - 10)}$$

$$7 \times 176(3y - 10) = 88 \times 175$$

$$6y = 45$$

$$\therefore y = 7\tfrac{1}{2}$$

$$x = \tfrac{176}{175}y = 7\tfrac{19}{35}$$

1 hour = 3600 seconds.

$$3600 \times 7\tfrac{19}{35} = 27154\tfrac{2}{7}$$

$$3600 \times 7\tfrac{1}{2} = 27000$$

Therefore A runs $7\frac{19}{35}$ yards per second, and B $7\frac{1}{2}$ yards per second.

In one hour A would run $27,154\frac{2}{7}$ yards, or $15\frac{3}{7}$ miles, and B would run 27,000 yards, or $15\frac{15}{44}$ miles.

40. In a mile race A gives B a start of 44 yards, and is beaten by 1 second. In a second trial A gives B a start of 6 seconds, and beats him by $9\frac{7}{9}$ yards. Find the number of yards each runs a second.

Let $x =$ the number of yards A runs in 1 second,
and $y =$ the number of yards B runs in 1 second.

Then,
$$\frac{1760}{x}=\frac{1716}{y}+1 \qquad (1)$$
$$\frac{1760}{x}=\frac{1760-9\frac{7}{9}}{y}-6 \qquad (2)$$

Subtract,
$$0=\frac{9\frac{7}{9}-44}{y}+7$$
$$7y=44-9\tfrac{7}{9}$$
$$\therefore\ y=\tfrac{44}{9}$$

Substitute value of y in (1).
$$\frac{1760}{x}=\frac{9\times 1716}{44}+1=352$$
$$\therefore\ x=5$$

Therefore A runs 5 yards per second, and B $4\frac{8}{9}$ yards per second.

41. An express train, after travelling an hour from A towards B, meets with an accident which delays it 15 minutes. It afterwards proceeds at two-thirds its usual rate, and arrives 24 minutes late. If the accident had happened 5 miles farther on, the train would have been only 21 minutes late. Find the usual rate of the train.

Let $x =$ the usual rate of the train in miles per hour,

and $y =$ the number of miles between A and B.

Then, $x =$ the number of miles the train has gone when the accident occurred.

$y-x =$ the number of miles remaining.

$\dfrac{y-x}{\frac{2}{3}x} =$ the time it takes the train to go the remaining distance.

$\dfrac{y-x}{x} =$ the time the train usually takes.

But the difference in running time $= 24-15 = 9$ min. $=\frac{3}{20}$ hr.
$$\therefore\ \frac{y-x}{\frac{2}{3}x}-\frac{y-x}{x}=\frac{3}{20} \qquad (1)$$

If the accident had happened 5 miles further on we should have
$$\frac{y-x-5}{\frac{2}{3}x}-\frac{y-x-5}{x}=\frac{1}{10} \qquad (2)$$

From (1),
$$\frac{y-x}{2x}=\frac{3}{20}$$
$$\frac{y}{2x}=\frac{13}{20}$$
$$y=\tfrac{13}{10}x$$

From (2), $$\frac{3y-3x-15}{2x}-\frac{2y-2x-10}{2x}=\frac{1}{10}$$

$$\frac{y-x-5}{2x}=\frac{1}{10}$$

$$12x-10y=-50$$

Put $\frac{13x}{10}$ for y, $$y=65$$

$$\therefore\ x=50$$

Therefore the usual rate of the train is 50 miles per hour.

42. A train, after running 2 hours from A towards B, meets with an accident which delays it 20 minutes. It afterwards proceeds at four-fifths its usual rate, and arrives 1 hour and 40 minutes late. If the accident had happened 40 miles nearer A, the train would have been 2 hours late. Find the usual rate of the train.

Let $5x =$ the usual rate of the train in miles per hour,

and $y =$ the number of miles between A and B.

Then, $10x =$ the number of miles the train has gone when the accident occurs.

$y-10x =$ the number of miles remaining.

$\frac{y-10x}{4x} =$ the time it takes the train to go the remaining distance.

$\frac{y-10x}{5x} =$ the time it usually takes the train.

$\therefore \frac{y-10x}{4x}-\frac{y-10x}{5x}=1\frac{1}{3}$ hours, the difference in *running time.*

If the accident had happened 40 miles nearer A we should have

$$\frac{y-10x+40}{4x}-\frac{y-10x+40}{5x}=1\frac{2}{3}$$

$$3y=110x$$

$$3y-130x=-120$$

Put $110x$ for $3y$, then $$110x-130x=-120$$

$$\therefore\ x=6$$

$$5x=30$$

$$y=220$$

Therefore the usual rate of the train is 30 miles an hour.

43. A and B can do a piece of work in $2\frac{1}{2}$ days, A and C in $3\frac{1}{3}$ days, B and C in $3\frac{3}{4}$ days. In what time can all three together, and each one separately, do the work?

Let $x =$ the number of days in which A can do the work,
$y =$ the number of days in which B can do the work,
and $z =$ the number of days in which C can do the work.

Then,
$$\frac{1}{x}+\frac{1}{y}=\frac{1}{2\frac{1}{2}}=\frac{2}{5} \qquad (1)$$
$$\frac{1}{x}+\frac{1}{z}=\frac{1}{3\frac{1}{3}}=\frac{3}{10} \qquad (2)$$
$$\frac{1}{y}+\frac{1}{z}=\frac{1}{3\frac{3}{4}}=\frac{4}{15} \qquad (3)$$

Add,
$$\frac{2}{x}+\frac{2}{y}+\frac{2}{z}=\frac{29}{30} \qquad (4)$$

$2 \times (1)$ is
$$\frac{2}{x}+\frac{2}{y}=\frac{4}{5}$$

Subtract,
$$\frac{2}{z}=\frac{1}{6}$$
$$\therefore z = 12$$

Substitute the value of z in (2) and (3).
$$\therefore x = \tfrac{60}{13}$$
$$y = \tfrac{60}{11}$$

From (4),
$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{29}{60}$$

Therefore A can do the work in $4\frac{8}{13}$ days, B in $5\frac{5}{11}$ days, C in 12 days, and all together in $2\frac{2}{29}$ days.

44. A sum of money, at interest, amounts in 8 months to $1488, and in 15 months to $1530. Find the principal and the rate of interest.

Let $x =$ the number of dollars in the principal,
and $y =$ the rate of interest in per cent.

Then,
$$x+\frac{2}{3}\times\frac{y}{100}x=1488 \qquad (1)$$
$$x+\frac{5}{4}\times\frac{y}{100}x=1530 \qquad (2)$$

Subtract,
$$\frac{7}{12}\times\frac{y}{100}x=42$$
$$\frac{xy}{100}=72$$

Substitute the value of $\frac{xy}{100}$ in (1).

$$x + 48 = 1488$$
$$\therefore\ x = 1440$$
$$y = 5$$

Therefore the principal is \$1440, and the rate of interest 5 %.

45. A number is expressed by two digits, the units' digit being the larger. If the number is divided by the sum of its digits, the quotient is 4. If the digits are reversed and the resulting number is divided by 2 more than the difference of the digits, the quotient is 14. Find the number.

Let $x =$ the tens' digit,
and $y =$ the units' digit.
Then, $10x + y =$ the number.

$$\therefore\ \frac{10x + y}{x + y} = 4 \qquad (1)$$
$$\frac{10y + x}{y - x + 2} = 14 \qquad (2)$$

From (1),
$$10x + y = 4x + 4y$$
$$6x = 3y$$
$$2x = y \qquad (3)$$

From (2),
$$10y + x = 14y - 14x + 28$$
$$15x - 4y = 28 \qquad (4)$$

Substitute the value of y from (3) in (4).

$$15x - 8x = 28$$
$$\therefore\ x = 4$$
$$y = 8$$

Therefore the number is 48.

46. A and B together can dig a well in 10 days. They work 4 days, and B finishes the work in 16 days. How long would it take each alone to dig the well?

Let $x =$ the number of days in which A can dig the well,
and $y =$ the number of days in which B can dig the well.
Then, $\frac{1}{x} + \frac{1}{y} =$ the part both can dig in one day.

$$\therefore\ 10\left(\frac{1}{x} + \frac{1}{y}\right) = 1 \qquad (1)$$
$$4\left(\frac{1}{x} + \frac{1}{y}\right) + \frac{16}{y} = 1 \qquad (2)$$

2 × (1) is $20\left(\frac{1}{x}+\frac{1}{y}\right)=2$

5 × (2) is $20\left(\frac{1}{x}+\frac{1}{y}\right)+\frac{80}{y}=5$

Subtract, $\frac{80}{y}=3$

$$\therefore y=\tfrac{80}{3}=26\tfrac{2}{3}$$

$$x=16$$

Therefore A can dig the well alone in 16 days, and B in $26\frac{2}{3}$ days.

47. The denominator of the greater of two fractions is 20. The fraction formed by taking for a numerator the sum of the numerators of the two fractions, and for a denominator the sum of the denominators, is equal to $\frac{2}{9}$. The fraction similarly formed with the difference of the numerators, and of the denominators, is equal to $\frac{1}{2}$. The sum of the numerators is twice the difference of the denominators. Find the fractions.

Let $\frac{x}{20}=$ the greater fraction,

and $\frac{y}{z}=$ the less fraction.

Then, $\frac{x+y}{20+z}=\frac{2}{9}$ (1)

$\frac{x-y}{20-z}=\frac{1}{2}$ (2)

$x+y=2(20-z)$ (3)

From (1), $9x+9y=40+2z$

$9x+9y-2z=40$ (4)

From (2), $2x-2y=20-z$

$2x-2y+z=20$ (5)

From (3), $x+y+2z=40$ (6)

(4) is $9x+9y-2z=40$

(6) is $x+y+2z=40$

Add, $10x+10y=80$

$x+y=8$ (7)

(4) is $9x+9y-2z=40$

2 × (5) is $4x-4y+2z=40$

Add, $13x+5y=80$ (8)

$5 \times (7)$ is $\qquad 5x + 5y = 40$

(8) is $\qquad 13x + 5y = 80$

Subtract, $\qquad 8x \qquad = 40$

$$\therefore x = 5$$

$$y = 3$$

$$z = 16$$

Therefore the fractions are $\frac{5}{20}$ and $\frac{3}{16}$.

48. A cistern can be filled in 5 hours by two pipes, A and B, together. Both are left open for 3 hours and 45 minutes, and then A is shut, and B takes 3 hours and 45 minutes longer to fill the cistern. How long would it take each pipe alone to fill the cistern?

Let $\qquad x =$ the number of hours it takes A to fill the cistern,

and $\qquad y =$ the number of hours it takes B to fill the cistern.

Then,
$$5\left(\frac{1}{x} + \frac{1}{y}\right) = 1 \qquad (1)$$

$$3\tfrac{3}{4}\left(\frac{1}{x} + \frac{1}{y}\right) + 3\tfrac{3}{4}\left(\frac{1}{y}\right) = 1 \qquad (2)$$

$3 \times (1)$ is
$$15\left(\frac{1}{x} + \frac{1}{y}\right) = 3$$

$4 \times (2)$ is
$$15\left(\frac{1}{x} + \frac{1}{y}\right) + \frac{15}{y} = 4$$

Subtract,
$$\frac{15}{y} = 1$$

$$\therefore y = 15$$

$$x = 7\tfrac{1}{2}$$

Therefore A can fill the cistern in $7\frac{1}{2}$ hours, and B in 15 hours.

49. A man put at interest \$20,000 in three sums, the first at 5 per cent, the second at $4\frac{1}{2}$ per cent, and the third at 4 per cent, receiving an income of \$905 a year. The sum at $4\frac{1}{2}$ per cent is one-third as much as the other two sums together. Find the three sums.

Let $\qquad x =$ the number of dollars in the first sum,

$\qquad y =$ the number of dollars in the second sum,

and $\qquad z =$ the number of dollars in the third sum.

Then,
$$x + y + z = 20000 \qquad (1)$$

$$\tfrac{5}{100}x + \tfrac{9}{200}y + \tfrac{4}{100}z = 905 \qquad (2)$$

$$y = \tfrac{1}{3}(x + z) \qquad (3)$$

From (3),
$$3y = x + z \qquad (4)$$

Substitute the value of $x + z$ in (1).

$$4y = 20000$$
$$\therefore y = 5000$$

From (2), $10x + 9y + 8z = 181000$

$8 \times (4)$ is $8x - 24y + 8z = 0$

Subtract, $2x + 33y = 181000$

$$\therefore x = 8000$$
$$z = 7000$$

Therefore the three sums are \$8000, \$5000, and \$7000.

50. An income of \$335 a year is obtained from two investments, one in $4\frac{1}{2}$ per cent stock and the other in 5 per cent stock. If the $4\frac{1}{2}$ per cent stock should be sold at 110, and the 5 per cent at 125, the sum realized from both stocks together would be \$8300. How much of each stock is there?

Let $x =$ the number of dollars in the par value of the one investment,

and $y =$ the number of dollars in the par value of the other investment.

Then, $\frac{9}{200}x + \frac{5}{100}y = 335$ (1)

$\frac{110}{100}x + \frac{125}{100}y = 8300$ (2)

From (1), $9x + 10y = 67000$ (3)

From (2), $110x + 125y = 830000$

$22x + 25y = 166000$ (4)

$5 \times (3)$ is $45x + 50y = 335000$

$2 \times (4)$ is $44x + 50y = 332000$

Subtract, $x = 3000$

$$\therefore y = 4000$$

Therefore the par value of the first stock is \$3000, and of the second stock \$4000.

51. A boy bought some apples at 3 for 5 cents, and some at 4 for 5 cents, paying for the whole \$1. He sold them at 2 cents apiece, and cleared 40 cents. How many of each kind did he buy?

Let $x =$ the number of apples of the first kind,

and $y =$ the number of apples of the second kind.

Then, $\frac{5}{3}x + \frac{5}{4}y =$ the number of cents all the apples cost.

$\therefore \frac{5}{3}x + \frac{5}{4}y = 100$ (1)

$2(x + y) = 140$ (2)

From (1), $4x + 3y = 240$
From (2), $4x + 4y = 280$
Subtract, $y = 40$
$\therefore\ x = 30$

Therefore he bought 30 apples of the first kind, and 40 of the second kind.

52. Find the area of a rectangular floor, such that if 3 feet were taken from the length and 3 feet added to the breadth, its area would be increased by 6 square feet, but if 5 feet were taken from the breadth and 3 feet added to the length, its area would be diminished by 90 square feet.

Let x = the number of feet in the length of the floor,
and y = the number of feet in the breadth of the floor.
Then, xy = the number of square feet in the area of the floor.
$(x-3)(y+3)$ = the number of square feet in the increased area of the floor.
$(x+3)(y-5)$ = the number of square feet in the diminished area of the floor.

$\therefore\ (x-3)(y+3) = xy + 6$ (1)
and $(x+3)(y-5) = xy - 90$ (2)
From (1), $3x - 3y - 9 = 6$
or $x - y = 5$ (3)
From (2), $-5x + 3y = -75$
or $5x - 3y = 75$ (4)
$5 \times (3)$ is $5x - 5y = 25$
$2y = 50$
$\therefore\ y = 25$
$x = 30$
$xy = 750$

Therefore the area of the floor is 750 square feet.

53. A courier was sent from A to B, a distance of 147 miles. After 7 hours, a second courier was sent from A, who overtook the first just as he was entering B. The time required by the first to travel 17 miles added to the time required by the second to travel 76 miles is 9 hours and 40 minutes. How many miles did each travel per hour?

Let $x =$ the number of miles the first courier travels per hour,

and $y =$ the number of miles the second courier travels per hour.

Then, $\frac{147}{x} =$ the number of hours it takes the first courier to travel from A to B.

$\frac{147}{y} =$ the number of hours it takes the second courier to travel from A to B.

But it takes the first courier 7 hours longer than the second.

$$\therefore \frac{147}{x} = \frac{147}{y} + 7 \qquad (1)$$

Also, $$\frac{17}{x} + \frac{76}{y} = 9\tfrac{2}{3} \qquad (2)$$

From (1), $$\frac{21}{x} - \frac{21}{y} = 1 \qquad (3)$$

21 × (2) is $$\frac{357}{x} + \frac{1596}{y} = 203$$

17 × (3) is $$\frac{357}{x} - \frac{357}{y} = 17$$

Subtract, $$\frac{1953}{y} = 186$$

$$\therefore y = \tfrac{1953}{186} = 10\tfrac{1}{2}$$
$$x = 7$$

Therefore the first courier travels 7 miles per hour, and the second courier $10\frac{1}{2}$ miles per hour.

54. A box contains a mixture of 6 quarts of oats and 9 of corn, and another box contains a mixture of 6 quarts of oats and 2 of corn. How many quarts must be taken from each box in order to have a mixture of 7 quarts, half oats and half corn?

Let $x =$ the number of quarts to be taken from the first box,

and $y =$ the number of quarts to be taken from the second box.

Then, $\frac{6}{15}x =$ the number of quarts of oats to be taken from the first box.

$\frac{9}{15}x =$ the number of quarts of corn to be taken from the first box.

$\frac{6}{8}y$ = the number of quarts of oats to be taken from the second box.

$\frac{2}{8}y$ = the number of quarts of corn to be taken from the second box.

$$\therefore \tfrac{6}{15}x + \tfrac{6}{8}y = \tfrac{7}{2} \qquad (1)$$

$$\tfrac{9}{15}x + \tfrac{2}{8}y = \tfrac{7}{2} \qquad (2)$$

Subtract, $$\frac{x}{5} - \frac{y}{2} = 0$$

$$\therefore x = \tfrac{5}{2}y$$

Substitute the value of y in (1).

$$y + \tfrac{6}{8}y = \tfrac{7}{2}$$

$$\therefore y = 2$$

$$x = 5$$

Therefore 5 quarts must be taken from the first box, and 2 quarts from the second box.

55. A train travelling 30 miles an hour takes 21 minutes longer to go from A to B than a train which travels 36 miles an hour. Find the distance from A to B.

Let x = the number of miles between A and B,
and y = the number of hours it takes the first train to go from A to B.

Then, $$30y = x \qquad (1)$$

$$36(y - \tfrac{21}{60}) = x \qquad (2)$$

Subtract, $$6y - \tfrac{63}{5} = 0$$

$$\therefore y = \tfrac{63}{30} = 2\tfrac{1}{10}$$

$$x = 30y = 63$$

Therefore the distance is 63 miles.

56. A man buys 570 oranges, some at 16 for 25 cents, and the rest at 18 for 25 cents. He sells them all at the rate of 15 for 25 cents, and gains 75 cents. How many of each kind does he buy?

Let x = the number he bought at 16 for 25 cents,
and y = the number he bought at 18 for 25 cents.

Then, $\frac{25x}{16}$ = the number of cents the first lot cost.

$\frac{25y}{18}$ = the number of cents the second lot cost.

$$\therefore x + y = 570 \qquad (1)$$

$$\frac{25x}{16} + \frac{25y}{18} = \frac{25}{15} \times 570 - 75 \qquad (2)$$

From (2), $\frac{x}{16}+\frac{y}{18}=38-3=35$

$$9x+8y=5040$$

8 × (1) is $8x+8y=4560$

Subtract, $x = 480$

$$\therefore y=90$$

Therefore he bought 480 oranges of the first kind and 90 of the second kind.

57. A and B run a mile race. In the first heat B receives 12 seconds' start, and is beaten by 44 yards. In the second heat B receives 165 yards' start, and arrives at the winning post 10 seconds before A. Find the time in which each can run a mile.

Let $x =$ the number of seconds in which A can run one mile,

and $y =$ the number of seconds in which B can run one mile.

$\frac{44}{1760}y =$ the number of seconds in which B can run 44 yards.

$\frac{165}{1760}y =$ the number of seconds in which B can run 165 yards.

In the first heat it takes B $12+\frac{44}{1760}y$ seconds longer to run the mile than it does A.

$$\therefore y=x+12+\frac{44}{1760}y \qquad (1)$$

In the second heat it takes B $\frac{165}{1760}y-10$ seconds longer to run the mile than it does A.

$$\therefore y=x+\frac{165}{1760}y-10 \qquad (2)$$

Subtract (2) from (1),

$$0=22-\frac{121}{1760}y$$

$$\therefore y=\frac{22\times 1760}{121}=320$$

$$x=300$$

Therefore A can run a mile in 5 minutes, and B in 5 minutes and 20 seconds.

Exercise 66. Page 194.

1. A number is expressed by two digits. If the number is divided by the last digit, the quotient is 15. Find the number.

Let $x =$ the tens' digit,
and $y =$ the units' digit.

$$10x + y = \text{the number.}$$

$$\therefore \frac{10x + y}{y} = 15$$

$$10x + y = 15y$$

$$10x = 14y$$

$$x = \tfrac{7}{5}y$$

Hence y must be divisible by 5, that is, it must be 5; and accordingly $x = \frac{7}{5}y = 7$.

Therefore the number is 75.

2. A number is expressed by three digits. The sum of the digits is 20. If 16 is subtracted from the number, and the remainder divided by 2, the digits will be reversed. Find the number.

Let $x =$ the hundreds' digit,
$y =$ the tens' digit,
and $z =$ the units' digit.

Then, $$x + y + z = 20 \qquad (1)$$

$$\frac{100x + 10y + z - 16}{2} = 100z + 10y + x \qquad (2)$$

From (2), $100x + 10y + z - 16 = 200z + 20y + 2x$

$$\begin{array}{lrl} & 98x - 10y - 199z = & 16 \\ 10 \times (1) \text{ is} & 10x + 10y + 10z = & 200 \\ \hline \text{Add,} & 108x \qquad - 189z = & 216 \end{array}$$

Divide by 27, $$4x - 7z = 8$$

$$4x = 7z + 8$$

$$x = \tfrac{7}{4}z + 2$$

Hence z must be divisible by 4; that is, z must be either 0, 4, or 8. These give for x the values 2, 9, 16, and for y the values 18, 7, -4. Of these three sets of values only the second is admissible.

Therefore the number is 974.

3. A man spends \$114 in buying calves at \$5 apiece, and pigs at \$3 apiece. How many did he buy of each?

Let $x =$ the number of calves,
and $y =$ the number of pigs.

Then, $5x + 3y = 114$

$$3y = 114 - 5x$$

$$y = 38 - \frac{5x}{3}$$

Hence x must be divisible by 3. Moreover, x must be less than 23, for otherwise the calves would cost more than $114.

The possible values of x are therefore 0, 3, 6, 9, 12, 15, 18, 21.

The corresponding values of y are 38, 33, 28, 23, 18, 13, 8, 3.

The problem admits of 7 solutions, as 0 for x is inadmissible.

4. In how many ways can a man pay a debt of $87 with five-dollar bills and two-dollar bills?

Let $x =$ the number of $5 bills,

and $y =$ the number of $2 bills.

Then, $5x + 2y = 87$

$$2y = 87 - 5x$$

$$y = 43 - 2x + \frac{1 - x}{2}$$

Hence $1 - x$ must be divisible by 2; that is, x must be an odd number. Also, x cannot exceed 17.

Hence x may have the following values, 1, 3, 5, 7, 9, 11, 13, 15, 17.

The corresponding values of y are 41, 36, 31, 26, 21, 16, 11, 6, 1.

Therefore there are 9 ways of paying the debt.

5. Find the smallest number which when divided by 5 or by 7 gives 4 for a remainder.

Let $n =$ the number.

Then, $\frac{n - 4}{5} = x$ (1)

$$\frac{n - 4}{7} = y \quad (2)$$

From (1), $n - 4 = 5x$

From (2), $n - 4 = 7y$

$$\therefore\ 5x = 7y$$

$$x = \frac{7y}{5}$$

Hence y is divisible by 5, and its lowest value is 5. Then, since $\frac{n - 4}{7} = 5$, $n = 39$.

Therefore the least value of n is 39.

6. A farmer sells 15 calves, 14 lambs, and 13 pigs for $200. Some days after, at the same price, he sells 7 calves, 11 lambs, and 16 pigs, for which he receives $141. What was the price of each?

Let $x =$ the number of dollars in the price of one calf,
$y =$ the number of dollars in the price of one lamb,
and $z =$ the number of dollars in the price of one pig.

Then,
$$15x + 14y + 13z = 200 \quad (1)$$
$$7x + 11y + 16z = 141 \quad (2)$$

$7 \times (1)$ is $105x + 98y + 91z = 1400$

$15 \times (2)$ is $105x + 165y + 240z = 2115$

Subtract, $67y + 149z = 715$

$$67y = 715 - 149z$$
$$y = 10 - 2z + \frac{45 - 15z}{67}$$

Hence $45 - 15z$ must be divisible by 67. Moreover z cannot exceed 4, for otherwise y would be negative. But $45 - 15z$ is only divisible, under these conditions, by 67, when $z = 3$. This gives us $y = 4$, $x = 7$.

Therefore the calves cost $7 apiece, the lambs $4 apiece, and the pigs $3 apiece.

Exercise 67. Page 196.

1. A train travelling b miles per hour is m hours in advance of a second train which travels a miles per hour. In how many hours will the second train overtake the first?

If the second train is to overtake the first, it must move faster than the first, that is, a must be greater than b. If this be the case the second train gains $a - b$ miles on the first in every hour, and as the first train has a start of bm miles, it will take the second train $\frac{bm}{a-b}$ hours to overtake the first. In this case we had $a > b$. If $a = b$, the two trains are moving at the same rate, and accordingly the second can never overtake the first. The expression $\frac{bm}{a-b}$ assumes in this case the form $\frac{bm}{0}$, which indicates the impossibility of the problem.

If $a < b$, $\frac{bm}{a-b}$ is negative. The second train is in this case continually falling behind the first. The trains were together before we begin to consider their motion, but are now continually separating.

The negative solution gives the time at which the trains *were* together, but this time is counted backward from the initial instant.

2. A man setting out on a journey drove at the rate of a miles an hour to the nearest railway station, distant b miles from his house. On arriving at the station he found that the train for his place of destination left c hours before. At what rate should he have driven in order to reach the station just in time for the train?

Since he drives a miles an hour, it takes him $\frac{b}{a}$ hours to reach the station. Since the train had been gone c hours when he reached the station, it left $\frac{b}{a}-c$ hours after he started from home. And since he had $\frac{b}{a}-c$ hours to drive b miles he should have driven $\frac{b}{\frac{b}{a}-c}$ or $\frac{ab}{b-ac}$ miles per hour.

If c be 0, the expression $\frac{ab}{b-ac}$ reduces to $\frac{ab}{b}$ or a. In this case he arrives just in time, so that the rate of a miles per hour is sufficient.

If $c=\frac{b}{a}$, that is, if $\frac{b}{a}-c=0$, the train left the station at the same time that he left his house. No rate of driving would have brought him to the station in time in this case. The expression $\frac{ab}{b-ac}$ becomes here $\frac{ab}{0}$, the symbol of impossibility.

If $c=-\frac{b}{a}$, $\frac{ab}{b-ac}$ becomes $\frac{ab}{2b}=\frac{a}{2}$. Also the time he had to reach the train, $\frac{b}{a}-c$, becomes $\frac{2b}{a}$. He need therefore drive only half as fast. The negative value of c indicates that the train *would not arrive* for c hours after he reached the station. The problem is therefore erroneously stated in this case.

3. A wine merchant has two kinds of wine which he sells, one at a dollars, and the other at b dollars per gallon. He wishes to make a mixture of l gallons, which shall cost him on the average m dollars a gallon. How many gallons must he take of each?

Let $x =$ the number of gallons of the first kind,
and $y =$ the number of gallons of the second kind.

Then, $$x + y = l \qquad (1)$$
$$ax + by = m(x + y) \qquad (2)$$

From (2), $$ax + by = ml$$
$a \times (1)$ is $$ax + ay = al$$

Subtract, $$(b - a)y = (m - a)l$$
$$\therefore\ y = \frac{(m-a)l}{b-a} = \frac{(a-m)l}{a-b}$$
$$x = \frac{m-b}{a-b}l$$

Therefore he must take $\frac{(m-b)l}{a-b}$ gallons of the first kind, and $\frac{(a-m)l}{a-b}$ gallons of the second kind.

We will suppose that, if the two prices are different, a is greater than b. Then it is clear that the price of any mixture must be less than a, but greater than b, dollars per gallon; that is, $a > m$ but $m > b$.

(1) If $a = b$, the price of *any* mixture will evidently be the same as that of either kind. The formulas become in this case $x = \frac{0}{0}$, $y = \frac{0}{0}$; that is, x and y are entirely indeterminate. For we may take as many gallons of each kind as we please, so long as we take only l gallons in all.

(2) If $m = a$, and a and b are different, then all the l gallons must be taken from the first kind. For any admixture of the second kind would change the price. The formulas in this case become $x = l$, $y = 0$. The same consideration applies when $m = b$.

(3) a cannot be greater than b and less than m, for then the price of the mixture would be higher than that of either of its parts. The formulas show that in this case y is negative; that is, such a problem is erroneous in statement.

(4) a and b cannot both be greater than m, for then the price of the mixture would be less than of either of the parts. In this case x comes out negative.

Exercise 68. Page 199.

1. Simplify $(x + 1)^2 < x^2 + 3x - 5$.

We have, $$(x + 1)^2 < x^2 + 3x - 5$$
Hence, $$x^2 + 2x + 1 < x^2 + 3x - 5$$
$$6 < x$$
$$x > 6$$

Therefore x is greater than 6.

2. Simplify $\frac{4x-2}{3} > \frac{3-5x}{7}$.

We have, $\frac{4x-2}{3} > \frac{3-5x}{7}$

Hence, $7(4x-2) > 3(3-5x)$

$28x - 14 > 9 - 15x$

$43x > 23$

$x > \frac{23}{43}$

Therefore x is greater than $\frac{23}{43}$.

3. Simplify $x + 2b > 7x$.

We have, $x + 2b > 7x$

Hence, $2b > 6x$

$b > 3x$

$3x < b$

$x < \frac{b}{3}$

Therefore x is less than $\frac{b}{3}$.

4. Simplify $3x - 2 < \frac{x}{2} + 7\frac{1}{3}$.

We have, $3x - 2 < \frac{x}{2} + 7\frac{1}{3}$

Hence, $3x - \frac{x}{2} < 2 + 7\frac{1}{3}$

$\frac{5x}{2} < 9\frac{1}{3}$

$x < \frac{2}{5} \times \frac{28}{3}$

$x < 3\frac{11}{15}$

Therefore x is less than $3\frac{11}{15}$.

5. Find the limiting values of x, given

$4x - 6 < 2x + 4$

and $2x + 4 > 16 - 2x$.

We have, $4x - 6 < 2x + 4$

Hence, $2x < 10$

$x < 5$

Again, $2x + 4 > 16 - 2x$

Hence, $4x > 12$

$x > 3$

Therefore x lies between 3 and 5.

6. Find the limiting values of x, given

$$\frac{ax}{5} + bx - ab > \frac{a^2}{5},$$

and
$$\frac{bx}{7} - ax + ab < \frac{b^2}{7}.$$

We have, $\frac{ax}{5} + bx - ab > \frac{a^2}{5}$

Hence, $ax + 5bx - 5ab > a^2$

$(a + 5b)x > a^2 + 5ab$

Dividing by $a + 5b$, $x > a$

Again, $\frac{bx}{7} - ax + ab < \frac{b^2}{7}$

Hence, $bx - 7ax + 7ab < b^2$

$(b - 7a)x < b^2 - 7ab$

Dividing by $b - 7a$, $x < b$

Therefore if $b > a$, x lies between a and b. But if $a > b$, there is no value of x which can satisfy the two inequalities.

7. Find the integral value of x, given

$$\tfrac{1}{4}(x + 2) + \tfrac{1}{3}x < \tfrac{1}{2}(x - 4) + 3,$$
$$\tfrac{1}{4}(x + 2) + \tfrac{1}{3}x > \tfrac{1}{2}(x + 1) + \tfrac{1}{3}.$$

We have, $\tfrac{1}{4}(x + 2) + \tfrac{1}{3}x < \tfrac{1}{2}(x - 4) + 3$

Hence, $3(x + 2) + 4x < 6(x - 4) + 36$

Therefore $x < 6$

Again, $\tfrac{1}{4}(x + 2) + \tfrac{1}{3}x > \tfrac{1}{2}(x + 1) + \tfrac{1}{3}$

$3(x + 2) + 4x > 6(x + 1) + 4$

Therefore $x > 4$

Therefore x lies between 4 and 6. The only possible integral value of x is therefore 5.

8. Twice a certain integral number increased by 7 is not greater than 19; and three times the number diminished by 5 is not less than 13. Find the number.

Let $x =$ the number.

Then, $2x + 7 < 19$ (1)

$3x - 5 > 13$ (2)

From (1), $2x < 12$

Therefore $x < 6$

From (2), $3x > 18$

Therefore, $x > 6$

Therefore x is not less and not greater than 6. That is, x is equal to 6. Note. — Throughout this problem the signs $>$ and $<$ are to be read "not less than" and "not greater than."

If the letters stand for unequal and positive numbers, show that

9. $a^2 + 3b^2 > 2b(a + b)$.

$a^2 + 3b^2$ is $>$ or $< 2b(a + b)$

as $a^2 + 3b^2$ is $>$ or $< 2ab + 2b^2$

as $a^2 - 2ab + b^2$ is $>$ or < 0

as $(a - b)^2$ is $>$ or < 0

But, $(a - b)^2 > 0$

Hence, $a^2 + 3b^2 > 2b(a + b)$

10. $a^3 + b^3 > a^2b + ab^2$.

$a^3 + b^3$ is $>$ or $< a^2b + ab^2$

as $(a + b)(a^2 - ab + b^2)$ is $>$ or $< ab(a + b)$

as $a^2 - ab + b^2$ is $>$ or $< ab$

as $a^2 - 2ab + b^2$ is $>$ or < 0

as $(a - b)^2$ is $>$ or < 0

But, $(a - b)^2 > 0$

Hence, $a^3 + b^3 > a^2b + ab^2$

11. $a^2 + b^2 + c^2 > ab + ac + bc$.

Show that $a^2 + b^2 + c^2 > ab + ac + bc$

Now $a^2 + b^2 > 2ab$

$b^2 + c^2 > 2bc$

$a^2 + c^2 > 2ac$

Adding, $2a^2 + 2b^2 + 2c^2 > 2ab + 2ac + 2bc$

Hence, $a^2 + b^2 + c^2 > ab + ac + bc$

12. $a^2b + a^2c + ab^2 + b^2c + ac^2 + bc^2 > 6abc$.

Show that $a^2b + a^2c + ab^2 + b^2c + ac^2 + bc^2 > 6abc$

Now $a^2 + b^2 > 2ab$

Hence, $c(a^2 + b^2) > 2abc$

$a^2c + b^2c > 2abc$

Similarly, $ab^2 + ac^2 > 2abc$

and $a^2b + bc^2 > 2abc$

Adding, $a^2b + a^2c + ab^2 + b^2c + ac^2 + bc^2 > 6abc$

12. $\frac{a}{b}+\frac{b}{a}>2.$

$\frac{a}{b}+\frac{b}{a}$ is $>$ or <2

as $\frac{a^2+b^2}{ab}$ is $>$ or <2

as a^2+b^2 is $>$ or $<2ab.$

But, $a^2+b^2>2ab$

Hence, $\frac{a}{b}+\frac{b}{a}>2$

14. $\frac{a+b}{2}>\frac{2ab}{a+b}.$

$\frac{a+b}{2}$ is $>$ or $<\frac{2ab}{a+b}$

as $(a+b)^2$ is $>$ or $<4ab$

as $a^2+2ab+b^2$ is $>$ or $<4ab$

as a^2+b^2 is $>$ or $<2ab.$

But, $a^2+b^2>2ab$

Hence, $\frac{a+b}{2}$ is $>\frac{2ab}{a+b}$

Exercise 69. Page 203.

Raise to the required power:

1. $(a^4)^3=a^{12}.$

2. $(a^2b^3)^5=a^{10}b^{15}.$

3. $\left(\frac{2xy^2}{3ab^3}\right)^4=\frac{2^4x^4y^8}{3^4a^4b^{12}}=\frac{16x^4y^8}{81a^4b^{12}}.$

4. $(-5ab^2c^3)^4=(-5)^4a^4b^8c^{12}=625a^4b^8c^{12}.$

5. $(-7x^2yz^3)^3=(-7)^3x^6y^3z^9=-343x^6y^3z^9.$

6. $\left(-\frac{3a^2b^3c^4}{2x^2y^5}\right)^5=-\frac{3^5a^{10}b^{15}c^{20}}{2^5x^{10}y^{25}}=-\frac{243a^{10}b^{15}c^{20}}{32x^{10}y^{25}}.$

7. $(-2x^2y^4)^6=(-2)^6x^{12}y^{24}=64x^{12}y^{24}.$

8. $(-3a^2b^3x^4)^5=(-3)^5a^{10}b^{15}x^{20}=-243a^{10}b^{15}x^{20}.$

9. $\left(-\frac{3x^2y^3}{4z^3}\right)^4=\frac{3^4x^8y^{12}}{4^4z^{12}}=\frac{81x^8y^{12}}{256z^{12}}.$

10. $(x+2)^5=x^5+5\times 2x^4+10\times 2^2x^3+10\times 2^3x^2+5\times 2^4x+2^5$
$=x^5+10x^4+40x^3+80x^2+80x+32.$

11. $(x^2-2)^4=(x^2)^4-4\times 2(x^2)^3+6\times 2^2(x^2)^2-4\times 2^3(x^2)+2^4$
$=x^8-8x^6+24x^4-32x^2+16.$

12. $(x+3)^5=x^5+5\times 3x^4+10\times 3^2x^3+10\times 3^3x^2+5\times 3^4x+3^5$
$=x^5+15x^4+90x^3+270x^2+405x+243.$

13. $(2x+1)^6$
$=(2x)^6+6(2x)^5+15(2x)^4+20(2x)^3+15(2x)^2+6(2x)+1$
$=64x^6+192x^5+240x^4+160x^3+60x^2+12x+1.$

14. $(2m^2-1)^3=(2m^2)^3-3(2m^2)^2+3(2m^2)-1$
$=8m^6-12m^4+6m^2-1.$

15. $(2x+3y)^5$
$=(2x)^5+5(2x)^4(3y)+10(2x)^3(3y)^2+10(2x)^2(3y)^3$
$+5(2x)(3y)^4+(3y)^5$
$=32x^5+240x^4y+720x^3y^2+1080x^2y^3+810xy^4+243y^5.$

16. $(2x-y)^6$
$=(2x)^6-6(2x)^5y+15(2x)^4y^2-20(2x)^3y^3+15(2x)^2y^4$
$-6(2x)y^5+y^6.$
$=64x^6-192x^5y+240x^4y^2-160x^3y^3+60x^2y^4-12xy^5+y^6.$

17. $(xy-2)^7$
$=(xy)^7-7\times2(xy)^6+21\times2^2(xy)^5-35\times2^3(xy)^4$
$+35\times2^4(xy)^3-21\times2^5(xy)^2+7\times2^6(xy)-2^7$
$=x^7y^7-14x^6y^6+84x^5y^5-280x^4y^4+560x^3y^3-672x^2y^2$
$+448xy-128.$

18. $(1-x+x^2)^2=[(1-x)+x^2]^2$
$=(1-x)^2+2x^2(1-x)+x^4$
$=1-2x+x^2+2x^2-2x^3+x^4$
$=1-2x+3x^2-2x^3+x^4.$

19. $(1-2x+3x^2)^2=[(1-2x)+3x^2]^2$
$=(1-2x)^2+6x^2(1-2x)+9x^4$
$=1-4x+4x^2+6x^2-12x^3+9x^4$
$=1-4x+10x^2-12x^3+9x^4.$

20. $(1-a+a^2)^3=[(1-a)+a^2]^3$
$=(1-a)^3+3a^2(1-a)^2+3a^4(1-a)+a^6$
$=1-3a+3a^2-a^3+3a^2-6a^3+3a^4+3a^4-3a^5+a^6$
$=1-3a+6a^2-7a^3+6a^4-3a^5+a^6.$

21. $(3-4x+5x^2)^2=[(3-4x)+5x^2]^2$
$=(3-4x)^2+10x^2(3-4x)+25x^4$
$=9-24x+16x^2+30x^2-40x^3+25x^4$
$=9-24x+46x^2-40x^3+25x^4.$

Exercise 70. Page 205.

Simplify:

1. $\sqrt{4x^2y^4} = 2xy^2$.
2. $\sqrt[3]{64x^9} = 4x^3$.
3. $\sqrt[4]{16x^8y^{12}} = 2x^2y^3$.
4. $\sqrt[5]{-32a^{10}} = -2a^2$.
5. $\sqrt[3]{-27x^3} = -3x$.
6. $\sqrt{25a^4} = 5a^2$.
7. $\sqrt[3]{-8a^3b^6} = -2ab^2$.
8. $\sqrt[6]{64x^{12}} = 2x^2$.
9. $\sqrt[3]{-216a^{12}} = -6a^4$.
10. $\sqrt[6]{729x^{18}} = 3x^3$.
11. $\sqrt[5]{243y^5z^{10}} = 3yz^2$.
12. $\sqrt[3]{-1728d^3} = -12d$.
13. $\sqrt[3]{-343a^6} = -7a^2$.
14. $\sqrt[4]{81a^{24}} = 3a^6$.
15. $\sqrt[3]{512a^{12}b^{15}} = 8a^4b^5$.
16. $\sqrt[3]{x^{3n}y^{12m}} = x^ny^{4m}$.
17. $\sqrt{\dfrac{9a^2b^6}{16x^4y^2}} = \dfrac{3ab^3}{4x^2y}$.
18. $\sqrt[3]{-\dfrac{8x^3y^6}{27z^9}} = -\dfrac{2xy^2}{3z^3}$.
19. $\sqrt[5]{-\dfrac{32a^{10}}{243x^{15}}} = -\dfrac{2a^2}{3x^3}$.
20. $\sqrt[4]{\dfrac{16x^4}{81a^8b^{12}}} = \dfrac{2x}{3a^2b^3}$.
21. $\sqrt[3]{\dfrac{125x^{21}}{216a^{24}}} = \dfrac{5x^7}{6a^8}$.

Exercise 71. Page 207.

Find the square root of

1. $x^4 - 8x^3 + 18x^2 - 8x + 1$.

$$
\begin{array}{r|l}
 & x^4 - 8x^3 + 18x^2 - 8x + 1 \;\underline{|\; x^2 - 4x + 1} \\
 & x^4 \\
2x^2 - 4x & -8x^3 + 18x^2 \\
 & \underline{-8x^3 + 16x^2} \\
2x^2 - 8x + 1 & 2x^2 - 8x + 1 \\
 & \underline{2x^2 - 8x + 1}
\end{array}
$$

2. $9a^4 - 6a^3 + 13a^2 - 4a + 4$.

$$
\begin{array}{r|l}
 & 9a^4 - 6a^3 + 13a^2 - 4a + 4 \;\underline{|\; 3a^2 - a + 2} \\
 & 9a^4 \\
6a^2 - a & -6a^3 + 13a^2 \\
 & \underline{-6a^3 + a^2} \\
6a^2 - 2a + 2 & 12a^2 - 4a + 4 \\
 & \underline{12a^2 - 4a + 4}
\end{array}
$$

3. $4x^4-12x^3y+29x^2y^2-30xy^3+25y^4$.

$$
\begin{array}{r|l}
\multicolumn{2}{l}{4x^4-12x^3y+29x^2y^2-30xy^3+25y^4\ \underline{\big|\ 2x^2-3xy+5y^2}}\\
\multicolumn{2}{l}{4x^4}\\
4x^2-3xy & -12x^3y+29x^2y^2\\
 & \underline{-12x^3y+9x^2y^2}\\
4x^2-6xy+5y^2 & 20x^2y^2-30xy^3+25y^4\\
 & \underline{20x^2y^2-30xy^3+25y^4}
\end{array}
$$

4. $1+4x+10x^2+12x^3+9x^4$.

$$
\begin{array}{r|l}
\multicolumn{2}{l}{1+4x+10x^2+12x^3+9x^4\ \underline{\big|\ 1+2x+3x^2}}\\
\multicolumn{2}{l}{1}\\
2+2x & 4x+10x^2\\
 & \underline{4x+4x^2}\\
2+4x+3x^2 & 6x^2+12x^3+9x^4\\
 & \underline{6x^2+12x^3+9x^4}
\end{array}
$$

5. $16-96x+216x^2-216x^3+81x^4$.

$$
\begin{array}{r|l}
\multicolumn{2}{l}{16-96x+216x^2-216x^3+81x^4\ \underline{\big|\ 4-12x+9x^2}}\\
\multicolumn{2}{l}{16}\\
8-12x & -96x+216x^2\\
 & \underline{-96x+144x^2}\\
8-24x+9x^2 & 72x^2-216x^3+81x^4\\
 & \underline{72x^2-216x^3+81x^4}
\end{array}
$$

6. $x^4-22x^3+95x^2+286x+169$.

$$
\begin{array}{r|l}
\multicolumn{2}{l}{x^4-22x^3+95x^2+286x+169\ \underline{\big|\ x^2-11x-13}}\\
\multicolumn{2}{l}{x^4}\\
2x^2-11x & -22x^3+95x^2\\
 & \underline{-22x^3+121x^2}\\
2x^2-22x-13 & -26x^2+286x+169\\
 & \underline{-26x^2+286x+169}
\end{array}
$$

7. $4x^4 - 11x^2 + 25 - 12x^3 + 30x.$

$$
\begin{array}{r|l}
 & 4x^4 - 12x^3 - 11x^2 + 30x + 25 \,\big|\, \underline{2x^2 - 3x - 5} \\
 & 4x^4 \\ \hline
4x^2 - 3x & -12x^3 - 11x^2 \\
 & -12x^3 + 9x^2 \\ \hline
4x^2 - 6x - 5 & -20x^2 + 30x + 25 \\
 & -20x^2 + 30x + 25 \\ \hline
\end{array}
$$

8. $9x^4 + 49 - 12x^3 - 28x + 46x^2.$

$$
\begin{array}{r|l}
 & 9x^4 - 12x^3 + 46x^2 - 28x + 49 \,\big|\, \underline{3x^2 - 2x + 7} \\
 & 9x^4 \\ \hline
6x^2 - 2x & -12x^3 + 46x^2 \\
 & -12x^3 + 4x^2 \\ \hline
6x^2 - 4x + 7 & 42x^2 - 28x + 49 \\
 & 42x^2 - 28x + 49 \\ \hline
\end{array}
$$

9. $49x^4 + 126x^3 + 121 - 73x^2 - 198x.$

$$
\begin{array}{r|l}
 & 49x^4 + 126x^3 - 73x^2 - 198x + 121 \,\big|\, \underline{7x^2 + 9x - 11} \\
 & 49x^4 \\ \hline
14x^2 + 9x & 126x^3 - 73x^2 \\
 & 126x^3 + 81x^2 \\ \hline
14x^2 + 18x - 11 & -154x^2 - 198x + 121 \\
 & -154x^2 - 198x + 121 \\ \hline
\end{array}
$$

10. $16x^4 - 30x - 31x^2 + 24x^3 + 25.$

$$
\begin{array}{r|l}
 & 16x^4 + 24x^3 - 31x^2 - 30x + 25 \,\big|\, \underline{4x^2 + 3x - 5} \\
 & 16x^4 \\ \hline
8x^2 + 3x & 24x^3 - 31x^2 \\
 & 24x^3 + 9x^2 \\ \hline
8x^2 + 6x - 5 & -40x^2 - 30x + 25 \\
 & -40x^2 - 30x + 25 \\ \hline
\end{array}
$$

11. $\frac{x^4}{a^4}-\frac{2x^3}{a^3}+\frac{3x^2}{a^2}-\frac{2x}{a}+1.$

$$\begin{array}{r|l}
 & \frac{x^4}{a^4}-\frac{2x^3}{a^3}+\frac{3x^2}{a^2}-\frac{2x}{a}+1 \;\Big|\; \frac{x^2}{a^2}-\frac{x}{a}+1\\
 & \frac{x^4}{a^4}\\
\frac{2x^2}{a^2}-\frac{x}{a} & -\frac{2x^3}{a^3}+\frac{3x^2}{a^2}\\
 & -\frac{2x^3}{a^3}+\frac{x^2}{a^2}\\ \hline
\frac{2x^2}{a^2}-\frac{2x}{a}+1 & \frac{2x^2}{a^2}-\frac{2x}{a}+1\\
 & \frac{2x^2}{a^2}-\frac{2x}{a}+1\\ \hline
\end{array}$$

12. $4x^4+4x^3-\frac{1}{2}x+\frac{1}{16}.$

$$\begin{array}{r|l}
 & 4x^4+4x^3-\frac{1}{2}x+\frac{1}{16} \;\Big|\; 2x^2+x-\frac{1}{4}\\
 & 4x^4\\
4x^2+x & 4x^3\\
 & 4x^3+x^2\\ \hline
4x^2+2x-\frac{1}{4} & -x^2-\frac{1}{2}x+\frac{1}{16}\\
 & -x^2-\frac{1}{2}x+\frac{1}{16}\\ \hline
\end{array}$$

13. $\frac{4a^2}{b^2}+8+\frac{4b^2}{a^2}.$

$$\begin{array}{r|l}
 & \frac{4a^2}{b^2}+8+\frac{4b^2}{a^2} \;\Big|\; \frac{2a}{b}+\frac{2b}{a}\\
 & \frac{4a^2}{b^2}\\
\frac{4a}{b}+\frac{2b}{a} & 8+\frac{4b^2}{a^2}\\
 & 8+\frac{4b^2}{a^2}\\ \hline
\end{array}$$

14. $a^4 - 2a^3 + \frac{3a^2}{2} - \frac{a}{2} + \frac{1}{16}$.

$$\begin{array}{r|l|l}
 & a^4 - 2a^3 + \frac{3a^2}{2} - \frac{a}{2} + \frac{1}{16} & a^2 - a + \frac{1}{4} \\
 & a^4 & \\
2a^2 - a & -2a^3 + \frac{3a^2}{2} & \\
 & -2a^3 + a^2 & \\
2a^2 - 2a + \frac{1}{4} & \frac{a^2}{2} - \frac{a}{2} + \frac{1}{16} & \\
 & \frac{a^2}{2} - \frac{a}{2} + \frac{1}{16} &
\end{array}$$

15. $x^4 + \frac{2x^3}{3} + \frac{10x^2}{9} + \frac{x}{3} + \frac{1}{4}$.

$$\begin{array}{r|l|l}
 & x^4 + \frac{2x^3}{3} + \frac{10x^2}{9} + \frac{x}{3} + \frac{1}{4} & x^2 + \frac{x}{3} + \frac{1}{2} \\
 & x^4 & \\
2x^2 + \frac{x}{3} & \frac{2x^3}{3} + \frac{10x^2}{9} & \\
 & \frac{2x^3}{3} + \frac{x^2}{9} & \\
2x^2 + \frac{2x}{3} + \frac{1}{2} & x^2 + \frac{x}{3} + \frac{1}{4} & \\
 & x^2 + \frac{x}{3} + \frac{1}{4} &
\end{array}$$

16. $\frac{4x^2}{y^2} + \frac{3x}{y} + \frac{41}{16} + \frac{3y}{4x} + \frac{y^2}{4x^2}$.

$$\begin{array}{r|l|l}
 & \frac{4x^2}{y^2} + \frac{3x}{y} + \frac{41}{16} + \frac{3y}{4x} + \frac{y^2}{4x^2} & \frac{2x}{y} + \frac{3}{4} + \frac{y}{2x} \\
 & \frac{4x^2}{y^2} & \\
\frac{4x}{y} + \frac{3}{4} & \frac{3x}{y} + \frac{41}{16} & \\
 & \frac{3x}{y} + \frac{9}{16} & \\
\frac{4x}{y} + \frac{3}{2} + \frac{y}{2x} & 2 + \frac{3y}{4x} + \frac{y^2}{4x^2} & \\
 & 2 + \frac{3y}{4x} + \frac{y^2}{4x^2} &
\end{array}$$

17. $\frac{a^2}{4} - ax + \frac{3a}{2} + x^2 - 3x + \frac{9}{4}$.

$$\begin{array}{r|l}
 & \frac{a^2}{4} - ax + x^2 + \frac{3a}{2} - 3x + \frac{9}{4} \;\Big|\; \underline{\frac{a}{2} - x + \frac{3}{2}} \\
 & \frac{a^2}{4} \\
\hline
a - x & -ax + x^2 \\
 & -ax + x^2 \\
\hline
a - 2x + \frac{3}{2} & \frac{3a}{2} - 3x + \frac{9}{4} \\
 & \frac{3a}{2} - 3x + \frac{9}{4} \\
\hline
\end{array}$$

18. $16x^4 + \frac{16}{3}x^2y + 8x^2 + \frac{4}{9}y^2 + \frac{4}{3}y + 1$.

$$\begin{array}{r|l}
 & 16x^4 + \frac{16}{3}x^2y + 8x^2 + \frac{4}{9}y^2 + \frac{4}{3}y + 1 \;\Big|\; \underline{4x^2 + \frac{2}{3}y + 1} \\
 & 16x^4 \\
\hline
8x^2 + \frac{2}{3}y & \frac{16}{3}x^2y + \frac{4}{9}y^2 \\
 & \frac{16}{3}x^2y + \frac{4}{9}y^2 \\
\hline
8x^2 + \frac{4}{3}y + 1 & 8x^2 + \frac{4}{3}y + 1 \\
 & 8x^2 + \frac{4}{3}y + 1 \\
\hline
\end{array}$$

19. $\frac{9x^4}{4} - \frac{3x^3}{2} + \frac{43x^2}{4} - \frac{7x}{2} + \frac{49}{4}$.

$$\begin{array}{r|l}
 & \frac{9x^4}{4} - \frac{3x^3}{2} + \frac{43x^2}{4} - \frac{7x}{2} + \frac{49}{4} \;\Big|\; \underline{\frac{3x^2}{2} - \frac{x}{2} + \frac{7}{2}} \\
 & \frac{9x^4}{4} \\
\hline
3x^2 - \frac{x}{2} & -\frac{3x^3}{2} + \frac{43x^2}{4} \\
 & -\frac{3x^3}{2} + \frac{x^2}{4} \\
\hline
3x^2 - x + \frac{7}{2} & \frac{21x^2}{2} - \frac{7x}{2} + \frac{49}{4} \\
 & \frac{21x^2}{2} - \frac{7x}{2} + \frac{49}{4} \\
\hline
\end{array}$$

20. $4a^2+\frac{9}{a^2}-\frac{6}{a}-11+4a.$

$$\begin{array}{r|l|l}
 & 4a^2+4a-11-\frac{6}{a}+\frac{9}{a^2} & \underline{2a+1-\frac{3}{a}} \\
 & 4a^2 & \\ \hline
4a+1 & 4a-11 & \\
 & 4a+\ \ 1 & \\ \hline
4a+2-\frac{3}{a} & -12-\frac{6}{a}+\frac{9}{a^2} & \\
 & -12-\frac{6}{a}+\frac{9}{a^2} & \\ \hline
\end{array}$$

Find to three terms the square root of

21. $a^2+b.$

$$\begin{array}{r|l|l}
 & a^2+b & \underline{a+\frac{b}{2a}-\frac{b^2}{8a^3}} \\
 & a^2 & \\ \hline
2a+\frac{b}{2a} & b & \\
 & b+\frac{b^2}{4a^2} & \\ \hline
2a+\frac{b}{a}-\frac{b^2}{8a^3} & -\frac{b^2}{4a^2} & \\
 & -\frac{b^2}{4a^2}-\frac{b^3}{8a^4}+\frac{b^4}{64a^6} & \\ \hline
 & \qquad\qquad \frac{b^3}{8a^4}-\frac{b^4}{64a^6} & \\
\end{array}$$

22. $x^2+\frac{1}{2}y.$

$$\begin{array}{r|l|l}
 & x^2+\frac{1}{2}y & \underline{x+\frac{y}{4x}-\frac{y^2}{32x^3}} \\
 & x^2 & \\ \hline
2x+\frac{y}{4x} & \frac{1}{2}y & \\
 & \frac{1}{2}y+\frac{y^2}{16x^2} & \\ \hline
2x+\frac{y}{2x}-\frac{y^2}{32x^3} & -\frac{y^2}{16x^2} & \\
 & -\frac{y^2}{16x^2}-\frac{y^3}{64x^4}+\frac{y^4}{1024x^6} & \\ \hline
 & \qquad\qquad \frac{y^3}{64x^4}-\frac{y^4}{1024x^6} & \\
\end{array}$$

23. $1+2a$.

$$\begin{array}{r|l|l}
 & 1+2a & 1+a-\frac{a^2}{2} \\
\cline{3-3}
 & 1 & \\
2+a & 2a & \\
 & 2a+a^2 & \\
\cline{2-2}
2+2a-\frac{a^2}{2} & -a^2 & \\
 & -a^2-a^3+\frac{a^4}{4} & \\
\cline{2-2}
 & a^3-\frac{a^4}{4} &
\end{array}$$

24. $1+a$.

$$\begin{array}{r|l|l}
 & 1+a & 1+\frac{a}{2}-\frac{a^2}{8} \\
\cline{3-3}
 & 1 & \\
2+\frac{a}{2} & a & \\
 & a+\frac{a^2}{4} & \\
\cline{2-2}
2+a-\frac{a^2}{8} & -\frac{a^2}{4} & \\
 & -\frac{a^2}{4}-\frac{a^3}{8}+\frac{a^4}{64} & \\
\cline{2-2}
 & \frac{a^3}{8}-\frac{a^4}{64} &
\end{array}$$

25. $1-2a$.

$$\begin{array}{r|l|l}
 & 1-2a & 1-a-\frac{a^2}{2} \\
\cline{3-3}
 & 1 & \\
2-a & -2a & \\
 & -2a+a^2 & \\
\cline{2-2}
2-2a-\frac{a^2}{2} & -a^2 & \\
 & -a^2+a^3+\frac{a^4}{4} & \\
\cline{2-2}
 & -a^3-\frac{a^4}{4} &
\end{array}$$

26. $4a^2+2b$.

$$\begin{array}{rll}
 & 4a^2+2b & \underline{\left|\,2a+\frac{b}{2a}-\frac{b^2}{16a^3}\right.} \\
 & \underline{4a^2} & \\
4a+\frac{b}{2a} & \left|\,2b\right. & \\
 & \left|\,\underline{2b+\frac{b^2}{4a^2}}\right. & \\
4a+\frac{b}{a}-\frac{b^2}{16a^3} & \left|\,-\frac{b^2}{4a^2}\right. & \\
 & \left|\,\underline{-\frac{b^2}{4a^2}-\frac{b^3}{16a^4}+\frac{b^4}{256a^6}}\right. & \\
 & \qquad\frac{b^3}{16a^4}-\frac{b^4}{256a^6} &
\end{array}$$

27. $4x^2+3$.

$$\begin{array}{rll}
 & 4x^2+3 & \underline{\left|\,2x+\frac{3}{4x}-\frac{9}{64x^3}\right.} \\
 & \underline{4x^2} & \\
4x+\frac{3}{4x} & \left|\,3\right. & \\
 & \left|\,\underline{3+\frac{9}{16x^2}}\right. & \\
4x+\frac{3}{2x}-\frac{9}{64x^3} & \left|\,-\frac{9}{16x^2}\right. & \\
 & \left|\,\underline{-\frac{9}{16x^2}-\frac{27}{128x^4}+\frac{81}{4096x^6}}\right. & \\
 & \qquad\frac{27}{128x^4}-\frac{81}{4096x^6} &
\end{array}$$

28. $4-3a$.

$$\begin{array}{rll}
 & 4-3a & \underline{\left|\,2-\frac{3a}{4}-\frac{9a^2}{64}\right.} \\
 & 4 & \\
4-\frac{3a}{4} & \left|\,-3a\right. & \\
 & \left|\,\underline{-3a+\frac{9a^2}{16}}\right. & \\
4-\frac{3a}{2}-\frac{9a^2}{64} & \left|\,-\frac{9a^2}{16}\right. & \\
 & \left|\,\underline{-\frac{9a^2}{16}+\frac{27a^3}{128}+\frac{81a^4}{4096}}\right. & \\
 & \qquad-\frac{27a^3}{128}-\frac{81a^4}{4096} &
\end{array}$$

29. $4a^2 - 1$.

$$
\begin{array}{r|l}
 & 4a^2 - 1 \qquad \left| \; 2a - \frac{1}{4a} - \frac{1}{64a^3} \right. \\
 & 4a^2 \\
\hline
4a - \frac{1}{4a} & -1 \\
 & -1 + \frac{1}{16a^2} \\
\hline
4a - \frac{1}{2a} - \frac{1}{64a^3} & -\frac{1}{16a^2} \\
 & -\frac{1}{16a^2} + \frac{1}{128a^4} + \frac{1}{4096a^6} \\
\hline
 & -\frac{1}{128a^4} - \frac{1}{4096a^6}
\end{array}
$$

Exercise 72. Page 211.

Find the square root of

1. 289.

$$
\begin{array}{rl}
 & 2\,89(17 \\
 & 1 \\
\hline
27) & 189 \\
 & \underline{189}
\end{array}
$$

2. 1225.

$$
\begin{array}{rl}
 & 12\,25(35 \\
 & 9 \\
\hline
65) & 325 \\
 & \underline{325}
\end{array}
$$

3. 12544.

$$
\begin{array}{rl}
 & 1\,25\,44(112 \\
 & 1 \\
\hline
21) & 25 \\
 & 21 \\
\hline
222) & 444 \\
 & \underline{444}
\end{array}
$$

4. 253009.

$$
\begin{array}{rl}
 & 25\,30\,09(503 \\
 & 25 \\
\hline
1003) & 3009 \\
 & \underline{3009}
\end{array}
$$

5. 529984.

$$
\begin{array}{rl}
 & 52\,99\,84(728 \\
 & 49 \\
\hline
142) & 399 \\
 & 284 \\
\hline
1448) & 11584 \\
 & \underline{11584}
\end{array}
$$

6. 150.0625.

$$
\begin{array}{rl}
 & 1\,50.06\,25(12.25 \\
 & 1 \\
\hline
22) & 50 \\
 & 44 \\
\hline
242) & 606 \\
 & 484 \\
\hline
2445) & 12225 \\
 & \underline{12225}
\end{array}
$$

7. 118.1569.

1 18.15 69(10.87
1
208)1815
1664
2167)15169
15169

8. 172.3969.

1 72.39 69(13.13
1
23)72
69
261)339
261
2623)7869
7869

9. 5200.140544.

52 00.14 05 44(72.112
49
142)300
284
1441)1614
1441
14421)17305
14421
144222)288444
288444

10. 1303.282201.

13 03.28 22 01(36.101
9
66)403
396
721)728
721
72201)72201
72201

11. 640.343025.

6 40.34 30 25(25.305
4
45)240
225
503)1534
1509
50605)253025
253025

12. 100.240144.

1 00.24 01 44(10.012
1
2001)002401
2001
20022)40044
40044

13. 316.021729.

3 16.02 17 29(17.777
1
27)216
189
347)2702
2429
3547)27317
24829
35547)248829
248829

14. 454.585041.

4 54.58 50 41(21.321
4
41)54
41
423)1358
1269
4262)8950
8524
42641)42641
42641

15. 5127.276025.

```
          51 27.27 60 25(71.605
          49
      141)227
          141
     1426)8627
          8556
   143205)716025
          716025
```

Find the square root to four decimal places of

16. 10.

```
         10(3.1622...
          9
      61)100
          61
     626)3900
         3756
    6322)14400
         12644
   63242)175600
```

17. 3.

```
         3(1.7320...
         1
      27)200
         189
     343)1100
         1029
    3462)7100
         6924
   34640)17600
```

18. 5.

```
         5(2.2360...
         4
      42)100
          84
     443)1600
         1329
    4466)27100
         26796
   44720)30400
```

19. 0.5.

```
          0.50(0.7071...
            49
     1407)10000
           9849
    14141)15100
```

20. 0.7.

```
          0.70(0.8366...
           .64
      163)600
          489
     1666)11100
          9996
    16726)110400
```

21. 0.9.

```
          0.90(0.9486...
            81
      184)900
          736
     1888)16400
          15104
    18966)129600
```

22. 0.607.

```
          0.60 70(0.7791...
            49
      147)1170
          1029
     1549)14100
          13941
    15581)15900
```

23. 0.521.

$$\begin{array}{rl} & 0.52\;10(0.7218\cdots \\ & \underline{49} \\ 142) & 310 \\ & \underline{284} \\ 1441) & 2600 \\ & \underline{1441} \\ 14427) & 115900 \end{array}$$

24. 0.687.

$$\begin{array}{rl} & 0.68\;70(0.8288\cdots \\ & \underline{64} \\ 162) & 470 \\ & \underline{324} \\ 1648) & 14600 \\ & \underline{13184} \\ 16568) & 141600 \end{array}$$

25. $\frac{2}{3}$.

$$\begin{array}{rl} & 0.66\;66\;66\;66(0.8164\cdots \\ & \underline{64} \\ 161) & 266 \\ & \underline{161} \\ 1626) & 10566 \\ & \underline{9756} \\ 16324) & 81066 \end{array}$$

26. $\frac{3}{4}$.

$$\begin{array}{rl} & 0.75(0.8660\cdots \\ & \underline{64} \\ 166) & 1100 \\ & \underline{996} \\ 1726) & 10400 \\ & \underline{10356} \\ 17320) & 4400 \end{array}$$

27. $\frac{4}{5}$.

$$\begin{array}{rl} & 0.80(0.8944\cdots \\ & \underline{64} \\ 169) & 1600 \\ & \underline{1521} \\ 1784) & 7900 \\ & \underline{7136} \\ 17884) & 76400 \end{array}$$

28. $\frac{6}{7}$.

$$\begin{array}{rl} & 0.85\;71\;42\;85(0.9258\cdot\cdot \\ & \underline{81} \\ 182) & 471 \\ & \underline{364} \\ 1845) & 10742 \\ & \underline{9225} \\ 18458) & 151785 \end{array}$$

29. $\frac{5}{8}$.

$$\begin{array}{rl} & 0.62\;50(0.7905\cdots \\ & \underline{49} \\ 149) & 1350 \\ & \underline{1341} \\ 15805) & 90000 \end{array}$$

30. $\frac{9}{11}$.

$$\begin{array}{rl} & 0.81\;81\;81\;81(0.9045\cdots \\ & \underline{81} \\ 1804) & 8181 \\ & \underline{7216} \\ 18085) & 96581 \end{array}$$

Exercise 73. Page 213.

Find the cube root of

1. $a^3 + 3a^2x + 3ax^2 + x^3$.

$$\begin{array}{l|l} & a^3 + 3a^2x + 3ax^2 + x^3 \;\underline{|\,a + x} \\ 3a^2 & a^3 \\ \underline{\qquad + 3ax + x^2} & 3a^2x + 3ax^2 + x^3 \\ 3a^2 + 3ax + x^2 & \underline{3a^2x + 3ax^2 + x^3} \end{array}$$

2. $8+12x+6x^2+x^3.$

$$
\begin{array}{lll}
 & 8+12x+6x^2+x^3 & \underline{|\,2+x} \\
12 & 8 & \\
\quad +6x+x^2 & \quad 12x+6x^2+x^3 & \\
\hline
12+6x+x^2 & \quad 12x+6x^2+x^3 &
\end{array}
$$

3. $x^6-3ax^5+5a^3x^3-3a^5x-a^6.$

$$
\begin{array}{lll}
 & x^6-3ax^5 \qquad +5a^3x^3-3a^5x-a^6 & \underline{|\,x^2-ax-a^2} \\
3x^4 & x^6 & \\
\quad -3ax^3+a^2x^2 & -3ax^5 \qquad +5a^3x^3 & \\
\hline
3x^4-3ax^3+a^2x^2 & -3ax^5+3a^2x^4-\ a^3x^3 & \\
\hline
3(x^2-ax)^2=3x^4-6ax^3+3a^2x^2 & -3a^2x^4+6a^3x^3-3a^5x-a^6 & \\
-3(x^2-ax)a^2= \qquad -3a^2x^2+3a^3x & & \\
\qquad\qquad +a^4 & & \\
\hline
3x^4-6ax^3 \qquad +3a^3x+a^4 & -3a^2x^4+6a^3x^3-3a^5x-a^6 &
\end{array}
$$

4. $1-6x+21x^2-44x^3+63x^4-54x^5+27x^6.$

$$
\begin{array}{lll}
 & 1-6x+21x^2-44x^3+63x^4-54x^5+27x^6 & \underline{|\,1-2x+3x^2} \\
3 & 1 & \\
\quad -6x+4x^2 & -6x+21x^2-44x^3 & \\
\hline
3-6x+4x^2 & -6x+12x^2-\ 8x^3 & \\
\hline
3(1-2x)^2=3-12x+12x^2 & 9x^2-36x^3+63x^4-54x^5+27x^6 & \\
9x^2(1-2x)\ = \qquad 9x^2-18x^3 & & \\
\qquad\qquad +9x^4 & & \\
\hline
3-12x+21x^2-18x^3+9x^4 & 9x^2-36x^3+63x^4-54x^5+27x^6 &
\end{array}
$$

5. $1-3x+6x^2-7x^3+6x^4-3x^5+x^6.$

$$\begin{array}{rr|l}
 & 1-3x+6x^2-7x^3+6x^4-3x^5+x^6 & \underline{1-x+x^2} \\
3 & 1 & \\
\underline{-3x+x^2} & -3x+6x^2-7x^3 & \\
3-3x+x^2 & \underline{-3x+3x^2-x^3} & \\
3(1-x)^2=3-6x+3x^2 & 3x^2-6x^3+6x^4-3x^5+x^6 & \\
3x^2(1-x)=\qquad 3x^2-3x^3 & & \\
\underline{+x^4} & & \\
3-6x+6x^2-3x^3+x^4 & \underline{3x^2-6x^3+6x^4-3x^5+x^6} &
\end{array}$$

6. $x^6+1-6x-6x^5+15x^2+15x^4-20x^3.$

$$\begin{array}{rr|l}
 & x^6-6x^5+15x^4-20x^3+15x^2-6x+1 & \underline{x^2-2x+1} \\
3x^4 & x^6 & \\
\underline{-6x^3+4x^2} & -6x^5+15x^4-20x^3 & \\
3x^4-6x^3+4x^2 & \underline{-6x^5+12x^4-8x^3} & \\
3(x^2-2x)^2=3x^4-12x^3+12x^2 & 3x^4-12x^3+15x^2-6x+1 & \\
3(x^2-2x)=\qquad 3x^2-6x & & \\
\underline{+1} & & \\
3x^4-12x^3+15x^2-6x+1 & \underline{3x^4-12x^3+15x^2-6x+1} &
\end{array}$$

7. $64x^6 - 144x^5 + 8 - 36x + 102x^2 - 171x^3 + 204x^4.$

$$
\begin{array}{lr|l}
 & 64x^6 - 144x^5 + 204x^4 - 171x^3 + 102x^2 - 36x + 8 & \underline{4x^2 - 3x + 2} \\
48x^4 & 64x^6 & \\
\cline{2-2}
\underline{\quad - 36x^3 + 9x^2} & -144x^5 + 204x^4 - 171x^3 & \\
48x^2 - 36x^3 + 9x^2 & -144x^5 + 108x^4 - 27x^3 & \\
\cline{2-2}
3(4x^2 - 3x)^2 = 48x^4 - 72x^3 + 27x^2 & 96x^4 - 144x^3 + 102x^2 - 36x + 8 & \\
6(4x^2 - 3x) = \qquad 24x^2 - 18x & & \\
\underline{\qquad\qquad\qquad\qquad + 4} & & \\
48x^4 - 72x^3 + 51x^2 - 18x + 4 & \underline{96x^4 - 144x^3 + 102x^2 - 36x + 8} &
\end{array}
$$

8. $27a^6 - 27a^5 - 18a^4 + 17a^3 + 6a^2 - 3a - 1.$

$$
\begin{array}{lr|l}
 & 27a^6 - 27a^5 - 18a^4 + 17a^3 + 6a^2 - 3a - 1 & \underline{3a^2 - a - 1} \\
27a^4 & 27a^6 & \\
\cline{2-2}
\underline{\quad - 9a^3 + a^2} & -27a^5 - 18a^4 + 17a^3 & \\
27a^4 - 9a^3 + a^2 & -27a^5 + 9a^4 - a^3 & \\
\cline{2-2}
3(3a^2 - a)^2 = 27a^4 - 18a^3 + 3a^2 & -27a^4 + 18a^3 + 6a^2 - 3a - 1 & \\
-3(3a^2 - a) = \qquad -9a^2 + 3a & & \\
\underline{\qquad\qquad\qquad\qquad + 1} & & \\
27a^4 - 18a^3 - 6a^2 + 3a + 1 & \underline{-27a^4 + 18a^3 + 6a^2 - 3a - 1} &
\end{array}
$$

9. $8x^6-36x^5+66x^4-63x^3+33x^2-9x+1.$

$$\begin{array}{r|l|l}
 & 8x^6-36x^5+66x^4-63x^3+33x^2-9x+1 & \underline{2x^2-3x+1} \\
12x^4 & 8x^6 & \\
\underline{-18x^3+9x^2} & -36x^5+66x^4-63x^3 & \\
12x^4-18x^3+9x^2 & \underline{-36x^5+54x^4-27x^3} & \\
3(2x^2-3x)^2=12x^4-36x^3+27x^2 & 12x^4-36x^3+33x^2-9x+1 & \\
3(2x^2-3x)\;=\qquad\qquad 6x^2-9x & & \\
\underline{+1} & & \\
12x^4-36x^3+33x^2-9x+1 & \underline{12x^4-36x^3+33x^2-9x+1} &
\end{array}$$

10. $27+108x+90x^2-80x^3-60x^4+48x^5-8x^6.$

$$\begin{array}{r|l|l}
 & 27+108x+90x^2-80x^3-60x^4+48x^5-8x^6 & \underline{3+4x-2x^2} \\
27 & 27 & \\
\underline{+36x+16x^2} & 108x+90x^2-80x^3 & \\
27+36x+16x^2 & \underline{108x+144x^2+64x^3} & \\
3(3+4x)^2=27+72x+48x^2 & -54x^2-144x^3-60x^4+48x^5-8x^6 & \\
-6x^2(3+4x)=\qquad -18x^2-24x^3 & & \\
\underline{+4x^4} & & \\
27+72x+30x^2-24x^3+4x^4 & \underline{-54x^2-144x^3-60x^4+48x^5-8x^6} &
\end{array}$$

11. $a^3 - a^2b + \frac{ab^2}{3} - \frac{b^3}{27}$.

$$\begin{array}{r|l|l}
 & a^3 - a^2b + \frac{ab^2}{3} - \frac{b^2}{27} & a - \frac{b}{3} \\ \cline{3-3}
3a^2 & a^3 & \\ \cline{2-2}
-ab + \frac{9}{b^2} & -a^2b + \frac{ab^2}{3} - \frac{b^3}{27} & \\ \hline
3a^2 - ab + \frac{b^2}{9} & -a^2b + \frac{ab^2}{3} - \frac{b^3}{27} & \\ \cline{2-2}
\end{array}$$

12. $\frac{x^9}{y^{15}} + \frac{3x^8}{y^{14}} - \frac{5x^6}{y^{12}} + \frac{3x^4}{y^{10}} - \frac{x^3}{y^9}$.

$$\begin{array}{r|l|l}
 & \frac{x^9}{y^{15}} + \frac{3x^8}{y^{14}} - \frac{5x^6}{y^{12}} + \frac{3x^4}{y^{10}} - \frac{x^3}{y^9} & \frac{x^3}{y^5} + \frac{x^2}{y^4} - \frac{x}{y^3} \\ \cline{3-3}
\frac{3x^6}{y^{10}} & \frac{x^9}{y^{15}} & \\ \cline{2-2}
+\frac{3x^5}{y^9} + \frac{x^4}{y^8} & \frac{3x^8}{y^{14}} - \frac{5x^6}{y^{12}} & \\ \hline
\frac{3x^6}{y^{10}} + \frac{3x^5}{y^9} + \frac{x^4}{y^8} & \frac{3x^8}{y^{14}} + \frac{3x^7}{y^{13}} + \frac{x^6}{y^{12}} & \\ \cline{2-2}
3\left(\frac{x^3}{y^5} + \frac{x^2}{y^4}\right)^2 = \frac{3x^6}{y^{10}} + \frac{6x^5}{y^9} + \frac{3x^4}{y^8} & -\frac{3x^7}{y^{13}} - \frac{6x^6}{y^{12}} + \frac{3x^4}{y^{10}} - \frac{x^3}{y^9} & \\
-\frac{3x}{y^3}\left(\frac{x^3}{y^5} + \frac{x^2}{y^4}\right) = \qquad -\frac{3x^4}{y^8} - \frac{3x^3}{y^7} & & \\
+\frac{x^2}{y^6} & & \\ \hline
\frac{3x^6}{y^{10}} + \frac{6x^5}{y^9} \qquad - \frac{3x^3}{y^7} + \frac{x^2}{y^6} & -\frac{3x^7}{y^{13}} - \frac{6x^6}{y^{12}} + \frac{3x^4}{y^{10}} + \frac{x^3}{y^9} & \\ \cline{2-2}
\end{array}$$

Exercise 74. Page 216.

Find the cube root of

1. 4913.

$$\begin{array}{rr|l}
 & & 4\,913 \,\underline{|\,17} \\
 & & 1 \\ \cline{3-3}
3\times(10)^2 = & 300 & 3913 \\
3(10\times7) = & 210 & \\
7^2 = & \underline{49} & \\
 & 559 & \underline{3913}
\end{array}$$

2. 42875.

$$\begin{array}{rr|l}
 & & 42\,875 \,\underline{|\,35} \\
 & & 27 \\ \cline{3-3}
3\times30^2 = & 2700 & 15875 \\
3(30\times5) = & 450 & \\
5^2 = & \underline{25} & \\
 & 3175 & \underline{15875}
\end{array}$$

3. 1404928.

$$\begin{array}{rr|l}
 & & 1\,404\,928 \,\underline{|\,112} \\
 & & 1 \\ \cline{3-3}
3\times10^2 = & 300 & 404 \\
3(10\times1) = & 30 & \\
1^2 = & \underline{1} & \\
 & \left.\begin{array}{r} 331 \\ 31 \end{array}\right\} & \begin{array}{l} 331 \\ \hline 73928 \end{array} \\ \cline{2-2}
3\times110^2 = & 36300 & \\
3(110\times2) = & 660 & \\
2^2 = & \underline{4} & \\
 & 36964 & \underline{73928}
\end{array}$$

4. 127263527.

$$\begin{array}{rr|l}
 & & 127\,263\,527 \,\underline{|\,503} \\
 & & 125 \\ \cline{3-3}
3\times500^2 = & 750000 & 2263527 \\
3(500\times3) = & 4500 & \\
3^2 = & \underline{9} & \\
 & 754509 & \underline{2263527}
\end{array}$$

5. 385828.352.

$$\begin{array}{rr|l}
 & & 385\ 828.352\ \underline{|\,72.8} \\
 & & 343 \\ \hline
3\times(70)^2 = & 14700 & 42828 \\
3(70\times 2) = & 420 & \\
\begin{array}{r}2^2 = \\ \\ \\ \end{array} & \left.\begin{array}{r}\underline{4} \\ 15124 \\ 424\end{array}\right\} & \begin{array}{l} \\ \underline{30248} \\ 12580352\end{array} \\ \cline{2-2}
3\times 720^2 = & 1555200 & \\
3(720\times 8) = & 17280 & \\
8^2 = & \underline{564} & \\
 & 1572544 & \underline{12580352}
\end{array}$$

6. 1838.265625.

$$\begin{array}{rr|l}
 & & \dot{1}\ 838.265\ 625\ \underline{|\,12.25} \\
 & & 1 \\ \hline
3\times 10^2 = & 300 & 838 \\
3(10\times 2) = & 60 & \\
\begin{array}{r}2^2 = \\ \\ \\ \end{array} & \left.\begin{array}{r}\underline{4} \\ 364 \\ 64\end{array}\right\} & \begin{array}{l} \\ \underline{728} \\ 110265\end{array} \\ \cline{2-2}
3\times 120^2 = & 43200 & \\
3(120\times 2) = & 720 & \\
\begin{array}{r}2^2 = \\ \\ \\ \end{array} & \left.\begin{array}{r}\underline{4} \\ 43924 \\ 724\end{array}\right\} & \begin{array}{l} \\ \underline{87848} \\ 22417625\end{array} \\ \cline{2-2}
3\times 1220^2 = & 4465200 & \\
3(1220\times 5) = & 18300 & \\
5^2 = & \underline{25} & \\
 & 4483525 & \underline{22417625}
\end{array}$$

Find to four decimal places the cube root of

7. 87.

$$\begin{array}{rr|l}
\multicolumn{2}{r}{} & 87\,\underline{|\,4.4310\ldots} \\
\multicolumn{2}{r}{} & \underline{64} \\
3\times 40^2 = & 4800 & 23000 \\
3(40\times 4) = & 480 & \\
4^2 = & \left.\begin{array}{r}\underline{16}\\ 5296\\ \underline{496}\end{array}\right\} & \begin{array}{l} \\ \underline{21184}\\ 1816000\end{array} \\
3\times 440^2 = & 580800 & \\
3(440\times 3) = & 3960 & \\
3^2 = & \left.\begin{array}{r}\underline{9}\\ 584769\\ \underline{3969}\end{array}\right\} & \begin{array}{l} \\ \underline{1754307}\\ 61693000\end{array} \\
3\times 4430^2 = & 58874700 & \\
3(4430\times 1) = & 13290 & \\
1^2 = & \left.\begin{array}{r}\underline{1}\\ 58887991\\ \underline{13291}\end{array}\right\} & \begin{array}{l} \\ \underline{58887991}\\ 2805009000\end{array} \\
3(44310)^2 = & 5890128300 &
\end{array}$$

8. 10.

$$\begin{array}{rr|l}
\multicolumn{2}{r}{} & 10\,\underline{|\,2.1544\ldots} \\
\multicolumn{2}{r}{} & \underline{8} \\
3\times 20^2 = & 1200 & 2000 \\
3(20\times 1) = & 60 & \\
1^2 = & \left.\begin{array}{r}\underline{1}\\ 1261\\ \underline{61}\end{array}\right\} & \begin{array}{l} \\ \underline{1261}\\ 739000\end{array} \\
3\times 210^2 = & 132300 & \\
3(210\times 5) = & 3150 & \\
5^2 = & \left.\begin{array}{r}\underline{25}\\ 135475\\ \underline{3175}\end{array}\right\} & \begin{array}{l} \\ \underline{677375}\\ 61625000\end{array} \\
3\times 2150^2 = & 13867500 & \\
3(2150\times 4) = & 25800 & \\
4^2 = & \left.\begin{array}{r}\underline{16}\\ 13893316\\ \underline{25816}\end{array}\right\} & \begin{array}{l} \\ \underline{55573264}\\ 6051736000\end{array} \\
3(21540)^2 = & 1391914800 &
\end{array}$$

9. 3.02.

$$
\begin{array}{r|l}
 & 3.020\,\underline{|\,1.4454\ldots} \\
 & 1 \\
3\times10^2 = 300 & \overline{2020} \\
3(10\times4) = 120 & \\
4^2 = \left.\begin{array}{r}\underline{16}\\ 436\\ \underline{136}\end{array}\right\} & \begin{array}{l} \\ \underline{1744}\\ 276000\end{array} \\
3\times140^2 = 58800 & \\
3(140\times4) = 1680 & \\
4^2 = \left.\begin{array}{r}\underline{16}\\ 60496\\ \underline{1696}\end{array}\right\} & \begin{array}{l} \\ \underline{241984}\\ 34016000\end{array} \\
3\times1440^2 = 6220800 & \\
3(1440\times5) = 21600 & \\
5^2 = \left.\begin{array}{r}\underline{25}\\ 6242425\\ \underline{21625}\end{array}\right\} & \begin{array}{l} \\ \underline{31212125}\\ 2803875000\end{array} \\
3(14450)^2 = 626407500 &
\end{array}
$$

10. 2.05.

$$
\begin{array}{r|l}
 & 2.050\,\underline{|\,1.2703\ldots} \\
 & 1 \\
3\times10^2 = 300 & \overline{1050} \\
3(10\times2) = 60 & \\
2^2 = \left.\begin{array}{r}\underline{4}\\ 364\\ \underline{64}\end{array}\right\} & \begin{array}{l} \\ \underline{728}\\ 322000\end{array} \\
3\times120^2 = 43200 & \\
3(120\times7) = 2520 & \\
7^2 = \left.\begin{array}{r}\underline{49}\\ 45769\\ \underline{2569}\end{array}\right\} & \begin{array}{l} \\ \underline{320383}\\ 1617000000\end{array} \\
3(1270)^2 = 483870000 &
\end{array}
$$

11. 0.05.

$$\begin{array}{rr|l}
 & & 0.050\,\underline{|0.3684\cdots}\\
 & & \underline{0.027}\\
3\times30^2= & 2700 & 23000\\
3(30\times6)= & 540 & \\
 & \left.\begin{array}{r}6^2=\underline{36}\\ 3276\\ \underline{576}\end{array}\right\} & \begin{array}{l}\\ \underline{19656}\\ 3344000\end{array}\\
3\times360^2= & 388800 & \\
3(360\times8)= & 8640 & \\
 & \left.\begin{array}{r}8^2=\underline{64}\\ 397504\\ \underline{8704}\end{array}\right\} & \begin{array}{l}\\ \underline{3180032}\\ 163968000\end{array}\\
3\times3760^2= & 40627200 &
\end{array}$$

12. 0.677.

$$\begin{array}{rr|l}
 & & 0.677\,\underline{|0.8780\cdots}\\
 & & \underline{0.512}\\
3\times80^2= & 19200 & 165000\\
3(80\times7)= & 1680 & \\
 & \left.\begin{array}{r}7^2=\underline{49}\\ 20929\\ \underline{1729}\end{array}\right\} & \begin{array}{l}\\ \underline{146503}\\ 18497000\end{array}\\
3\times870^2= & 2270700 & \\
3(870\times8)= & 20880 & \\
 & \left.\begin{array}{r}8^2=\underline{64}\\ 2291644\\ \underline{20944}\end{array}\right\} & \begin{array}{l}\\ \underline{18333152}\\ 163848000\end{array}\\
3\times8780^2= & 231265200 &
\end{array}$$

13. $\frac{2}{3}$.

$$\begin{array}{rr|l}
 & & 0.666\ 666\ 666\ 666\,\underline{|0.8735\cdots}\\
 & & \underline{0.512}\\
3\times80^2= & 19200 & 154666\\
3(80\times7)= & 1680 & \\
 & \left.\begin{array}{r}7^2=\underline{49}\\ 20929\\ \underline{1729}\end{array}\right\} & \begin{array}{l}\\ \underline{146503}\\ 8163666\end{array}\\
3\times870^2= & 2270700 & \\
3(870\times3)= & 7830 & \\
 & \left.\begin{array}{r}3^2=\underline{9}\\ 2278539\\ \underline{7839}\end{array}\right\} & \begin{array}{l}\\ \underline{6835617}\\ 1328049666\end{array}\\
3\times8730^2= & 228638700 &
\end{array}$$

14. $\frac{3}{4}$.

$$\begin{array}{rr|r}
 & & 0.750\,\underline{|\,0.9085\cdots} \\
 & & \underline{0.729} \\
3 \times 900^2 = & 2430000 & 21000000 \\
3(900 \times 8) = & 21600 & \\
8^2 = & \underline{64} & \\
 & 2451664 & \underline{19613312} \\
 & \underline{21664} & 1386688000 \\
3 \times 9080^2 = & 247339200 &
\end{array}$$

15. $\frac{9}{11}$.

$$\begin{array}{rr|r}
 & & 0.818\ 181\ 818\ 181\,\underline{|\,0.9352\cdots} \\
 & & \underline{0.729} \\
3 \times 90^2 = & 24300 & 89181 \\
3(90 \times 3) = & 810 & \\
3^2 = & \underline{9} & \\
 & 25119 & \underline{75357} \\
 & \underline{819} & 13824818 \\
3 \times 930^2 = & 2594700 & \\
3(930 \times 5) = & 13950 & \\
5^2 = & \underline{25} & \\
 & 2608675 & \underline{13043375} \\
 & \underline{13975} & 781443181 \\
3 \times 9350^2 = & 262267500 &
\end{array}$$

16. $\frac{1}{13}$.

$$\begin{array}{rr|r}
 & & 0.076\ 923\ 076\ 923\,\underline{|\,0.4252\cdots} \\
 & & \underline{0.064} \\
3 \times 40^2 = & 4800 & 12923 \\
3(40 \times 2) = & 240 & \\
2^2 = & \underline{4} & \\
 & 5044 & \underline{10088} \\
 & \underline{244} & 2835076 \\
3 \times 420^2 = & 529200 & \\
3(420 \times 5) = & 6300 & \\
5^2 = & \underline{25} & \\
 & 535525 & \underline{2677625} \\
 & \underline{6325} & 157451923 \\
3 \times 4250^2 = & 54187500 &
\end{array}$$

Exercise 75. Page 216.

Find the fourth root of

1. $81x^4+108x^3+54x^2+12x+1$.

$$\begin{array}{rl}
 & 81x^4+108x^3+54x^2+12x+1\;\underline{|\,9x^2+6x+1} \\
 & 81x^4 \\ \hline
18x^2+6x & \multicolumn{1}{|l}{108x^3+54x^2} \\
 & \multicolumn{1}{|l}{108x^3+36x^2} \\ \hline
18x^2+12x+1 & \multicolumn{1}{|l}{18x^2+12x+1} \\
 & \multicolumn{1}{|l}{18x^2+12x+1} \\ \hline
\end{array}$$

$$\begin{array}{rl}
 & 9x^2+6x+1\;\underline{|\,3x+1} \\
 & 9x^2 \\ \hline
6x+1 & \multicolumn{1}{|l}{6x+1} \\
 & \multicolumn{1}{|l}{6x+1} \\ \hline
\end{array}$$

2. $16x^4-32ax^3+24a^2x^2-8a^3x+a^4$.

$$\begin{array}{rl}
 & 16x^4-32ax^3+24a^2x^2-8a^3x+a^4\;\underline{|\,4x^2-4ax+a^2} \\
 & 16x^4 \\ \hline
8x^2-4ax & \multicolumn{1}{|l}{-32ax^3+24a^2x^2} \\
 & \multicolumn{1}{|l}{-32ax^3+16a^2x^2} \\ \hline
8x^2-8ax+a^2 & \multicolumn{1}{|l}{8a^2x^2-8a^3x+a^4} \\
 & \multicolumn{1}{|l}{8a^2x^2-8a^3x+a^4} \\ \hline
\end{array}$$

$$\begin{array}{rl}
 & 4x^2-4ax+a^2\;\underline{|\,2x-a} \\
 & 4x^2 \\ \hline
4x-a & \multicolumn{1}{|l}{-4ax+a^2} \\
 & \multicolumn{1}{|l}{-4ax+a^2} \\ \hline
\end{array}$$

3. $1+4x+x^8+4x^7+10x^6+16x^3+10x^2+19x^4+16x^5$.

$$\begin{array}{rl}
 & 1+4x+10x^2+16x^3+19x^4+16x^5+10x^6+4x^7+x^8\;\underline{|\,1+2x+3x^2+2x^3+x^4} \\
 & 1 \\ \hline
2+2x & \multicolumn{1}{|l}{4x+10x^2} \\
 & \multicolumn{1}{|l}{4x+4x^2} \\ \hline
2+4x+3x^2 & \multicolumn{1}{|l}{6x^2+16x^3+19x^4} \\
 & \multicolumn{1}{|l}{6x^2+12x^3+9x^4} \\ \hline
2+4x+6x^2+2x^3 & \multicolumn{1}{|l}{4x^3+10x^4+16x^5+10x^6} \\
 & \multicolumn{1}{|l}{4x^3+8x^4+12x^5+4x^6} \\ \hline
2+4x+6x^2+4x^3+x^4 & \multicolumn{1}{|l}{2x^4+4x^5+6x^6+4x^7+x^8} \\
 & \multicolumn{1}{|l}{2x^4+4x^5+6x^6+4x^7+x^8} \\ \hline
\end{array}$$

$$
\begin{array}{r|l}
\multicolumn{2}{l}{1+2x+3x^2+2x^3+x^4 \,\underline{|\,1+x+x^2}} \\
\multicolumn{2}{l}{1} \\
\cline{2-2}
2+x & 2x+3x^2 \\
 & 2x+x^2 \\
\cline{2-2}
2+2x+x^2 & 2x^2+2x^3+x^4 \\
 & 2x^2+2x^3+x^4 \\
\cline{2-2}
\end{array}
$$

Find the sixth root of

4. $1+6d+d^6+6d^5+15d^4+20d^3+15d^2$.

$$
\begin{array}{r|l}
\multicolumn{2}{l}{1+6d+15d^2+20d^3+15d^4+6d^5+d^6 \,\underline{|\,1+3d+3d^2+d^3}} \\
\multicolumn{2}{l}{1} \\
\cline{2-2}
2+3d & 6d+15d^2 \\
 & 6d+9d^2 \\
\cline{2-2}
2+6d+3d^2 & 6d^2+20d^3+15d^4 \\
 & 6d^2+18d^3+9d^4 \\
\cline{2-2}
2+6d+6d^2+d^3 & 2d^3+6d^4+6d^5+d^6 \\
 & 2d^3+6d^4+6d^5+d^6 \\
\cline{2-2}
\end{array}
$$

$$1+3d+3d^2+d^3=(1+d)^3.$$

Hence the required 6th root is $1+d$.

5. $729-1458x+1215x^2-540x^3+135x^4-18x^5+x^6$.

$$
\begin{array}{r|l}
\multicolumn{2}{l}{729-1458x+1215x^2-540x^3+135x^4-18x^5+x^6 \,\underline{|\,27-27x+9x^2-x^3}} \\
\multicolumn{2}{l}{729} \\
\cline{2-2}
54-27x & -1458x+1215x^2 \\
 & -1458x+729x^2 \\
\cline{2-2}
54-54x+9x^2 & 486x^2-540x^3+135x^4 \\
 & 486x^2-486x^3+81x^4 \\
\cline{2-2}
54-54x+18x^2-x^3 & -54x^3+54x^4-18x^5+x^6 \\
 & -54x^3+54x^4-18x^5+x^6 \\
\cline{2-2}
\end{array}
$$

$$
\begin{array}{r|l}
\multicolumn{2}{l}{27-27x+9x^2-x^3 \,\underline{|\,3-x}} \\
\multicolumn{2}{l}{27} \\
\cline{2-2}
3\times 3^2=27 & -27x+9x^2-x^3 \\
-9x+x^2 & \\
\cline{1-1}
27-9x+x^2 & -27x+9x^2-x^3 \\
\cline{2-2}
\end{array}
$$

6. $1 - 18y + 135y^2 - 540y^3 + 1215y^4 - 1458y^5 + 729y^6.$

$$\begin{array}{r|l}
 & 1-18y+135y^2-540y^3+1215y^4-1458y^5+729y^6 \;\underline{|\,1-9y+27y^2-27y^3} \\
 & 1 \\
2-9y & -18y+135y^2 \\
 & \underline{-18y+81y^2} \\
2-18y+27y^2 & 54y^2-540y^3+1215y^4 \\
 & \underline{54y^2-486y^3+729y^4} \\
2-18y+54y^2-27y^3 & -54y^3+486y^4-1458y^5+729y^6 \\
 & \underline{-54y^3+486y^4-1458y^5+729y^6}
\end{array}$$

$$\begin{array}{r|l}
 & 1-9y+27y^2-27y^3 \;\underline{|\,1-3y} \\
 & 1 \\
3 & -9y+27y^2-27y^3 \\
\underline{-9y+9y^2} & \\
3-9y+9y^2 & \underline{-9y+27y^2-27y^3}
\end{array}$$

Exercise 76. Page 221.

Express with fractional exponents:

1. $\sqrt[3]{a^2} = a^{\frac{2}{3}}.$

2. $\sqrt{a^3} = a^{\frac{3}{2}}.$

3. $\sqrt[5]{a^7} = a^{\frac{7}{5}}.$

4. $\sqrt[3]{-8} = -2.$

5. $\sqrt[4]{a^3} = a^{\frac{3}{4}}.$

6. $\sqrt[3]{a^5} = a^{\frac{5}{3}}.$

7. $\sqrt{a} + \sqrt[3]{x} + \sqrt[4]{16b^2} = a^{\frac{1}{2}} + x^{\frac{1}{3}} + 2b^{\frac{1}{2}}.$

8. $\sqrt[5]{a^2x^5} + \sqrt[3]{a^3c} = a^{\frac{2}{5}}x^{\frac{5}{5}} + a^{\frac{3}{3}}c^{\frac{1}{3}} = a^{\frac{2}{5}}x + ac^{\frac{1}{3}}.$

Express with root-signs:

9. $a^{\frac{3}{4}} = \sqrt[4]{a^3}.$

10. $c^{\frac{2}{3}} = \sqrt[3]{c^2}.$

11. $a^{\frac{1}{2}}b^{\frac{1}{3}} = \sqrt{a}\sqrt[3]{b}.$

12. $a^{\frac{1}{3}}b^{\frac{2}{3}} = \sqrt[3]{ab^2}.$

13. $x^{\frac{1}{3}}y^{-\frac{1}{3}} = \sqrt[3]{\frac{x}{y}}.$

14. $3x^{\frac{2}{3}}y^{-\frac{3}{4}} = \frac{3\sqrt[3]{x^2}}{\sqrt[4]{y^3}}.$

15. $a^{\frac{1}{2}} - x^{\frac{1}{3}}c^{\frac{1}{3}} = \sqrt{a} - \sqrt[3]{cx}.$

16. $a^{\frac{2}{3}} + x^{\frac{2}{3}}c^{\frac{2}{3}} = \sqrt[3]{a^2} + \sqrt[3]{c^2x^2}.$

Express with positive exponents:

17. $a^{-3} = \frac{1}{a^3}.$

18. $a^{-\frac{2}{3}} = \frac{1}{a^{\frac{2}{3}}}.$

19. $3x^{-2}y^3 = \frac{3y^3}{x^2}.$

20. $4xy^{-7} = \frac{4x}{y^7}$.

21. $4x^{-3}y^{-2} = \frac{4}{x^3y^2}$.

22. $3a^{-4}b^{\frac{3}{2}} = \frac{3b^{\frac{3}{2}}}{a^4}$.

23. $\frac{2a^{-1}x^2}{3^{-2}b^2y^{-5}} = \frac{3^2 \times 2x^2y^5}{ab^2}$.

Write without denominators:

24. $\frac{3y^2z^3}{x^{-5}} = 3x^5y^2z^3$.

25. $\frac{x^2z}{x^{-3}z^{-2}} = x^2zx^3z^2 = x^5z^3$.

26. $\frac{abc}{a^{-2}bc^{-2}d} = abca^2b^{-1}c^2d^{-1} = a^3c^3d^{-1}$.

27. $\frac{a^{-1}b^{-2}c^{-3}}{a^{-2}b^{-2}c^{-4}} = a^{-1}b^{-2}c^{-3}a^2b^2c^4 = ac$.

28. $\frac{x^{-2}y^{\frac{2}{3}}}{a^{-2}b^{-\frac{6}{3}}} = x^{-2}y^{\frac{2}{3}}a^2b^{\frac{6}{3}} = a^2b^2x^{-2}y^{\frac{2}{3}}$.

29. $\frac{a^{-4}b^{-5}c^{-6}}{a^{-7}b^{-5}c^{-3}} = a^{-4}b^{-5}c^{-6}a^7b^5c^3 = a^3c^{-3}$.

Find the value of

30. $8^{\frac{5}{3}} = (\sqrt[3]{8})^5 = 2^5 = 32$.

31. $16^{-\frac{5}{4}} = \left(\frac{1}{\sqrt[4]{16}}\right)^5 = \frac{1}{2^5} = \frac{1}{32}$.

32. $27^{\frac{2}{3}} = (\sqrt[3]{27})^2 = 3^2 = 9$.

33. $(-8)^{-\frac{2}{3}} = \left(\frac{1}{\sqrt[3]{-8}}\right)^2 = \left(\frac{1}{-2}\right)^2 = \frac{1}{4}$.

34. $36^{\frac{3}{2}} = (\sqrt{36})^3 = 6^3 = 216$.

35. $(-27)^{\frac{5}{3}} = (\sqrt[3]{-27})^5 = (-3)^5 = -243$.

36. $(-27)^{\frac{1}{3}} \times 25^{\frac{5}{2}} = \sqrt[3]{-27} \times (\sqrt{25})^5 = -3 \times 5^5 = -9375$.

37. $81^{-\frac{3}{2}} \times 16^{\frac{3}{4}} = \left(\frac{1}{\sqrt{81}}\right)^3 \times (\sqrt[4]{16})^3 = \left(\frac{1}{9}\right)^3 \times 2^3 = \frac{8}{729}$.

Simplify:

38. $8^{\frac{2}{3}} \times 4^{-\frac{1}{2}} = (\sqrt[3]{8})^2 \times \frac{1}{\sqrt{4}} = \frac{2^2}{2} = 2.$

39. $\left(\frac{1}{25}\right)^{\frac{1}{2}} \times 16^{-\frac{3}{4}} = \frac{1}{5} \times \left(\frac{1}{\sqrt[4]{16}}\right)^3 = \frac{1}{5} \times \frac{1}{2^3} = \frac{1}{5 \times 8} = \frac{1}{40}.$

40. $\left(\frac{1}{125}\right)^{-\frac{2}{3}} \times \left(\frac{1}{36}\right)^{\frac{1}{2}} = 125^{\frac{2}{3}} \times \frac{1}{6} = (\sqrt[3]{125})^2 \times \frac{1}{6} = \frac{25}{6}.$

41. $a^{-\frac{3}{4}}b^{\frac{2}{3}} \times a^{\frac{1}{4}}b^{\frac{1}{3}} = a^{-\frac{3}{4}+\frac{1}{4}}b^{\frac{2}{3}+\frac{1}{3}} = a^{-\frac{1}{2}}b = \frac{b}{\sqrt{a}}.$

42. $(a^{-\frac{2}{3}}b^3)^{-\frac{3}{2}} = a^{(-\frac{2}{3})\times(-\frac{3}{2})}b^{3\times(-\frac{3}{2})} = ab^{-\frac{9}{2}} = \frac{a}{b^{\frac{9}{2}}}.$

43. $(a^{-\frac{1}{2}}b^{-1})^{-2} = a^{(-\frac{1}{2})\times(-2)}b^{(-1)\times(-2)} = ab^2.$

If $a = 4,\ b = 2,\ c = 1,$ **find the value of**

44. $a^{\frac{1}{2}}b^{-1} = 4^{\frac{1}{2}}2^{-1} = \frac{\sqrt{4}}{2} = \frac{2}{2} = 1.$

45. $ab^{-2} = 4 \times 2^{-2} = \frac{4}{2^2} = \frac{4}{4} = 1.$

46. $a^{-\frac{1}{2}}b^2 = 4^{-\frac{1}{2}} \times 2^2 = \frac{2^2}{\sqrt{4}} = \frac{4}{2} = 2.$

47. $a^{-\frac{1}{2}}c^{-\frac{2}{3}} = 4^{-\frac{1}{2}} \times 1^{-\frac{2}{3}} = \frac{1}{\sqrt{4} \times (\sqrt[3]{1})^2} = \frac{1}{2}.$

48. $8(ab)^{\frac{1}{3}} = 3(4 \times 2)^{\frac{1}{3}} = 3 \times 8^{\frac{1}{3}} = 3 \times 2 = 6.$

49. $2(ab)^{-\frac{1}{3}} = 2(4 \times 2)^{-\frac{1}{3}} = 2 \times 8^{-\frac{1}{3}} = \frac{2}{8^{\frac{1}{3}}} = \frac{2}{2} = 1.$

50. $(ab^2c)^{\frac{1}{4}} = (4 \times 2^2 \times 1)^{\frac{1}{4}} = 16^{\frac{1}{4}} = 2.$

51. $(ab^2c)^{-\frac{1}{4}} = (4 \times 2^2 \times 1)^{-\frac{1}{4}} = 16^{-\frac{1}{4}} = \frac{1}{16^{\frac{1}{4}}} = \frac{1}{2}.$

Exercise 77. Page 223.

Multiply:

1. $a^{\frac{1}{2}}+b^{\frac{1}{2}}$ by $a^{\frac{1}{2}}-b^{\frac{1}{2}}$.

$$\begin{array}{llll}
a^{\frac{1}{2}}+b^{\frac{1}{2}} & & & \\
a^{\frac{1}{2}}-b^{\frac{1}{2}} & & & \\
\hline
a & +a^{\frac{1}{2}}b^{\frac{1}{2}} & & \\
 & -a^{\frac{1}{2}}b^{\frac{1}{2}} & -b & \\
\hline
a & & -b &
\end{array}$$

2. $a^{\frac{1}{2}}+b^{\frac{1}{2}}$ by $a^{\frac{1}{2}}+b^{\frac{1}{2}}$.

$$\begin{array}{lll}
a^{\frac{1}{2}}+b^{\frac{1}{2}} & & \\
a^{\frac{1}{2}}+b^{\frac{1}{2}} & & \\
\hline
a & +\ a^{\frac{1}{2}}b^{\frac{1}{2}} & \\
 & +\ a^{\frac{1}{2}}b^{\frac{1}{2}} & +b \\
\hline
a & +2a^{\frac{1}{2}}b^{\frac{1}{2}} & +b
\end{array}$$

3. $a^{\frac{1}{2}}-b^{\frac{1}{2}}$ by $a^{\frac{1}{2}}-b^{\frac{1}{2}}$.

$$\begin{array}{lll}
a^{\frac{1}{2}}-b^{\frac{1}{2}} & & \\
a^{\frac{1}{2}}-b^{\frac{1}{2}} & & \\
\hline
a & -\ a^{\frac{1}{2}}b^{\frac{1}{2}} & \\
 & -\ a^{\frac{1}{2}}b^{\frac{1}{2}} & +b \\
\hline
a & -2a^{\frac{1}{2}}b^{\frac{1}{2}} & +b
\end{array}$$

4. $x^{\frac{3}{2}}+2x$ by $x^{\frac{3}{2}}-2x$.

$$\begin{array}{lll}
x^{\frac{3}{2}}+2x & & \\
x^{\frac{3}{2}}-2x & & \\
\hline
x^3 & +2x^{\frac{5}{2}} & \\
 & -2x^{\frac{5}{2}} & -4x^2 \\
\hline
x^3 & & -4x^2
\end{array}$$

5. $x^{-2}+x^{-1}y^{-1}+y^{-2}$ by $x^{-2}-x^{-1}y^{-1}+y^{-2}$.

$$\begin{array}{lllll}
x^{-2}+x^{-1}y^{-1}+y^{-2} & & & & \\
x^{-2}-x^{-1}y^{-1}+y^{-2} & & & & \\
\hline
x^{-4} & +x^{-3}y^{-1} & +x^{-2}y^{-2} & & \\
 & -x^{-3}y^{-1} & -x^{-2}y^{-2} & -x^{-1}y^{-3} & \\
 & & +x^{-2}y^{-2} & +x^{-1}y^{-3} & +y^{-4} \\
\hline
x^{-4} & & +x^{-2}y^{-2} & & +y^{-4}
\end{array}$$

6. $x^{\frac{2}{3}}-x^{\frac{1}{3}}y^{\frac{1}{3}}+y^{\frac{2}{3}}$ by $x^{\frac{1}{3}}+y^{\frac{1}{3}}$.

$$\begin{array}{llll}
x^{\frac{2}{3}}-x^{\frac{1}{3}}y^{\frac{1}{3}}+y^{\frac{2}{3}} & & & \\
x^{\frac{1}{3}}+y^{\frac{1}{3}} & & & \\
\hline
x & -x^{\frac{2}{3}}y^{\frac{1}{3}} & +x^{\frac{1}{3}}y^{\frac{2}{3}} & \\
 & +x^{\frac{2}{3}}y^{\frac{1}{3}} & -x^{\frac{1}{3}}y^{\frac{2}{3}} & +y \\
\hline
x & & & +y
\end{array}$$

7. $x^{\frac{2}{3}}+x^{\frac{1}{3}}y^{\frac{1}{3}}+y^{\frac{2}{3}}$ by $x^{\frac{1}{3}}-y^{\frac{1}{3}}$.

$$\begin{array}{llll}
x^{\frac{2}{3}}+x^{\frac{1}{3}}y^{\frac{1}{3}}+y^{\frac{2}{3}} & & & \\
x^{\frac{1}{3}}-y^{\frac{1}{3}} & & & \\
\hline
x & +x^{\frac{2}{3}}y^{\frac{1}{3}} & +x^{\frac{1}{3}}y^{\frac{2}{3}} & \\
 & -x^{\frac{2}{3}}y^{\frac{1}{3}} & -x^{\frac{1}{3}}y^{\frac{2}{3}} & -y \\
\hline
x & & & -y
\end{array}$$

8. $1+b^{-1}+b^{-2}$ by $1-b^{-1}+b^{-2}$.

$$\begin{array}{rrrrr}
1 & +b^{-1} & +b^{-2} & & \\
1 & -b^{-1} & +b^{-2} & & \\ \hline
1 & +b^{-1} & +b^{-2} & & \\
 & -b^{-1} & -b^{-2} & -b^{-3} & \\
 & & +b^{-2} & +b^{-3} & +b^{-4} \\ \hline
1 & & +b^{-2} & & +b^{-4}
\end{array}$$

9. $x^{\frac{1}{2}}+x^{\frac{1}{4}}y^{\frac{1}{4}}+y^{\frac{1}{2}}$ by $x^{\frac{1}{2}}-x^{\frac{1}{4}}y^{\frac{1}{4}}+y^{\frac{1}{2}}$.

$$\begin{array}{rrrrr}
x^{\frac{1}{2}} & +x^{\frac{1}{4}}y^{\frac{1}{4}} & +y^{\frac{1}{2}} & & \\
x^{\frac{1}{2}} & -x^{\frac{1}{4}}y^{\frac{1}{4}} & +y^{\frac{1}{2}} & & \\ \hline
x & +x^{\frac{3}{4}}y^{\frac{1}{4}} & +x^{\frac{1}{2}}y^{\frac{1}{2}} & & \\
 & -x^{\frac{3}{4}}y^{\frac{1}{4}} & -x^{\frac{1}{2}}y^{\frac{1}{2}} & -x^{\frac{1}{4}}y^{\frac{3}{4}} & \\
 & & +x^{\frac{1}{2}}y^{\frac{1}{2}} & +x^{\frac{1}{4}}y^{\frac{3}{4}} & +y \\ \hline
x & & +x^{\frac{1}{2}}y^{\frac{1}{2}} & & +y
\end{array}$$

10. $a^{\frac{2}{3}}b^{-\frac{1}{2}}+2a^{\frac{1}{3}}-3b^{\frac{1}{2}}$ by $2b^{-\frac{1}{2}}-4a^{-\frac{1}{3}}-6a^{-\frac{2}{3}}b^{\frac{1}{2}}$.

$$\begin{array}{rrrrr}
a^{\frac{2}{3}}b^{-\frac{1}{2}} & +2a^{\frac{1}{3}} & -3b^{\frac{1}{2}} & & \\
2b^{-\frac{1}{2}} & -4a^{-\frac{1}{3}} & -6a^{-\frac{2}{3}}b^{\frac{1}{2}} & & \\ \hline
2a^{\frac{2}{3}}b^{-1} & +4a^{\frac{1}{3}}b^{-\frac{1}{2}} & -\ 6 & & \\
 & -4a^{\frac{1}{3}}b^{-\frac{1}{2}} & -\ 8 & +12a^{-\frac{1}{3}}b^{\frac{1}{2}} & \\
 & & -\ 6 & -12a^{-\frac{1}{3}}b^{\frac{1}{2}} & +18a^{-\frac{2}{3}}b \\ \hline
2a^{\frac{2}{3}}b^{-1} & & -20 & & +18a^{-\frac{2}{3}}b
\end{array}$$

Divide:

11. $a-b$ by $a^{\frac{1}{3}}-b^{\frac{1}{3}}$.

$$\begin{array}{ll}
a-b & \underline{|\,a^{\frac{1}{3}}-b^{\frac{1}{3}}} \\
\underline{a-a^{\frac{2}{3}}b^{\frac{1}{3}}} & a^{\frac{2}{3}}+a^{\frac{1}{3}}b^{\frac{1}{3}}+b^{\frac{2}{3}} \\
\quad a^{\frac{2}{3}}b^{\frac{1}{3}}-b & \\
\quad \underline{a^{\frac{2}{3}}b^{\frac{1}{3}}-a^{\frac{1}{3}}b^{\frac{2}{3}}} & \\
\qquad\qquad a^{\frac{1}{3}}b^{\frac{2}{3}}-b & \\
\qquad\qquad \underline{a^{\frac{1}{3}}b^{\frac{2}{3}}-b} &
\end{array}$$

12. $a+b$ by $a^{\frac{1}{3}}+b^{\frac{1}{3}}$.

$$\begin{array}{ll}
a+b & \underline{|\,a^{\frac{1}{3}}+b^{\frac{1}{3}}} \\
\underline{a+a^{\frac{2}{3}}b^{\frac{1}{3}}} & a^{\frac{2}{3}}-a^{\frac{1}{3}}b^{\frac{1}{3}}+b^{\frac{2}{3}} \\
\quad -a^{\frac{2}{3}}b^{\frac{1}{3}}+b & \\
\quad \underline{-a^{\frac{2}{3}}b^{\frac{1}{3}}-a^{\frac{1}{3}}b^{\frac{2}{3}}} & \\
\qquad\qquad a^{\frac{1}{3}}b^{\frac{2}{3}}+b & \\
\qquad\qquad \underline{a^{\frac{1}{3}}b^{\frac{2}{3}}+b} &
\end{array}$$

13. $a-b$ by $a^{\frac{1}{4}}-b^{\frac{1}{4}}$.

$$\begin{array}{l|l}
a-b & \underline{a^{\frac{1}{4}}-b^{\frac{1}{4}}} \\
\underline{a-a^{\frac{3}{4}}b^{\frac{1}{4}}} & a^{\frac{3}{4}}+a^{\frac{1}{2}}b^{\frac{1}{4}}+a^{\frac{1}{4}}b^{\frac{1}{2}}+b^{\frac{3}{4}} \\
\quad a^{\frac{3}{4}}b^{\frac{1}{4}}-b & \\
\quad \underline{a^{\frac{3}{4}}b^{\frac{1}{4}}-a^{\frac{1}{2}}b^{\frac{1}{2}}} & \\
\qquad a^{\frac{1}{2}}b^{\frac{1}{2}}-b & \\
\qquad \underline{a^{\frac{1}{2}}b^{\frac{1}{2}}-a^{\frac{1}{4}}b^{\frac{3}{4}}} & \\
\qquad\quad a^{\frac{1}{4}}b^{\frac{3}{4}}-b & \\
\qquad\quad \underline{a^{\frac{1}{4}}b^{\frac{3}{4}}-b} &
\end{array}$$

14. $a+b$ by $a^{\frac{1}{5}}+b^{\frac{1}{5}}$.

$$\begin{array}{l|l}
a+b & \underline{a^{\frac{1}{5}}+b^{\frac{1}{5}}} \\
\underline{a+a^{\frac{4}{5}}b^{\frac{1}{5}}} & a^{\frac{4}{5}}-a^{\frac{3}{5}}b^{\frac{1}{5}}+a^{\frac{2}{5}}b^{\frac{2}{5}}-a^{\frac{1}{5}}b^{\frac{3}{5}}+b^{\frac{4}{5}} \\
\quad -a^{\frac{4}{5}}b^{\frac{1}{5}}+b & \\
\quad \underline{-a^{\frac{4}{5}}b^{\frac{1}{5}}-a^{\frac{3}{5}}b^{\frac{2}{5}}} & \\
\qquad a^{\frac{3}{5}}b^{\frac{2}{5}}+b & \\
\qquad \underline{a^{\frac{3}{5}}b^{\frac{2}{5}}+a^{\frac{2}{5}}b^{\frac{3}{5}}} & \\
\qquad\quad -a^{\frac{2}{5}}b^{\frac{3}{5}}+b & \\
\qquad\quad \underline{-a^{\frac{2}{5}}b^{\frac{3}{5}}-a^{\frac{1}{5}}b^{\frac{4}{5}}} & \\
\qquad\qquad a^{\frac{1}{5}}b^{\frac{4}{5}}+b & \\
\qquad\qquad \underline{a^{\frac{1}{5}}b^{\frac{4}{5}}+b} &
\end{array}$$

15. $2x^{-2}+6x^{-1}y^{-1}-16x^2y^{-4}$ by $2x+2x^2y^{-1}+4x^3y^{-2}$.

$$\begin{array}{l|l}
2x^{-2}+6x^{-1}y^{-1}-16x^2y^{-4} & \underline{2x+2x^2y^{-1}+4x^3y^{-2}} \\
\underline{2x^{-2}+2x^{-1}y^{-1}+4y^{-2}} & x^{-3}+2x^{-2}y^{-1}-4x^{-1}y^{-2} \\
\quad 4x^{-1}y^{-1}-4y^{-2}-16x^2y^{-4} & \\
\quad \underline{4x^{-1}y^{-1}+4y^{-2}+8xy^{-3}} & \\
\qquad -8y^{-2}-8xy^{-3}-16x^2y^{-4} & \\
\qquad \underline{-8y^{-2}-8xy^{-3}-16x^2y^{-4}} &
\end{array}$$

16. $x + y + z - 3x^{\frac{1}{3}}y^{\frac{1}{3}}z^{\frac{1}{3}}$ by $x^{\frac{1}{3}} + y^{\frac{1}{3}} + z^{\frac{1}{3}}$.

$$\begin{array}{l|l}
x + y + z - 3x^{\frac{1}{3}}y^{\frac{1}{3}}z^{\frac{1}{3}} & x^{\frac{1}{3}} + y^{\frac{1}{3}} + z^{\frac{1}{3}} \\ \cline{2-2}
\underline{x + x^{\frac{2}{3}}(y^{\frac{1}{3}} + z^{\frac{1}{3}})} & x^{\frac{2}{3}} - x^{\frac{1}{3}}(y^{\frac{1}{3}} + z^{\frac{1}{3}}) + (y^{\frac{2}{3}} - y^{\frac{1}{3}}z^{\frac{1}{3}} + z^{\frac{2}{3}}) \\
\quad -x^{\frac{2}{3}}(y^{\frac{1}{3}} + z^{\frac{1}{3}}) + y + z - 3x^{\frac{1}{3}}y^{\frac{1}{3}}z^{\frac{1}{3}} & \\
\quad \underline{-x^{\frac{2}{3}}(y^{\frac{1}{3}} + z^{\frac{1}{3}}) - x^{\frac{1}{3}}y^{\frac{2}{3}} - x^{\frac{1}{3}}z^{\frac{2}{3}} - 2x^{\frac{1}{3}}y^{\frac{1}{3}}z^{\frac{1}{3}}} & \\
\qquad\qquad x^{\frac{1}{3}}(y^{\frac{2}{3}} - y^{\frac{1}{3}}z^{\frac{1}{3}} + z^{\frac{2}{3}}) + y + z & \\
\qquad\qquad \underline{x^{\frac{1}{3}}(y^{\frac{2}{3}} - y^{\frac{1}{3}}z^{\frac{1}{3}} + z^{\frac{2}{3}}) + y + z} &
\end{array}$$

17. $x - 3x^{\frac{2}{3}} + 3x^{\frac{1}{3}} - 1$ by $x^{\frac{1}{3}} - 1$.

$$\begin{array}{l|l}
x - 3x^{\frac{2}{3}} + 3x^{\frac{1}{3}} - 1 & x^{\frac{1}{3}} - 1 \\ \cline{2-2}
\underline{x - x^{\frac{2}{3}}} \phantom{+ 3x^{\frac{1}{3}} - 1} & x^{\frac{2}{3}} - 2x^{\frac{1}{3}} + 1 \\
\quad -2x^{\frac{2}{3}} + 3x^{\frac{1}{3}} & \\
\quad \underline{-2x^{\frac{2}{3}} + 2x^{\frac{1}{3}}} & \\
\qquad\qquad x^{\frac{1}{3}} - 1 & \\
\qquad\qquad \underline{x^{\frac{1}{3}} - 1} &
\end{array}$$

18. $x + x^{\frac{1}{2}}y^{\frac{1}{2}} + y$ by $x^{\frac{1}{2}} - x^{\frac{1}{4}}y^{\frac{1}{4}} + y^{\frac{1}{2}}$.

$$\begin{array}{l|l}
x + x^{\frac{1}{2}}y^{\frac{1}{2}} + y & x^{\frac{1}{2}} - x^{\frac{1}{4}}y^{\frac{1}{4}} + y^{\frac{1}{2}} \\ \cline{2-2}
\underline{x - x^{\frac{3}{4}}y^{\frac{1}{4}} + x^{\frac{1}{2}}y^{\frac{1}{2}}} & x^{\frac{1}{2}} + x^{\frac{1}{4}}y^{\frac{1}{4}} + y^{\frac{1}{2}} \\
\quad x^{\frac{3}{4}}y^{\frac{1}{4}} + y & \\
\quad \underline{x^{\frac{3}{4}}y^{\frac{1}{4}} - x^{\frac{1}{2}}y^{\frac{1}{2}} + x^{\frac{1}{4}}y^{\frac{3}{4}}} & \\
\qquad\qquad x^{\frac{1}{2}}y^{\frac{1}{2}} - x^{\frac{1}{4}}y^{\frac{3}{4}} + y & \\
\qquad\qquad \underline{x^{\frac{1}{2}}y^{\frac{1}{2}} - x^{\frac{1}{4}}y^{\frac{3}{4}} + y} &
\end{array}$$

19. $x^{\frac{2}{3}} - 4x^{\frac{1}{3}} + 1 + 6x^{-\frac{1}{3}}$ by $x^{\frac{1}{3}} - 2$.

$$\begin{array}{l|l}
x^{\frac{2}{3}} - 4x^{\frac{1}{3}} + 1 + 6x^{-\frac{1}{3}} & x^{\frac{1}{3}} - 2 \\ \cline{2-2}
\underline{x^{\frac{2}{3}} - 2x^{\frac{1}{3}}} & x^{\frac{1}{3}} - 2 - 3x^{-\frac{1}{3}} \\
\qquad -2x^{\frac{1}{3}} + 1 + 6x^{-\frac{1}{3}} & \\
\qquad \underline{-2x^{\frac{1}{3}} + 4 \qquad\quad} & \\
\qquad\qquad\qquad -3 + 6x^{-\frac{1}{3}} & \\
\qquad\qquad\qquad \underline{-3 + 6x^{-\frac{1}{3}}} &
\end{array}$$

20. $9x - 12x^{\frac{1}{2}} - 2 + 4x^{-\frac{1}{2}} + x^{-1}$ by $3x^{\frac{1}{2}} - 2 - x^{-\frac{1}{2}}$.

$$\begin{array}{l|l}
9x - 12x^{\frac{1}{2}} - 2 + 4x^{-\frac{1}{2}} + x^{-1} & 3x^{\frac{1}{2}} - 2 - x^{-\frac{1}{2}} \\ \cline{2-2}
\underline{9x - 6x^{\frac{1}{2}} - 3} & 3x^{\frac{1}{2}} - 2 - x^{-\frac{1}{2}} \\
\qquad - 6x^{\frac{1}{2}} + 1 + 4x^{-\frac{1}{2}} + x^{-1} & \\
\qquad \underline{- 6x^{\frac{1}{2}} + 4 + 2x^{-\frac{1}{2}} \qquad\;\;} & \\
\qquad\qquad\qquad -3 + 2x^{-\frac{1}{2}} + x^{-1} & \\
\qquad\qquad\qquad \underline{-3 + 2x^{-\frac{1}{2}} + x^{-1}} &
\end{array}$$

Find the square root of

21. $x^{\frac{2}{3}} + 2x^{\frac{1}{3}} + 1$.

$$\begin{array}{r|l}
\multicolumn{2}{l}{\qquad\qquad x^{\frac{2}{3}} + 2x^{\frac{1}{3}} + 1 \,\lfloor\, \underline{x^{\frac{1}{3}} + 1}} \\
\multicolumn{2}{l}{\qquad\qquad x^{\frac{2}{3}}} \\ \cline{2-2}
2x^{\frac{1}{3}} + 1 & 2x^{\frac{1}{3}} + 1 \\
 & \underline{2x^{\frac{1}{3}} + 1}
\end{array}$$

22. $4a^{\frac{2}{3}} - 4a^{\frac{1}{3}}b^{\frac{1}{4}} + b^{\frac{1}{2}}$.

$$\begin{array}{r|l}
\multicolumn{2}{l}{\qquad\quad 4a^{\frac{2}{3}} - 4a^{\frac{1}{3}}b^{\frac{1}{4}} + b^{\frac{1}{2}} \,\lfloor\, \underline{2a^{\frac{1}{3}} - b^{\frac{1}{4}}}} \\
\multicolumn{2}{l}{\qquad\quad 4a^{\frac{2}{3}}} \\ \cline{2-2}
4a^{\frac{1}{3}} - b^{\frac{1}{4}} & -4a^{\frac{1}{3}}b^{\frac{1}{4}} + b^{\frac{1}{2}} \\
 & \underline{-4a^{\frac{1}{3}}b^{\frac{1}{4}} + b^{\frac{1}{2}}}
\end{array}$$

23. $x^{\frac{2}{3}} - 4x^{\frac{1}{3}} + 4$.

$$\begin{array}{r|l}
\multicolumn{2}{l}{\qquad\quad x^{\frac{2}{3}} - 4x^{\frac{1}{3}} + 4 \,\lfloor\, \underline{x^{\frac{1}{3}} - 2}} \\
\multicolumn{2}{l}{\qquad\quad x^{\frac{2}{3}}} \\ \cline{2-2}
2x^{\frac{1}{3}} - 2 & -4x^{\frac{1}{3}} + 4 \\
 & \underline{-4x^{\frac{1}{3}} + 4}
\end{array}$$

24. $4a^{-2} + 4a^{-1} + 1$.

$$\begin{array}{r|l}
\multicolumn{2}{l}{\quad 4a^{-2} + 4a^{-1} + 1 \,\lfloor\, \underline{2a^{-1} + 1}} \\
\multicolumn{2}{l}{\quad 4a^{-2}} \\ \cline{2-2}
4a^{-1} + 1 & 4a^{-1} + 1 \\
 & \underline{4a^{-1} + 1}
\end{array}$$

25. $9a - 12a^{\frac{1}{2}} + 10 - 4a^{-\frac{1}{2}} + a^{-1}$.

$$\begin{array}{r|l|l}
 & 9a - 12a^{\frac{1}{2}} + 10 - 4a^{-\frac{1}{2}} + a^{-1} & \underline{3a^{\frac{1}{2}} - 2 + a^{-\frac{1}{2}}} \\
 & 9a & \\
6a^{\frac{1}{2}} - 2 & -12a^{\frac{1}{2}} + 10 & \\
 & \underline{-12a^{\frac{1}{2}} + 4} & \\
6a^{\frac{1}{2}} - 4 + a^{-\frac{1}{2}} & 6 - 4a^{-\frac{1}{2}} + a^{-1} & \\
 & \underline{6 - 4a^{-\frac{1}{2}} + a^{-1}} &
\end{array}$$

26. $49x^{\frac{4}{3}} - 28x + 18x^{\frac{2}{3}} - 4x^{\frac{1}{3}} + 1$.

$$\begin{array}{r|l|l}
 & 49x^{\frac{4}{3}} - 28x + 18x^{\frac{2}{3}} - 4x^{\frac{1}{3}} + 1 & \underline{7x^{\frac{2}{3}} - 2x^{\frac{1}{3}} + 1} \\
 & 49x^{\frac{4}{3}} & \\
14x^{\frac{2}{3}} - 2x^{\frac{1}{3}} & -28x + 18x^{\frac{2}{3}} & \\
 & \underline{-28x + 4x^{\frac{2}{3}}} & \\
14x^{\frac{2}{3}} - 4x^{\frac{1}{3}} + 1 & 14x^{\frac{2}{3}} - 4x^{\frac{1}{3}} + 1 & \\
 & \underline{14x^{\frac{2}{3}} - 4x^{\frac{1}{3}} + 1} &
\end{array}$$

27. $m^2 + 2m - 1 - 2m^{-1} + m^{-2}$.

$$\begin{array}{r|l|l}
 & m^2 + 2m - 1 - 2m^{-1} + m^{-2} & \underline{m + 1 - m^{-1}} \\
 & m^2 & \\
2m + 1 & 2m - 1 & \\
 & \underline{2m + 1} & \\
2m + 2 - m^{-1} & -2 - 2m^{-1} + m^{-2} & \\
 & \underline{-2 - 2m^{-1} + m^{-2}} &
\end{array}$$

28. $1+4y^{-\frac{1}{3}}-2y^{-\frac{2}{3}}-4y^{-1}+25y^{-\frac{4}{3}}-24y^{-\frac{5}{3}}+16y^{-2}.$

$$\begin{array}{r|l}
 & 1+4y^{-\frac{1}{3}}-2y^{-\frac{2}{3}}-4y^{-1}+25y^{-\frac{4}{3}}-24y^{-\frac{5}{3}}+16y^{-2} \quad \underline{|1+2y^{-\frac{1}{3}}-3y^{-\frac{2}{3}}+4y^{-1}} \\
 & 1 \\ \hline
2+2y^{-\frac{1}{3}} & 4y^{-\frac{1}{3}}-2y^{-\frac{2}{3}} \\
 & 4y^{-\frac{1}{3}}+4y^{-\frac{2}{3}} \\ \hline
2+4y^{-\frac{1}{3}}-3y^{-\frac{2}{3}} & -6y^{-\frac{2}{3}}-4y^{-1}+25y^{-\frac{4}{3}} \\
 & -6y^{-\frac{2}{3}}-12y^{-1}+9y^{-\frac{4}{3}} \\ \hline
2+4y^{-\frac{1}{3}}-6y^{-\frac{2}{3}}+4y^{-1} & 8y^{-1}+16y^{-\frac{4}{3}}-24y^{-\frac{5}{3}}+16y^{-2} \\
 & 8y^{-1}+16y^{-\frac{4}{3}}-24y^{-\frac{5}{3}}+16y^{-2} \\ \hline
\end{array}$$

Expand:

29. $(x-x^{-1})^3=x^3-3x^2x^{-1}+3xx^{-2}-x^{-3}$

$=x^3-3x+3x^{-1}-x^{-3}.$

30. $(\sqrt{x}^3-\frac{1}{2}x)^4=(x^{\frac{1}{3}}-\frac{1}{2}x)^4$

$=x^{\frac{4}{3}}-4(\frac{1}{2})x^{\frac{3}{3}}x+6(\frac{1}{2})^2x^{\frac{2}{3}}x^2-4(\frac{1}{2})^3x^{\frac{1}{3}}x^3+(\frac{1}{2})^4x^4$

$=x^{\frac{4}{3}}-2x^2+\frac{3}{2}x^{\frac{8}{3}}-\frac{1}{2}x^{\frac{10}{3}}+\frac{1}{16}x^4.$

31. $(2a^{\frac{1}{2}}-a^{-\frac{1}{2}})^3=(2a^{\frac{1}{2}})^3-3(2a^{\frac{1}{2}})^2(a^{-\frac{1}{2}})+3(2a^{\frac{1}{2}})(a^{-\frac{1}{2}})^2-(a^{-\frac{1}{2}})^3$

$=8a^{\frac{3}{2}}-12a^{\frac{1}{2}}+6a^{-\frac{1}{2}}-a^{-\frac{3}{2}}.$

32. $(2x^{-2}+x^{\frac{1}{2}})^5=(2x^{-2})^5+5(2x^{-2})^4(x^{\frac{1}{2}})+10(2x^{-2})^3(x^{\frac{1}{2}})^2$

$+10(2x^{-2})^2(x^{\frac{1}{2}})^3+5(2x^{-2})(x^{\frac{1}{2}})^4+(x^{\frac{1}{2}})^5$

$=32x^{-10}+80x^{-\frac{15}{2}}+80x^{-5}+40x^{-\frac{5}{2}}+10+x^{\frac{5}{2}}.$

33. $(\frac{1}{2}\sqrt{x}-\frac{1}{3}\sqrt[3]{x})^4=(\frac{1}{2}x^{\frac{1}{2}}-\frac{1}{3}x^{\frac{1}{3}})^4$

$=(\frac{1}{2}x^{\frac{1}{2}})^4-4(\frac{1}{2}x^{\frac{1}{2}})^3(\frac{1}{3}x^{\frac{1}{3}})+6(\frac{1}{2}x^{\frac{1}{2}})^2(\frac{1}{3}x^{\frac{1}{3}})^2$

$-4(\frac{1}{2}x^{\frac{1}{2}})(\frac{1}{3}x^{\frac{1}{3}})^3+(\frac{1}{3}x^{\frac{1}{3}})^4$

$=\frac{1}{16}x^2-\frac{1}{6}x^{\frac{11}{6}}+\frac{1}{6}x^{\frac{5}{3}}-\frac{2}{27}x^{\frac{3}{2}}+\frac{1}{81}x^{\frac{4}{3}}.$

34. $(\frac{1}{3}\sqrt{x^{-3}} + \frac{1}{4}x^{-1})^3 = (\frac{1}{3}x^{-\frac{3}{2}} + \frac{1}{4}x^{-1})^3$
$= (\frac{1}{3}x^{-\frac{3}{2}})^3 + 3(\frac{1}{3}x^{-\frac{3}{2}})^2(\frac{1}{4}x^{-1}) + 3(\frac{1}{3}x^{-\frac{3}{2}})(\frac{1}{4}x^{-1})^2 + (\frac{1}{4}x^{-1})^3$
$= \frac{1}{27}x^{-\frac{9}{2}} + \frac{1}{12}x^{-4} + \frac{1}{16}x^{-\frac{7}{2}} + \frac{1}{64}x^{-3}$.

Exercise 78. Page 226.

Simplify:

1. $\sqrt[4]{25} = \sqrt[4]{5^2} = 5^{\frac{2}{4}} = 5^{\frac{1}{2}} = \sqrt{5}$.
2. $\sqrt[8]{16} = \sqrt[8]{2^4} = 2^{\frac{4}{8}} = 2^{\frac{1}{2}} = \sqrt{2}$.
3. $\sqrt[6]{27} = \sqrt[6]{3^3} = 3^{\frac{3}{6}} = 3^{\frac{1}{2}} = \sqrt{3}$.
4. $\sqrt[4]{49} = \sqrt[4]{7^2} = 7^{\frac{2}{4}} = 7^{\frac{1}{2}} = \sqrt{7}$.
5. $\sqrt[6]{64} = \sqrt[6]{2^6} = 2^{\frac{6}{6}} = 2$.
6. $\sqrt[6]{a^2b^2} = a^{\frac{2}{6}}b^{\frac{2}{6}} = a^{\frac{1}{3}}b^{\frac{1}{3}} = \sqrt[3]{ab}$.
7. $\sqrt[4]{a^2b^2} = a^{\frac{2}{4}}b^{\frac{2}{4}} = a^{\frac{1}{2}}b^{\frac{1}{2}} = \sqrt{ab}$.
8. $\sqrt[8]{a^4b^4} = a^{\frac{4}{8}}b^{\frac{4}{8}} = a^{\frac{1}{2}}b^{\frac{1}{2}} = \sqrt{ab}$
9. $\sqrt[6]{27\,a^3b^6} = \sqrt[6]{3^3\,a^3b^6} = 3^{\frac{3}{6}}\,a^{\frac{3}{6}}b^{\frac{6}{6}} = 3^{\frac{1}{2}}\,a^{\frac{1}{2}}b = b\sqrt{3\,a}$.
10. $\sqrt[8]{16\,a^4b^4} = \sqrt[8]{2^4\,a^4b^4} = 2^{\frac{4}{8}}\,a^{\frac{4}{8}}b^{\frac{4}{8}} = 2^{\frac{1}{2}}\,a^{\frac{1}{2}}b^{\frac{1}{2}} = \sqrt{2\,ab}$.
11. $\sqrt[6]{\dfrac{25\,a^2}{64\,b^2}} = \sqrt[6]{\dfrac{5^2\,a^2}{2^6\,b^2}} = \dfrac{5^{\frac{2}{6}}\,a^{\frac{2}{6}}}{2^{\frac{6}{6}}\,b^{\frac{2}{6}}} = \dfrac{5^{\frac{1}{3}}\,a^{\frac{1}{3}}}{2\,b^{\frac{1}{3}}} = \dfrac{1}{2}\sqrt[3]{\dfrac{5\,a}{b}}$.
12. $\sqrt[4]{\dfrac{16\,x^2}{(x-3)^2}} = \sqrt[4]{\dfrac{2^4\,x^2}{(x-3)^2}} = \dfrac{2^{\frac{4}{4}}\,x^{\frac{2}{4}}}{(x-3)^{\frac{2}{4}}} = \dfrac{2\,x^{\frac{1}{2}}}{(x-3)^{\frac{1}{2}}} = 2\sqrt{\dfrac{x}{x-3}}$.
13. $\sqrt[6]{\dfrac{a^3b^6}{8\,x^3y^3}} = \sqrt[6]{\dfrac{a^3b^6}{2^3\,x^3y^3}} = \dfrac{a^{\frac{3}{6}}b^{\frac{6}{6}}}{2^{\frac{3}{6}}\,x^{\frac{3}{6}}y^{\frac{3}{6}}} = \dfrac{a^{\frac{1}{2}}b}{2^{\frac{1}{2}}\,x^{\frac{1}{2}}y^{\frac{1}{2}}} = b\sqrt{\dfrac{a}{2\,xy}}$.

Exercise 79. Page 227.

Simplify·

1. $\sqrt{28} = \sqrt{4 \times 7} = 2\sqrt{7}$.
2. $\sqrt{72} = \sqrt{36 \times 2} = 6\sqrt{2}$.
3. $\sqrt[3]{72} = \sqrt[3]{8 \times 9} = 2\sqrt[3]{9}$.
4. $\sqrt[3]{500} = \sqrt[3]{125 \times 4} = 5\sqrt[3]{4}$.
5. $\sqrt[3]{432} = \sqrt[3]{216 \times 2} = 6\sqrt[3]{2}$.
6. $\sqrt[3]{192} = \sqrt[3]{64 \times 3} = 4\sqrt[3]{3}$.

7. $\sqrt[5]{128} = \sqrt[5]{32 \times 4} = 2\sqrt[5]{4}$.

8. $\sqrt[4]{243} = \sqrt[4]{81 \times 3} = 3\sqrt[4]{3}$.

9. $\sqrt[4]{176} = \sqrt[4]{16 \times 11} = 2\sqrt[4]{11}$.

10. $\sqrt{405} = \sqrt{81 \times 5} = 9\sqrt{5}$.

11. $2\sqrt[4]{112} = 2\sqrt[4]{16 \times 7} = 4\sqrt[4]{7}$.

12. $3\sqrt[3]{864} = 3\sqrt[3]{216 \times 4} = 18\sqrt[3]{4}$.

13. $7\sqrt[4]{144} = 7\sqrt[4]{16 \times 9} = 14\sqrt[4]{9} = 14\sqrt{3}$.

14. $8\sqrt{m^2n} = 8m\sqrt{n}$.

15. $3\sqrt[4]{b^8a^3} = 3b^2\sqrt[4]{a^3}$.

16. $2\sqrt[5]{a^{18}c^4} = 2\sqrt[5]{a^{15} \times a^3c^4} = 2a^3\sqrt[5]{a^3c^4}$.

17. $11\sqrt[6]{a^{12}b^{15}} = 11\sqrt[6]{a^{12}b^{12} \times b^3} = 11a^2b^2\sqrt[6]{b^3} = 11a^2b^2\sqrt{b}$.

18. $7\sqrt[3]{8a^3b} = 7 \times 2a\sqrt[3]{b} = 14a\sqrt[3]{b}$.

19. $6\sqrt[3]{27m^2n^3} = 6\sqrt[3]{27n^3 \times m^2} = 18n\sqrt[3]{m^2}$.

20. $4\sqrt[4]{x^7y^8} = 4\sqrt[4]{x^4y^8 \times x^3} = 4xy^2\sqrt[4]{x^3}$.

21. $\sqrt[3]{1029} = \sqrt[3]{343 \times 3} = 7\sqrt[3]{3}$.

22. $\sqrt[3]{-2187} = \sqrt[3]{(-729) \times 3} = -9\sqrt[3]{3}$.

23. $\sqrt[4]{1250} = \sqrt[4]{625 \times 2} = 5\sqrt[4]{2}$.

24. $4\sqrt[3]{648} = 4\sqrt[3]{216 \times 3} = 24\sqrt[3]{3}$.

25. $\sqrt[3]{\dfrac{64x^6y}{27m^3n^3}} = \sqrt[3]{\dfrac{64x^6}{27m^3n^3} \times y} = \dfrac{4x^2}{3mn}\sqrt[3]{y}$

26. $\sqrt{\dfrac{4a^3b}{9}} = \sqrt{\left(\dfrac{4a^2}{9}\right)ab} = \dfrac{2a}{3}\sqrt{ab}$.

27. $\sqrt[3]{\dfrac{125x^3}{216y^3}} = \dfrac{5x}{6y}$.

28. $2\sqrt[5]{\dfrac{m^6n}{243}} = 2\sqrt[5]{\left(\dfrac{m^5}{243}\right)mn} = \dfrac{2m}{3}\sqrt[5]{mn}$.

29. $\sqrt[4]{\frac{x^5y^7}{1296}} = \sqrt[4]{\left(\frac{x^4y^4}{1296}\right)xy^3} = \frac{xy}{6}\sqrt[4]{xy^3}.$

30. $\sqrt[3]{\frac{(x-y)^3z^3}{512}} = \frac{(x-y)z}{8}.$

31. $\frac{3ab}{2c}\sqrt{\frac{20c^2}{9a^2b^2}} = \frac{3ab}{2c}\sqrt{\frac{4c^2}{9a^2b^2}\times 5} = \frac{3ab}{2c}\times\frac{2c}{3ab}\sqrt{5} = \sqrt{5}.$

Exercise 80. Page 228.

Simplify:

1. $2\sqrt{\tfrac{1}{2}} = 2\sqrt{\tfrac{2}{4}} = \tfrac{2}{2}\sqrt{2} = \sqrt{2}.$

2. $\tfrac{3}{4}\sqrt{\tfrac{2}{3}} = \tfrac{3}{4}\sqrt{\tfrac{6}{9}} = \tfrac{3}{4}\times\tfrac{1}{3}\sqrt{6} = \tfrac{1}{4}\sqrt{6}.$

3. $\tfrac{1}{4}\sqrt{\tfrac{1}{5}} = \tfrac{1}{4}\sqrt{\tfrac{5}{25}} = \tfrac{1}{4}\times\tfrac{1}{5}\sqrt{5} = \tfrac{1}{20}\sqrt{5}.$

4. $7\sqrt{\frac{4}{5}} = 7\sqrt{\frac{4\times 5}{25}} = 7\times\frac{2}{5}\sqrt{5} = \frac{14}{5}\sqrt{5}.$

5. $\sqrt[4]{\frac{25}{16}} = \sqrt[4]{\frac{5^2}{2^4}} = \frac{1}{2}\sqrt[4]{5^2} = \frac{1}{2}\sqrt{5}.$

6. $3\sqrt{\frac{9}{80}} = 3\sqrt{\frac{9}{16\times 5}} = \frac{9}{4}\sqrt{\frac{1}{5}} = \frac{9}{4}\sqrt{\frac{5}{25}} = \frac{9}{20}\sqrt{5}.$

7. $\sqrt[3]{\tfrac{8}{3}} = 2\sqrt[3]{\tfrac{1}{3}} = 2\sqrt[3]{\tfrac{9}{27}} = \tfrac{2}{3}\sqrt[3]{9}.$

8. $\sqrt[3]{\tfrac{9}{8}} = \tfrac{1}{2}\sqrt[3]{9}.$

9. $\sqrt[3]{\frac{24}{343}} = \sqrt[3]{\frac{8\times 3}{343}} = \frac{2}{7}\sqrt[3]{3}.$

10. $2\sqrt[3]{\frac{5}{3}} = 2\sqrt[3]{\frac{5\times 9}{27}} = \frac{2}{3}\sqrt[3]{45}.$

11. $3\sqrt[5]{\frac{2}{81}} = 3\sqrt[5]{\frac{2\times 3}{243}} = \frac{3}{3}\sqrt[5]{6} = \sqrt[5]{6}.$

12. $2\sqrt[5]{\frac{3}{16}} = 2\sqrt[5]{\frac{3\times 2}{32}} = \frac{2}{2}\sqrt[5]{6} = \sqrt[5]{6}.$

13. $\sqrt{\frac{a^4c}{b^3}}=\sqrt{\left(\frac{a^4}{b^4}\right)bc}=\frac{a^2}{b^2}\sqrt{bc}.$

14. $\sqrt[4]{\frac{b^4}{a^3}}=\sqrt[4]{\left(\frac{b^4}{a^4}\right)a}=\frac{b}{a}\sqrt[4]{a}.$

15. $\sqrt[3]{\frac{ax^4}{b^2}}=\sqrt[3]{\left(\frac{x^3}{b^3}\right)abx}=\frac{x}{b}\sqrt[3]{abx}.$

16. $\sqrt[3]{\frac{7a}{125x}}=\sqrt[3]{\frac{7ax^2}{125x^3}}=\frac{1}{5x}\sqrt[3]{7ax^2}.$

17. $\sqrt{\frac{a^2cy^2}{bd^2}}=\sqrt{\left(\frac{a^2y^2}{b^2d^2}\right)bc}=\frac{ay}{bd}\sqrt{bc}.$

18. $2\sqrt[3]{\frac{2a^3b^2c}{3x^2yz^3}}=2\sqrt[3]{\frac{a^3}{27x^3y^3z^3}\times 18b^2cxy^2}=\frac{2a}{3xyz}\sqrt[3]{18b^2cxy^2}.$

Exercise 81. Page 229.

Express as entire surds:

1. $5\sqrt{5}=\sqrt{25}\sqrt{5}=\sqrt{125}.$

2. $3\sqrt{11}=\sqrt{9}\sqrt{11}=\sqrt{99}.$

3. $3\sqrt[3]{3}=\sqrt[3]{27}\sqrt[3]{3}=\sqrt[3]{81}.$

4. $2\sqrt[3]{4}=\sqrt[3]{8}\sqrt[3]{4}=\sqrt[3]{32}.$

5. $2\sqrt[4]{3}=\sqrt[4]{16}\sqrt[4]{3}=\sqrt[4]{48}.$

6. $3\sqrt[5]{2}=\sqrt[5]{243}\sqrt[5]{2}=\sqrt[5]{486}.$

7. $2\sqrt[6]{2}=\sqrt[6]{64}\sqrt[6]{2}=\sqrt[6]{128}.$

8. $2\sqrt[4]{4}=2\sqrt{2}=\sqrt{4}\sqrt{2}=\sqrt{8}.$

9. $-2\sqrt[3]{y}=\sqrt[3]{-8}\sqrt[3]{y}=\sqrt[3]{-8y}.$

10. $-3\sqrt{y^3}=-\sqrt{9}\sqrt{y^3}=-\sqrt{9y^3}.$

11. $-m\sqrt[6]{10}=-\sqrt[6]{m^6}\sqrt[6]{10}=-\sqrt[6]{10m^6}.$

12. $-2\sqrt[4]{x}=-\sqrt[4]{16}\sqrt[4]{x}=-\sqrt[4]{16x}.$

13. $\frac{1}{2}\sqrt{a}=\sqrt{\frac{1}{4}}\sqrt{a}=\sqrt{\frac{a}{4}}.$

14. $-\frac{2}{3}\sqrt[3]{a^2}=\sqrt[3]{-\frac{8}{27}}\sqrt[3]{a^2}=\sqrt[3]{-\frac{8a^2}{27}}.$

15. $\frac{3}{4}\sqrt{m^3}=\sqrt{\frac{9}{16}}\sqrt{m^3}=\sqrt{\frac{9m^3}{16}}.$

16. $-\frac{1}{2}\sqrt[4]{m^7}=-\sqrt[4]{\frac{1}{16}}\sqrt[4]{m^7}=-\sqrt[4]{\frac{m^7}{16}}.$

Exercise 82. Page 230

Reduce to surds of the same order:

1. $\sqrt[4]{3}$ and $\sqrt[6]{5}$.

$$\sqrt[4]{3} = 3^{\frac{1}{4}} = 3^{\frac{3}{12}} = \sqrt[12]{27}$$
$$\sqrt[6]{5} = 5^{\frac{1}{6}} = 5^{\frac{2}{12}} = \sqrt[12]{25}$$

2. $\sqrt[3]{14}$ and $\sqrt{6}$.

$$\sqrt[3]{14} = 14^{\frac{1}{3}} = 14^{\frac{2}{6}} = \sqrt[6]{196}$$
$$\sqrt{6} = 6^{\frac{1}{2}} = 6^{\frac{1}{2}} = \sqrt[6]{216}$$

3. $\sqrt{2}$ and $\sqrt[3]{4}$.

$$\sqrt{2} = 2^{\frac{1}{2}} = 2^{\frac{3}{6}} = \sqrt[6]{8}$$
$$\sqrt[3]{4} = 4^{\frac{1}{3}} = 4^{\frac{2}{6}} = \sqrt[6]{16}$$

4. $\sqrt{a}$ and $\sqrt[3]{b^2}$.

$$\sqrt{a} = a^{\frac{1}{2}} = a^{\frac{3}{6}} = \sqrt[6]{a^3}$$
$$\sqrt[3]{b^2} = b^{\frac{2}{3}} = b^{\frac{4}{6}} = \sqrt[6]{b^4}$$

5. $\sqrt{5}$ and $\sqrt[6]{75}$.

$$\sqrt{5} = 5^{\frac{1}{2}} = 5^{\frac{3}{6}} = \sqrt[6]{125}$$
$$\sqrt[6]{75} = \sqrt[6]{75}$$

6. $2^{\frac{1}{2}}$, $2^{\frac{2}{3}}$, and $2^{\frac{3}{4}}$.

$$2^{\frac{1}{2}} = 2^{\frac{6}{12}} = \sqrt[12]{64}$$
$$2^{\frac{2}{3}} = 2^{\frac{8}{12}} = \sqrt[12]{256}$$
$$2^{\frac{3}{4}} = 2^{\frac{9}{12}} = \sqrt[12]{512}$$

7. $\sqrt{2}$, $\sqrt[3]{3}$, and $\sqrt[4]{5}$.

$$\sqrt{2} = 2^{\frac{1}{2}} = 2^{\frac{6}{12}} = \sqrt[12]{64}$$
$$\sqrt[3]{3} = 3^{\frac{1}{3}} = 3^{\frac{4}{12}} = \sqrt[12]{81}$$
$$\sqrt[4]{5} = 5^{\frac{1}{4}} = 5^{\frac{3}{12}} = \sqrt[12]{125}$$

8. $\sqrt[6]{a^2}$, $\sqrt[3]{b}$, and $\sqrt{c}$.

$$\sqrt[6]{a^2} = \sqrt[6]{a^2}$$
$$\sqrt[3]{b} = b^{\frac{1}{3}} = b^{\frac{2}{6}} = \sqrt[6]{b^2}$$
$$\sqrt{c} = c^{\frac{1}{2}} = c^{\frac{3}{6}} = \sqrt[6]{c^3}$$

9. $\sqrt[5]{a^4}$, $\sqrt[10]{c^3}$, and $\sqrt{x^3}$.

$$\sqrt[5]{a^4} = a^{\frac{4}{5}} = a^{\frac{8}{10}} = \sqrt[10]{a^8}$$
$$\sqrt[10]{c^3} = \sqrt[10]{c^3}$$
$$\sqrt{x^3} = x^{\frac{3}{2}} = x^{\frac{15}{10}} = \sqrt[10]{x^{15}}$$

10. $\sqrt[4]{x^2y}$, $\sqrt[3]{abc}$, and $\sqrt[6]{2z}$.

$$\sqrt[4]{x^2y} = x^{\frac{2}{4}}y^{\frac{1}{4}} = x^{\frac{6}{12}}y^{\frac{3}{12}} = \sqrt[12]{x^6y^3}$$
$$\sqrt[3]{abc} = a^{\frac{1}{3}}b^{\frac{1}{3}}c^{\frac{1}{3}} = a^{\frac{4}{12}}b^{\frac{4}{12}}c^{\frac{4}{12}} = \sqrt[12]{a^4b^4c^4}$$
$$\sqrt[6]{2z} = 2^{\frac{1}{6}}z^{\frac{1}{6}} = 2^{\frac{2}{12}}z^{\frac{2}{12}} = \sqrt[12]{2^2z^2} = \sqrt[12]{4z^2}$$

11. $\sqrt[6]{x-y}$ and $\sqrt[4]{x+y}$.

$$\sqrt[6]{x-y}=\sqrt[12]{(x-y)^2}$$
$$\sqrt[4]{x+y}=\sqrt[12]{(x+y)^3}$$

12. $\sqrt[3]{a+b}$ and $\sqrt{a-b}$.

$$\sqrt[3]{a+b}=\sqrt[6]{(a+b)^2}$$
$$\sqrt{a-b}=\sqrt[6]{(a-b)^3}$$

Arrange in order of magnitude:

13. $\sqrt[3]{15}$ and $\sqrt{6}$.

$$\sqrt[3]{15}=\sqrt[6]{15^2}=\sqrt[6]{225}$$
$$\sqrt{6}\ =\sqrt[6]{6^3}\ =\sqrt[6]{216}$$

Therefore the ascending order of magnitude is $\sqrt{6}$, $\sqrt[3]{15}$.

14. $\sqrt[3]{4}$ and $\sqrt{3}$.

$$\sqrt[3]{4}=\sqrt[6]{4^2}=\sqrt[6]{16}$$
$$\sqrt{3}=\sqrt[6]{3^3}=\sqrt[6]{27}$$

Therefore the ascending order of magnitude is $\sqrt[3]{4}$, $\sqrt{3}$.

15. $\sqrt[6]{80}$, $\sqrt[3]{9}$, and $\sqrt{8}$.

$$\sqrt[6]{80}=\sqrt[6]{80}$$
$$\sqrt[3]{9}\ =\sqrt[6]{9^2}=\sqrt[6]{81}$$
$$\sqrt{8}\ =\sqrt[6]{8^3}=\sqrt[6]{512}$$

Therefore the ascending order of magnitude is $\sqrt[6]{80}$, $\sqrt[3]{9}$, $\sqrt{8}$.

16. $\sqrt{3}$, $\sqrt[3]{5}$, $\sqrt[4]{7}$.

$$\sqrt{3}=\sqrt[12]{3^6}=\sqrt[12]{729}$$
$$\sqrt[3]{5}=\sqrt[12]{5^4}=\sqrt[12]{625}$$
$$\sqrt[4]{7}=\sqrt[12]{7^3}=\sqrt[12]{343}$$

Therefore the ascending order of magnitude is $\sqrt[4]{7}$, $\sqrt[3]{5}$, $\sqrt{3}$.

Exercise 83. Page 231.

Simplify:

1. $4\sqrt{11}+3\sqrt{11}-5\sqrt{11}=(4+3-5)\sqrt{11}=2\sqrt{11}.$

2. $2\sqrt{3}-5\sqrt{3}+9\sqrt{3}=(2-5+9)\sqrt{3}=6\sqrt{3}.$

3.
$$\begin{aligned}5\sqrt[3]{4}+2\sqrt[3]{32}-\sqrt[3]{108}&=5\sqrt[3]{4}+2\sqrt[3]{8\times 4}-\sqrt[3]{27\times 4}\\&=5\sqrt[3]{4}+4\sqrt[3]{4}-3\sqrt[3]{4}\\&=6\sqrt[3]{4}.\end{aligned}$$

4.
$$\begin{aligned}3\sqrt[5]{2}+4\sqrt[5]{2}-\sqrt[5]{64}&=3\sqrt[5]{2}+4\sqrt[5]{2}-\sqrt[5]{32\times 2}\\&=3\sqrt[5]{2}+4\sqrt[5]{2}-2\sqrt[5]{2}\\&=5\sqrt[5]{2}.\end{aligned}$$

5. $$\begin{aligned}\tfrac{1}{2}\sqrt[3]{5}+2\tfrac{1}{2}\sqrt[3]{5}+\tfrac{1}{4}\sqrt[3]{40}&=\tfrac{1}{2}\sqrt[3]{5}+2\tfrac{1}{2}\sqrt[3]{5}+\tfrac{1}{4}\sqrt[3]{8\times 5}\\&=\tfrac{1}{2}\sqrt[3]{5}+2\tfrac{1}{2}\sqrt[3]{5}+\tfrac{1}{2}\sqrt[3]{5}\\&=3\tfrac{1}{2}\sqrt[3]{5}.\end{aligned}$$

6. $$\begin{aligned}3\sqrt[4]{3}-5\sqrt[4]{48}+\sqrt[4]{243}&=3\sqrt[4]{3}-5\sqrt[4]{16\times 3}+\sqrt[4]{81\times 3}\\&=3\sqrt[4]{3}-10\sqrt[4]{3}+3\sqrt[4]{3}\\&=-4\sqrt[4]{3}.\end{aligned}$$

7. $$\begin{aligned}\sqrt{27}+\sqrt{48}+\sqrt{75}&=\sqrt{9\times 3}+\sqrt{16\times 3}+\sqrt{25\times 3}\\&=3\sqrt{3}+4\sqrt{3}+5\sqrt{3}\\&=12\sqrt{3}.\end{aligned}$$

8. $$\begin{aligned}4\sqrt{147}+3\sqrt{75}+\sqrt{192}&=4\sqrt{49\times 3}+3\sqrt{25\times 3}+\sqrt{64\times 3}\\&=28\sqrt{3}+15\sqrt{3}+8\sqrt{3}\\&=51\sqrt{3}.\end{aligned}$$

9. $$\sqrt{a}+\tfrac{1}{2}\sqrt{a}+\tfrac{3}{2}\sqrt{a}=(1+\tfrac{1}{2}+\tfrac{3}{2})\sqrt{a}=3\sqrt{a}.$$

10. $$\sqrt[3]{a^2}+\tfrac{1}{2}\sqrt[3]{a^2}-3\sqrt[3]{27a^2}=\sqrt[3]{a^2}+\tfrac{1}{2}\sqrt[3]{a^2}-9\sqrt[3]{a^2}=-\tfrac{15}{2}\sqrt[3]{a^2}.$$

11. $$\sqrt{a^3}+b\sqrt{a}-3\sqrt{a}=a\sqrt{a}+b\sqrt{a}-3\sqrt{a}=(a+b-3)\sqrt{a}.$$

12. $$\sqrt{25b}+2\sqrt{9b}-3\sqrt{4b}=5\sqrt{b}+6\sqrt{b}-6\sqrt{b}=5\sqrt{b}.$$

13. $$\begin{aligned}2\sqrt{175}-3\sqrt{63}+5\sqrt{28}&=2\sqrt{25\times 7}-3\sqrt{9\times 7}+5\sqrt{4\times 7}\\&=10\sqrt{7}-9\sqrt{7}+10\sqrt{7}\\&=11\sqrt{7}.\end{aligned}$$

14. $$\begin{aligned}&\sqrt{2}+3\sqrt{32}+\tfrac{1}{2}\sqrt{128}-6\sqrt{18}\\&=\sqrt{2}+3\sqrt{16\times 2}+\tfrac{1}{2}\sqrt{64\times 2}-6\sqrt{9\times 2}\\&=\sqrt{2}+12\sqrt{2}+4\sqrt{2}-18\sqrt{2}\\&=-\sqrt{2}.\end{aligned}$$

15. $$\begin{aligned}&\sqrt{75}+\sqrt{48}-\sqrt{147}+\sqrt{300}\\&=\sqrt{25\times 3}+\sqrt{16\times 3}-\sqrt{49\times 3}+\sqrt{100\times 3}\\&=5\sqrt{3}+4\sqrt{3}-7\sqrt{3}+10\sqrt{3}\\&=12\sqrt{3}.\end{aligned}$$

16. $20\sqrt{245} - \sqrt{5} + \sqrt{125} - 2\frac{1}{2}\sqrt{180}$
$= 20\sqrt{49 \times 5} - \sqrt{5} + \sqrt{25 \times 5} - 2\frac{1}{2}\sqrt{36 \times 5}$
$= 140\sqrt{5} - \sqrt{5} + 5\sqrt{5} - 15\sqrt{5}$
$= 129\sqrt{5}.$

17. $2\sqrt{20} + \frac{1}{2}\sqrt{12} - 2\sqrt{27} + 5\sqrt{45} - 9\sqrt{12}$
$= 2\sqrt{4 \times 5} + \frac{1}{2}\sqrt{4 \times 3} - 2\sqrt{9 \times 3} + 5\sqrt{9 \times 5} - 9\sqrt{4 \times 3}$
$= 4\sqrt{5} + \sqrt{3} - 6\sqrt{3} + 15\sqrt{5} - 18\sqrt{3}$
$= 19\sqrt{5} - 23\sqrt{3}.$

18. $7\sqrt{25} + 4\sqrt{45} - \sqrt{9} - 2\sqrt{80} + \sqrt{20} - 4\sqrt{64}$
$= 7\sqrt{5^2} + 4\sqrt{3^2 \times 5} - \sqrt{9} - 2\sqrt{16 \times 5} + \sqrt{4 \times 5} - 4\sqrt{8^2}$
$= 35 + 12\sqrt{5} - 3 - 8\sqrt{5} + 2\sqrt{5} - 32$
$= 6\sqrt{5}.$

19. $\sqrt[3]{54} + \sqrt{\frac{1}{2}} - \sqrt[3]{250} - \frac{3}{4}\sqrt{\frac{2}{9}} = \sqrt[3]{27 \times 2} + \sqrt{\frac{2}{4}} - \sqrt[3]{125 \times 2} - \frac{3}{4}\,\frac{\sqrt{2}}{3}$
$= 3\sqrt[3]{2} + \frac{1}{2}\sqrt{2} - 5\sqrt[3]{2} - \frac{1}{4}\sqrt{2}$
$= -2\sqrt[3]{2} + \frac{1}{4}\sqrt{2}.$

20. $2\sqrt{\frac{5}{3}} + \sqrt{60} - \sqrt{15} + \sqrt{\frac{3}{5}} + \sqrt{\frac{4}{15}}$
$= 2\sqrt{\frac{15}{9}} + \sqrt{4 \times 15} - \sqrt{15} + \sqrt{\frac{15}{25}} + \sqrt{\frac{4 \times 15}{15^2}}$
$= \frac{2}{3}\sqrt{15} + 2\sqrt{15} - \sqrt{15} + \frac{1}{5}\sqrt{15} + \frac{2}{15}\sqrt{15}$
$= 2\sqrt{15}.$

21. $\sqrt[3]{27c^4} - \sqrt[3]{8c^4} + \sqrt[3]{125c} = \sqrt[3]{27c^3 \times c} - \sqrt[3]{8c^3 \times c} + \sqrt[3]{125c}$
$= 3c\sqrt[3]{c} - 2c\sqrt[3]{c} + 5\sqrt[3]{c}$
$= (c + 5)\sqrt[3]{c}.$

22. $\sqrt[5]{a^5b} - \sqrt[5]{b^6} + \sqrt[5]{32b} = a\sqrt[5]{b} - b\sqrt[5]{b} + 2\sqrt[5]{b}$
$= (a - b + 2)\sqrt[5]{b}.$

23. $\sqrt{a^4x} + \sqrt{b^4x} - \sqrt{4a^2b^2x} = a^2\sqrt{x} + b^2\sqrt{x} - 2ab\sqrt{x}$
$= (a^2 - 2ab + b^2)\sqrt{x}$
$= (a - b)^2\sqrt{x}.$

24.

$$\begin{aligned}\sqrt{4x^2y^2z}+\sqrt{y^4z}+\sqrt{x^4z}&=2xy\sqrt{z}+y^2\sqrt{z}+x^2\sqrt{z}\\&=(x^2+2xy+y^2)\sqrt{z}\\&=(x+y)^2\sqrt{z}.\end{aligned}$$

25.

$$\begin{aligned}\sqrt{a^2b^2c}-a\sqrt{4c}+b\sqrt{a^2c}&=ab\sqrt{c}-2a\sqrt{c}+ab\sqrt{c}\\&=(2ab-2a)\sqrt{c}.\end{aligned}$$

26.

$$\begin{aligned}\sqrt[4]{81a^5}-\sqrt[4]{16a}+\sqrt[4]{256a^5}&=3a\sqrt[4]{a}-2\sqrt[4]{a}+4a\sqrt[4]{a}\\&=(7a-2)\sqrt[4]{a}.\end{aligned}$$

27.

$$\begin{aligned}\sqrt[3]{27m^4}-\sqrt[3]{125m}+\sqrt[3]{216m}&=3m\sqrt[3]{m}-5\sqrt[3]{m}+6\sqrt[3]{m}\\&=(3m+1)\sqrt[3]{m}.\end{aligned}$$

28.

$$\begin{aligned}\sqrt{8a}-\sqrt{50a^3}-3\sqrt{18a}&=2\sqrt{2a}-5a\sqrt{2a}-9\sqrt{2a}\\&=(-7-5a)\sqrt{2a}.\end{aligned}$$

29.

$$\begin{aligned}&6a\sqrt{63ab^3}-3\sqrt{112a^3b^3}+2ab\sqrt{343ab}\\&=6a\sqrt{9b^2\times 7ab}-3\sqrt{16a^2b^2\times 7ab}+2ab\sqrt{49\times 7ab}\\&=18ab\sqrt{7ab}-12ab\sqrt{7ab}+14ab\sqrt{7ab}\\&=20ab\sqrt{7ab}.\end{aligned}$$

30.

$$\begin{aligned}&3\sqrt{125m^3n^2}+n\sqrt{20m^3}-\sqrt{500m^3n^2}\\&=3\sqrt{25m^2n^2\times 5m}+n\sqrt{4m^2\times 5m}-\sqrt{100m^2n^2\times 5m}\\&=15mn\sqrt{5m}+2mn\sqrt{5m}-10mn\sqrt{5m}\\&=7mn\sqrt{5m}.\end{aligned}$$

31.

$$\begin{aligned}&\sqrt{32a^4b^5}+6\sqrt{72b}+3\sqrt{128a^2b^3}\\&=\sqrt{16a^4b^4\times 2b}+6\sqrt{36\times 2b}+3\sqrt{64a^2b^2\times 2b}\\&=4a^2b^2\sqrt{2b}+36\sqrt{2b}+24ab\sqrt{2b}\\&=(4a^2b^2+24ab+36)\sqrt{2b}\\&=(2ab+6)^2\sqrt{2b}.\end{aligned}$$

32.

$$\begin{aligned}&2\sqrt[3]{a^6b}-3a^2\sqrt[3]{64b}+5a\sqrt[3]{a^3b}+2a^2\sqrt[3]{125b}\\&=2a^2\sqrt[3]{b}-12a^2\sqrt[3]{b}+5a^2\sqrt[3]{b}+10a^2\sqrt[3]{b}\\&=5a^2\sqrt[3]{b}.\end{aligned}$$

Exercise 84. Page 233.

1. $\sqrt{3} \times \sqrt{27} = \sqrt{81} = 9.$

2. $\sqrt{5} \times \sqrt{20} = \sqrt{100} = 10.$

3. $\sqrt{2} \times \sqrt{18} = \sqrt{36} = 6.$

4. $\sqrt[3]{3} \times \sqrt[3]{9} = \sqrt[3]{27} = 3.$

5. $\sqrt[3]{2} \times \sqrt[3]{32} = \sqrt[3]{64} = 4.$

6. $\sqrt[4]{27} \times \sqrt[4]{3} = \sqrt[4]{81} = 3.$

7. $\sqrt[5]{4} \times \sqrt[5]{8} = \sqrt[5]{32} = 2.$

8. $\sqrt[5]{27} \times \sqrt[5]{9} = \sqrt[5]{243} = 3.$

9. $\sqrt{2} \times \sqrt{12} = \sqrt{24} = \sqrt{4 \times 6} = 2\sqrt{6}.$

10. $\sqrt{3} \times \sqrt{6} = \sqrt{18} = \sqrt{9 \times 2} = 3\sqrt{2}.$

11. $\sqrt[3]{3} + \sqrt[3]{18} = \sqrt[3]{54} = \sqrt[3]{27 \times 2} = 3\sqrt[3]{2}.$

12. $\sqrt[4]{6} \times \sqrt[4]{8} = \sqrt[4]{48} = \sqrt[4]{16 \times 3} = 2\sqrt[4]{3}.$

13. $\sqrt[5]{54} \times \sqrt[5]{9} = \sqrt[5]{486} = \sqrt[5]{243 \times 2} = 3\sqrt[5]{2}.$

14. $2\sqrt{8} \times \sqrt{2} = 2\sqrt{16} = 8.$

15. $\sqrt[5]{8} \times \sqrt[5]{-4} = \sqrt[5]{-32} = -2.$

16. $\sqrt[3]{7} \times \sqrt[3]{-49} = \sqrt[3]{-343} = -7.$

17. $\sqrt[3]{81} \times \sqrt[3]{-45} = 3\sqrt[3]{3} \times \sqrt[3]{-45} = 3\sqrt[3]{-135} = 3\sqrt[3]{-27 \times 5} = -9\sqrt[3]{5}$

18. $\frac{2}{3}\sqrt[3]{18} \times \frac{3}{4}\sqrt[3]{3} = \frac{2}{3} \times \frac{3}{4} \times \sqrt[3]{18} \times \sqrt[3]{3} = \frac{1}{2}\sqrt[3]{54} = \frac{3}{2}\sqrt[3]{2}.$

19.
$$\begin{aligned}(\sqrt{18} + 2\sqrt{72} - 3\sqrt{8}) \times \sqrt{2} &= \sqrt{18 \times 2} + 2\sqrt{72 \times 2} - 3\sqrt{8 \times 2}\\ &= 6 + 24 - 12\\ &= 18.\end{aligned}$$

20.
$$\begin{aligned}(\sqrt[3]{32} - \tfrac{1}{2}\sqrt[3]{864} + 3\sqrt[3]{4}) \times \sqrt[3]{2} &= \sqrt[3]{64} - \tfrac{1}{2}\sqrt[3]{1728} + 3\sqrt[3]{8}\\ &= 4 - 6 + 6\\ &= 4.\end{aligned}$$

21.
$$\begin{aligned}&(\tfrac{1}{3}\sqrt{27} - \tfrac{1}{4}\sqrt{2187} + \tfrac{1}{8}\sqrt{432}) \times \sqrt{3}\\ &\quad = (\tfrac{1}{3}\sqrt{9 \times 3} - \tfrac{1}{4}\sqrt{729 \times 3} + \tfrac{1}{8}\sqrt{144 \times 3}) \times \sqrt{3}\\ &\quad = (\sqrt{3} - \tfrac{27}{4}\sqrt{3} + \tfrac{3}{2}\sqrt{3}) \times \sqrt{3}\\ &\quad = (-\tfrac{17}{4}\sqrt{3}) \times \sqrt{3} = -\tfrac{51}{4}.\end{aligned}$$

22. $\sqrt{5} \times \sqrt[3]{4} = \sqrt[6]{5^3} \times \sqrt[6]{4^2} = \sqrt[6]{125 \times 16} = \sqrt[6]{2000}.$

23. $\sqrt[3]{16} \times \sqrt[6]{250} = 2\sqrt[3]{2} \times \sqrt[6]{250} = 2\sqrt[6]{2^2} \times \sqrt[6]{250} = 2\sqrt[6]{1000} = 2\sqrt{10}.$

24. $\sqrt[4]{64} \times \sqrt[8]{16} = \sqrt[4]{2^6} \times \sqrt[8]{2^4} = \sqrt{2^3} \times \sqrt{2} = \sqrt{2^4} = 2^2 = 4.$

25. $\sqrt{3} \times \sqrt[3]{72} = \sqrt{3} \times \sqrt[3]{2^3 \times 3^2} = \sqrt{3} \times 2\sqrt[3]{3^2}$

$= \sqrt[6]{3^3} \times 2\sqrt[6]{3^4} = 2\sqrt[6]{3^7} = 6\sqrt[6]{3}.$

26. $\sqrt{\frac{2}{9}} \times \sqrt[3]{\frac{1}{2}} = \frac{1}{3}\sqrt{2} \times \frac{1}{2}\sqrt[3]{4} = \frac{1}{6}\sqrt{2} \times \sqrt[3]{4}$

$= \frac{1}{6}\sqrt[6]{2^3} \times \sqrt[6]{2^4} = \frac{1}{6}\sqrt[6]{2^7} = \frac{1}{3}\sqrt[6]{2}.$

27. $\sqrt{\frac{7}{12}} \times \sqrt[3]{\frac{3}{7}} = \sqrt[6]{\frac{7^3}{12^3}} \times \sqrt[6]{\frac{3^2}{7^2}} = \sqrt[6]{\frac{7^3 \times 3^2}{12^3 \times 7^2}} = \sqrt[6]{\frac{7}{3 \times 4^3}} = \sqrt[6]{\frac{7}{192}}.$

28. $\sqrt[3]{81} \times \sqrt{3} = \sqrt[3]{3^4} \times \sqrt{3} = \sqrt[6]{3^8} \times \sqrt[6]{3^3} = \sqrt[6]{3^{11}} = 3\sqrt[6]{3^5}.$

29. $\sqrt[3]{\frac{5}{6}} \times \sqrt{\frac{3}{5}} = \sqrt[6]{\frac{5^2}{6^2}} \times \sqrt[6]{\frac{3^3}{5^3}} = \sqrt[6]{\frac{5^2 \times 3^3}{6^2 \times 5^3}} = \sqrt[6]{\frac{3}{2^2 \times 5}} = \sqrt[6]{\frac{3}{20}}.$

30. $\sqrt[4]{\frac{2}{3}} \times \sqrt[3]{\frac{1}{2}} = \sqrt[12]{\frac{2^3}{3^3}} \times \sqrt[12]{\frac{1}{2^4}} = \sqrt[12]{\frac{2^3}{3^3 \times 2^4}} = \sqrt[12]{\frac{1}{3^3 \times 2}} = \sqrt[12]{\frac{1}{54}}.$

31. $\sqrt[3]{2x} \times \sqrt[4]{x^3} = \sqrt[12]{2^4x^4} \times \sqrt[12]{x^9} = \sqrt[12]{2^4x^{13}} = x\sqrt[12]{16x}.$

32. $\sqrt[5]{y^4} \times \sqrt[7]{2y^3} = \sqrt[35]{y^{28}} \times \sqrt[35]{2^5y^{15}} = \sqrt[35]{2^5y^{43}} = y\sqrt[35]{32y^8}.$

33. $\sqrt[3]{7} \times \sqrt{5} = \sqrt[6]{7^2} \times \sqrt[6]{5^3} = \sqrt[6]{7^2 \times 5^3} = \sqrt[6]{6125}.$

Exercise 85. Page 234.

Multiply:

1. $\sqrt{5 + \sqrt{4}}$ by $\sqrt{5 - \sqrt{4}}$.

$\sqrt{5 + \sqrt{4}} \times \sqrt{5 - \sqrt{4}} = \sqrt{7} \times \sqrt{3} = \sqrt{21}.$

2. $\sqrt{9 - \sqrt{17}}$ by $\sqrt{9 + \sqrt{17}}$.

$\sqrt{9 - \sqrt{17}} \times \sqrt{9 + \sqrt{17}} = \sqrt{(9 - \sqrt{17})(9 + \sqrt{17})} = \sqrt{81 - 17}$

$= \sqrt{64} = 8.$

3. $3+2\sqrt{5}$ by $2-\sqrt{5}$.

$$\begin{array}{r}
3+2\sqrt{5} \\
2-\ \ \sqrt{5} \\
\hline
6+4\sqrt{5} \\
-3\sqrt{5}-10 \\
\hline
-4+\ \ \sqrt{5}
\end{array}$$

4. $8+3\sqrt{2}$ by $2-\sqrt{2}$.

$$\begin{array}{r}
8+3\sqrt{2} \\
2-\ \ \sqrt{2} \\
\hline
16+6\sqrt{2} \\
-8\sqrt{2}-6 \\
\hline
10-2\sqrt{2}
\end{array}$$

5. $5+2\sqrt{3}$ by $3-5\sqrt{3}$.

$$\begin{array}{r}
5+2\sqrt{3} \\
3-5\sqrt{3} \\
\hline
15+\ \ 6\sqrt{3} \\
-25\sqrt{3}-30 \\
\hline
-15-19\sqrt{3}
\end{array}$$

6. $3-\sqrt{6}$ by $6-3\sqrt{6}$.

$$\begin{array}{r}
3-\ \ \sqrt{6} \\
6-3\sqrt{6} \\
\hline
18-\ \ 6\sqrt{6} \\
-\ \ 9\sqrt{6}+18 \\
\hline
36-15\sqrt{6}
\end{array}$$

7. $2\sqrt{6}-3\sqrt{5}$ by $\sqrt{3}+2\sqrt{2}$.

$$\begin{array}{l}
2\sqrt{6}-3\sqrt{5} \\
\ \ \sqrt{3}+2\sqrt{2} \\
\hline
4\sqrt{12}-6\sqrt{10} \\
\qquad\qquad\qquad +2\sqrt{18}-3\sqrt{15} \\
\hline
4\sqrt{12}-6\sqrt{10}+2\sqrt{18}-3\sqrt{15} \\
=8\sqrt{3}-6\sqrt{10}+6\sqrt{2}-3\sqrt{15}.
\end{array}$$

8. $7-\sqrt{3}$ by $\sqrt{2}+\sqrt{5}$.

$$\begin{array}{l}
\ \ 7-\sqrt{3} \\
\sqrt{2}+\sqrt{5} \\
\hline
7\sqrt{2}-\sqrt{6} \\
\qquad\qquad +7\sqrt{5}-\sqrt{15} \\
\hline
7\sqrt{2}-\sqrt{6}+7\sqrt{5}-\sqrt{15}
\end{array}$$

9. $\sqrt[3]{9}-2\sqrt[3]{4}$ by $4\sqrt[3]{3}+\sqrt[3]{2}$.

$$\begin{array}{l}
\ \ \sqrt[3]{9}-2\sqrt[3]{4} \\
4\sqrt[3]{3}+\ \ \sqrt[3]{2} \\
\hline
4\sqrt[3]{27}-8\sqrt[3]{12} \\
\qquad\qquad\qquad +\sqrt[3]{18}-2\sqrt[3]{8} \\
\hline
4\sqrt[3]{27}-8\sqrt[3]{12}+\sqrt[3]{18}-2\sqrt[3]{8}
\end{array}$$

$$=12-8\sqrt[3]{12}+\sqrt[3]{18}-4=8-8\sqrt[3]{12}+\sqrt[3]{18}.$$

10. $2\sqrt{30}-3\sqrt{5}+5\sqrt{3}$ by $\sqrt{8}+\sqrt{3}-\sqrt{5}$.

$$\begin{aligned}(2\sqrt{30}-3\sqrt{5}+5\sqrt{3})(\sqrt{8}+\sqrt{3}-\sqrt{5})\\ &=2\sqrt{240}-3\sqrt{40}+5\sqrt{24}+2\sqrt{90}-3\sqrt{15}+15\\ &\qquad-2\sqrt{150}+15-5\sqrt{15}\\ &=8\sqrt{15}-6\sqrt{10}+10\sqrt{6}+6\sqrt{10}-3\sqrt{15}+15\\ &\qquad-10\sqrt{6}+15-5\sqrt{15}\\ &=30.\end{aligned}$$

11. $3\sqrt{5}-2\sqrt{3}+4\sqrt{7}$ by $3\sqrt{7}-4\sqrt{5}-5\sqrt{3}$.

$$\begin{aligned}(3\sqrt{5}-2\sqrt{3}+4\sqrt{7})(3\sqrt{7}-4\sqrt{5}-5\sqrt{3})\\ &=9\sqrt{35}-6\sqrt{21}+84-60+8\sqrt{15}-16\sqrt{35}-15\sqrt{15}+30-20\sqrt{21}\\ &=54-7\sqrt{35}-26\sqrt{21}-7\sqrt{15}.\end{aligned}$$

12. $4\sqrt{8}+\frac{1}{2}\sqrt{12}-\frac{1}{4}\sqrt{32}$ by $8\sqrt{32}-4\sqrt{50}-2\sqrt{2}$.

$$4\sqrt{8}+\tfrac{1}{2}\sqrt{12}-\tfrac{1}{4}\sqrt{32}=8\sqrt{2}+\sqrt{3}-\sqrt{2}=7\sqrt{2}+\sqrt{3}$$

$$8\sqrt{32}-4\sqrt{50}-2\sqrt{2}=32\sqrt{2}-20\sqrt{2}-2\sqrt{2}=10\sqrt{2}$$

$$\begin{aligned}\therefore\ (4\sqrt{8}+\tfrac{1}{2}\sqrt{12}-\tfrac{1}{4}\sqrt{32})(8\sqrt{32}-4\sqrt{50}-2\sqrt{2})\\ &=(7\sqrt{2}+\sqrt{3})\times 10\sqrt{2}=140+10\sqrt{6}.\end{aligned}$$

13. $\sqrt[3]{6}-\sqrt[3]{3}+\sqrt[3]{16}$ by $\sqrt[3]{36}+\sqrt[3]{9}-\sqrt[3]{4}$.

$$\begin{aligned}(\sqrt[3]{6}-\sqrt[3]{3}+\sqrt[3]{16})(\sqrt[3]{36}+\sqrt[3]{9}-\sqrt[3]{4})\\ &=\sqrt[3]{216}+\sqrt[3]{54}-\sqrt[3]{24}-\sqrt[3]{108}-\sqrt[3]{27}+\sqrt[3]{12}+\sqrt[3]{576}+\sqrt[3]{144}-\sqrt[3]{64}\\ &=6+3\sqrt[3]{2}-2\sqrt[3]{3}-3\sqrt[3]{4}-3+\sqrt[3]{12}+4\sqrt[3]{9}+2\sqrt[3]{18}-4\\ &=-1+3\sqrt[3]{2}-2\sqrt[3]{3}-3\sqrt[3]{4}+\sqrt[3]{12}+4\sqrt[3]{9}+2\sqrt[3]{18}.\end{aligned}$$

14. $2\sqrt{\frac{2}{3}}-8\sqrt{\frac{3}{8}}+3\sqrt{\frac{3}{2}}$ by $3\sqrt{\frac{2}{3}}-\sqrt{12}-\sqrt{6}$.

$$2\sqrt{\tfrac{2}{3}}-8\sqrt{\tfrac{3}{8}}+3\sqrt{\tfrac{3}{2}}=\tfrac{2}{3}\sqrt{6}-2\sqrt{6}+\tfrac{3}{2}\sqrt{6}=\tfrac{1}{6}\sqrt{6}$$

$$3\sqrt{\tfrac{2}{3}}-\sqrt{12}-\sqrt{6}=\sqrt{6}-\sqrt{12}-\sqrt{6}=-2\sqrt{3}$$

$$\begin{aligned}\therefore\ (2\sqrt{\tfrac{2}{3}}-8\sqrt{\tfrac{3}{8}}+3\sqrt{\tfrac{3}{2}})(3\sqrt{\tfrac{2}{3}}-\sqrt{12}-\sqrt{6})&=(\tfrac{1}{6}\sqrt{6})\times(-2\sqrt{3})\\ &=-\sqrt{2}.\end{aligned}$$

15. $2\sqrt{\frac{5}{6}}-4\sqrt{\frac{3}{2}}-7\sqrt{\frac{6}{5}}$ by $3\sqrt{\frac{5}{6}}-5\sqrt{30}-2\sqrt{\frac{15}{2}}$.

$$\begin{aligned}&(2\sqrt{\tfrac{5}{6}}-4\sqrt{\tfrac{3}{2}}-7\sqrt{\tfrac{6}{5}})(3\sqrt{\tfrac{5}{6}}-5\sqrt{30}-2\sqrt{\tfrac{15}{2}})\\ &\quad=(2\sqrt{\tfrac{30}{36}}-4\sqrt{\tfrac{6}{4}}-7\sqrt{\tfrac{30}{25}})(3\sqrt{\tfrac{30}{36}}-5\sqrt{30}-2\sqrt{\tfrac{30}{4}})\\ &\quad=(\tfrac{1}{3}\sqrt{30}-2\sqrt{6}-\tfrac{7}{5}\sqrt{30})(\tfrac{1}{2}\sqrt{30}-5\sqrt{30}-\sqrt{30})\\ &\quad=(-\tfrac{16}{15}\sqrt{30}-2\sqrt{6})(-\tfrac{11}{2}\sqrt{30})\\ &\quad=\tfrac{88}{15}\times 30+11\sqrt{180}\\ &\quad=176+66\sqrt{5}.\end{aligned}$$

16. $2\sqrt{12}+3\sqrt{3}+6\sqrt{\frac{1}{3}}$ by $2\sqrt{12}+3\sqrt{3}+6\sqrt{\frac{1}{3}}$.

$$\begin{aligned}&(2\sqrt{12}+3\sqrt{3}+6\sqrt{\tfrac{1}{3}})(2\sqrt{12}+3\sqrt{3}+6\sqrt{\tfrac{1}{3}})\\ &\quad=(4\sqrt{3}+3\sqrt{3}+2\sqrt{3})(4\sqrt{3}+3\sqrt{3}+2\sqrt{3})\\ &\quad=(9\sqrt{3})^2\\ &\quad=243.\end{aligned}$$

Exercise 86 Page 235.

Divide:

1. $\sqrt{243}$ by $\sqrt{3}$.

$$\frac{\sqrt{243}}{\sqrt{3}}=\sqrt{\frac{243}{3}}=\sqrt{81}=9.$$

2. $\sqrt[3]{81}$ by $\sqrt[3]{3}$.

$$\frac{\sqrt[3]{81}}{\sqrt[3]{3}}=\sqrt[3]{27}=3.$$

3. $\sqrt{3a^7}$ by $\sqrt{a^8}$.

$$\frac{\sqrt{3a^7}}{\sqrt{a^8}}=\sqrt{\frac{3a^7}{a^8}}=\sqrt{3a^4}=a^2\sqrt{3}.$$

4. $\sqrt{\frac{1}{2}}$ by $\sqrt{\frac{2}{9}}$.

$$\frac{\sqrt{\frac{1}{2}}}{\sqrt{\frac{2}{9}}}=\sqrt{\frac{\frac{1}{2}}{\frac{2}{9}}}=\sqrt{\tfrac{9}{4}}=\tfrac{3}{2}.$$

5. $\sqrt{\frac{5}{6}}$ by $\sqrt{\frac{3}{5}}$.

$$\frac{\sqrt{\frac{5}{6}}}{\sqrt{\frac{3}{5}}}=\sqrt{\frac{\frac{5}{6}}{\frac{3}{5}}}=\sqrt{\tfrac{25}{18}}=\sqrt{\tfrac{50}{36}}=\tfrac{5}{6}\sqrt{2}.$$

6. $\sqrt{\frac{7}{12}}$ by $\sqrt{\frac{3}{7}}$.

$$\frac{\sqrt{\frac{7}{12}}}{\sqrt{\frac{3}{7}}}=\sqrt{\frac{\frac{7}{12}}{\frac{3}{7}}}=\sqrt{\tfrac{49}{36}}=\tfrac{7}{6}.$$

7. $\sqrt{\frac{14}{15}}$ by $\sqrt{\frac{24}{35}}$.

$$\frac{\sqrt{\frac{14}{15}}}{\sqrt{\frac{24}{35}}}=\sqrt{\frac{\frac{14}{15}}{\frac{24}{35}}}=\sqrt{\tfrac{49}{36}}=\tfrac{7}{6}.$$

8. $\sqrt{\frac{22}{27}}$ by $\sqrt{\frac{6}{11}}$.

$$\frac{\sqrt{\frac{22}{27}}}{\sqrt{\frac{6}{11}}}=\sqrt{\tfrac{121}{81}}=\tfrac{11}{9}.$$

9. $\sqrt{\frac{52}{85}}$ by $\sqrt{\frac{17}{65}}$.

$$\frac{\sqrt{\frac{52}{85}}}{\sqrt{\frac{17}{65}}}=\sqrt{\frac{\frac{52}{85}}{\frac{17}{65}}}=\sqrt{\frac{52\times 65}{85\times 17}}=\sqrt{\frac{2^2\times 13^2}{17^2}}=\frac{26}{17}.$$

10. $3\sqrt{6}+45\sqrt{2}$ by $3\sqrt{3}$.

$$(3\sqrt{6}+45\sqrt{2})\div 3\sqrt{3}$$
$$=\frac{3\sqrt{6}}{3\sqrt{3}}+\frac{45\sqrt{2}}{3\sqrt{3}}=\sqrt{\tfrac{6}{3}}+15\sqrt{\tfrac{2}{3}}=\sqrt{2}+5\sqrt{6}.$$

11. $42\sqrt{5}-30\sqrt{3}$ by $2\sqrt{15}$.

$$(42\sqrt{5}-30\sqrt{3})\div 2\sqrt{15}$$
$$=\frac{42\sqrt{5}}{2\sqrt{15}}-\frac{30\sqrt{3}}{2\sqrt{15}}=21\sqrt{\tfrac{5}{15}}-15\sqrt{\tfrac{3}{15}}$$
$$=\frac{21}{\sqrt{3}}-\frac{15}{\sqrt{5}}=7\sqrt{3}-3\sqrt{5}.$$

12. $84\sqrt{15}+168\sqrt{6}$ by $3\sqrt{21}$.

$$(84\sqrt{15}+168\sqrt{6})\div 3\sqrt{21}$$
$$=\frac{84\sqrt{15}}{3\sqrt{21}}+\frac{168\sqrt{6}}{3\sqrt{21}}=28\sqrt{\tfrac{15}{21}}+56\sqrt{\tfrac{6}{21}}$$
$$=28\sqrt{\tfrac{5}{7}}+56\sqrt{\tfrac{2}{7}}=4\sqrt{35}+8\sqrt{14}.$$

13. $30\sqrt[3]{4}-36\sqrt[3]{10}+30\sqrt[3]{90}$ by $3\sqrt[3]{20}$.

$$(30\sqrt[3]{4}-36\sqrt[3]{10}+30\sqrt[3]{90})\div 3\sqrt[3]{20}$$
$$=\frac{30\sqrt[3]{4}}{3\sqrt[3]{20}}-\frac{36\sqrt[3]{10}}{3\sqrt[3]{20}}+\frac{30\sqrt[3]{90}}{3\sqrt[3]{20}}=10\sqrt[3]{\tfrac{1}{5}}-12\sqrt[3]{\tfrac{1}{2}}+10\sqrt[3]{\tfrac{9}{2}}$$
$$=2\sqrt[3]{25}-6\sqrt[3]{4}+5\sqrt[3]{36}.$$

14. $50\sqrt[3]{18}+18\sqrt[3]{20}-48\sqrt[3]{5}$ by $2\sqrt[3]{30}$.

$$(50\sqrt[3]{18}+18\sqrt[3]{20}-48\sqrt[3]{5})\div 2\sqrt[3]{30}$$
$$=\frac{50\sqrt[3]{18}}{2\sqrt[3]{30}}+\frac{18\sqrt[3]{20}}{2\sqrt[3]{30}}-\frac{48\sqrt[3]{5}}{2\sqrt[3]{30}}=25\sqrt[3]{\tfrac{3}{5}}+9\sqrt[3]{\tfrac{2}{3}}-24\sqrt[3]{\tfrac{1}{6}}$$
$$=5\sqrt[3]{75}+3\sqrt[3]{18}-4\sqrt[3]{36}.$$

15. $\sqrt{54}$ by $\sqrt[4]{36}$.

$$\frac{\sqrt{54}}{\sqrt[4]{36}}=\frac{\sqrt{54}}{\sqrt{6}}=\sqrt{9}=3.$$

16. $\sqrt[3]{49}$ by $\sqrt{7}$.

$$\frac{\sqrt[3]{49}}{\sqrt{7}}=\frac{\sqrt[6]{7^4}}{\sqrt[6]{7^3}}=\sqrt[6]{7}.$$

17. $\sqrt[3]{12}$ by $\sqrt{6}$.

$$\frac{\sqrt[3]{12}}{\sqrt{6}}=\frac{\sqrt[6]{144}}{\sqrt[6]{216}}=\sqrt[6]{\tfrac{2}{3}}.$$

18. $\sqrt{\frac{8}{45}}$ by $\sqrt[4]{6\frac{1}{4}}$.

$$\frac{\sqrt{\frac{8}{45}}}{\sqrt[4]{6\frac{1}{4}}}=\frac{\sqrt{\frac{8}{45}}}{\sqrt[4]{\frac{25}{4}}}=\frac{\sqrt{\frac{8}{45}}}{\sqrt{\frac{5}{2}}}=\sqrt{\tfrac{16}{225}}=\tfrac{4}{15}.$$

19. $\sqrt{\frac{3}{5}}$ by $\sqrt[6]{3\frac{3}{8}}$.

$$\frac{\sqrt{\frac{3}{5}}}{\sqrt[6]{3\frac{3}{8}}}=\frac{\sqrt[6]{\frac{27}{125}}}{\sqrt[6]{\frac{27}{8}}}=\sqrt[6]{\tfrac{8}{125}}=\sqrt{\tfrac{2}{5}}.$$

20. $\sqrt[3]{2x}$ by $\sqrt{x^3}$.

$$\frac{\sqrt[3]{2x}}{\sqrt{x^3}}=\frac{\sqrt[6]{4x^2}}{\sqrt[6]{x^9}}=\sqrt[6]{\frac{4}{x^7}}=\frac{1}{x}\sqrt[6]{\frac{4}{x}}.$$

21. $\sqrt[6]{0.064}$ by $\sqrt{10}$.

$$\frac{\sqrt[6]{0.064}}{\sqrt{10}}=\frac{\sqrt[6]{\frac{64}{1000}}}{\sqrt{10}}=\frac{\frac{2}{\sqrt[6]{1000}}}{\sqrt{10}}=\frac{\frac{2}{\sqrt{10}}}{\sqrt{10}}=\tfrac{2}{10}=\tfrac{1}{5}.$$

22. $\sqrt{x^2-y^2}$ by $x+y$.

$$\frac{\sqrt{x^2-y^2}}{x+y}=\frac{\sqrt{x+y}\times\sqrt{x-y}}{x+y}=\frac{\sqrt{x-y}}{\sqrt{x+y}}.$$

Exercise 87. Page 236.

Divide:

1. $\sqrt{a}+\sqrt{b}$ by $\sqrt{ab}$.

$$\frac{\sqrt{a}+\sqrt{b}}{\sqrt{ab}}=\frac{(\sqrt{a}+\sqrt{b})\sqrt{ab}}{ab}=\frac{a\sqrt{b}+b\sqrt{a}}{ab}.$$

2. $\sqrt{125}$ by $5\sqrt{65}$.

$$\frac{\sqrt{125}}{5\sqrt{65}}=\frac{5\sqrt{5}}{5\sqrt{65}}=\frac{1}{\sqrt{13}}=\frac{\sqrt{13}}{13}.$$

3. 3 by $11+3\sqrt{7}$.

$$\frac{3}{11+3\sqrt{7}}=\frac{3(11-3\sqrt{7})}{(11+3\sqrt{7})(11-3\sqrt{7})}=\frac{3(11-3\sqrt{7})}{121-63}=\frac{33-9\sqrt{7}}{58}.$$

4. $3\sqrt{2}-1$ by $3\sqrt{2}+1$.

$$\frac{3\sqrt{2}-1}{3\sqrt{2}+1}=\frac{(3\sqrt{2}-1)^2}{(3\sqrt{2}+1)(3\sqrt{2}-1)}=\frac{19-6\sqrt{2}}{18-1}=\frac{19-6\sqrt{2}}{17}.$$

5. 17 by $3\sqrt{7}+2\sqrt{3}$.

$$\frac{17}{3\sqrt{7}+2\sqrt{3}}=\frac{17(3\sqrt{7}-2\sqrt{3})}{(3\sqrt{7}+2\sqrt{3})(3\sqrt{7}-2\sqrt{3})}=\frac{17(3\sqrt{7}-2\sqrt{3})}{51}$$
$$=\frac{3\sqrt{7}-2\sqrt{3}}{3}.$$

6. 1 by $\sqrt{2}+\sqrt{3}$.

$$\frac{1}{\sqrt{2}+\sqrt{3}}=\frac{\sqrt{2}-\sqrt{3}}{(\sqrt{2}+\sqrt{3})(\sqrt{2}-\sqrt{3})}=\frac{\sqrt{2}-\sqrt{3}}{-1}=\sqrt{3}-\sqrt{2}.$$

7. $3+5\sqrt{7}$ by $3-5\sqrt{7}$.

$$\frac{3+5\sqrt{7}}{3-5\sqrt{7}}=\frac{(3+5\sqrt{7})^2}{(3-5\sqrt{7})(3+5\sqrt{7})}=\frac{184+30\sqrt{7}}{-166}=-\frac{92+15\sqrt{7}}{83}.$$

8. $21\sqrt{3}$ by $4\sqrt{3}-3\sqrt{2}$.

$$\frac{21\sqrt{3}}{4\sqrt{3}-3\sqrt{2}}=\frac{21\sqrt{3}(4\sqrt{3}+3\sqrt{2})}{(4\sqrt{3}-3\sqrt{2})(4\sqrt{3}+3\sqrt{2})}=\frac{84\times 3+63\sqrt{6}}{30}$$
$$=\frac{84+21\sqrt{6}}{10}.$$

9. $75\sqrt{14}$ by $8\sqrt{2}+2\sqrt{7}$.

$$\frac{75\sqrt{14}}{8\sqrt{2}+2\sqrt{7}}=\frac{75\sqrt{14}(8\sqrt{2}-2\sqrt{7})}{(8\sqrt{2}+2\sqrt{7})(8\sqrt{2}-2\sqrt{7})}=\frac{600\sqrt{28}-150\sqrt{98}}{100}$$
$$=\frac{120\sqrt{7}-105\sqrt{2}}{10}=\frac{24\sqrt{7}-21\sqrt{2}}{2}.$$

10. $\sqrt{5}-\sqrt{3}$ by $\sqrt{5}+\sqrt{3}$.

$$\frac{\sqrt{5}-\sqrt{3}}{\sqrt{5}+\sqrt{3}}=\frac{(\sqrt{5}-\sqrt{3})^2}{(\sqrt{5}+\sqrt{3})(\sqrt{5}-\sqrt{3})}=\frac{8-2\sqrt{15}}{2}=4-\sqrt{15}.$$

11. $\sqrt{8}+\sqrt{7}$ by $\sqrt{7}-\sqrt{2}$.

$$\frac{\sqrt{8}+\sqrt{7}}{\sqrt{7}-\sqrt{2}}=\frac{(\sqrt{8}+\sqrt{7})(\sqrt{7}+\sqrt{2})}{(\sqrt{7}-\sqrt{2})(\sqrt{7}+\sqrt{2})}=\frac{\sqrt{56}+7+\sqrt{16}+\sqrt{14}}{7-2}$$

$$=\frac{2\sqrt{14}+7+4+\sqrt{14}}{5}=\frac{11+3\sqrt{14}}{5}.$$

12. $7-3\sqrt{10}$ by $5+4\sqrt{5}$.

$$\frac{7-3\sqrt{10}}{5+4\sqrt{5}}=\frac{(7-3\sqrt{10})(5-4\sqrt{5})}{(5+4\sqrt{5})(5-4\sqrt{5})}=\frac{35-28\sqrt{5}-15\sqrt{10}+12\sqrt{50}}{25-80}$$

$$=\frac{35-28\sqrt{5}-15\sqrt{10}+60\sqrt{2}}{-55}.$$

Given $\sqrt{2}=1.41421$, $\sqrt{3}=1.73205$, $\sqrt{5}=2.23607$; find to four places of decimals the value of

13. $\frac{10}{\sqrt{2}}=\frac{10\sqrt{2}}{2}=5\sqrt{2}=5\times 1.41421=7.07105.$

14. $\frac{8}{\sqrt{3}}=\frac{8\sqrt{3}}{3}=\frac{8\times 1.73205}{3}=4.61880.$

15. $\frac{12}{\sqrt{5}}=\frac{12\sqrt{5}}{5}=\frac{12\times 2.23607}{5}=5.36656$

16. $\frac{1}{\sqrt{500}}=\frac{1}{10\sqrt{5}}=\frac{\sqrt{5}}{50}=\frac{2.23607}{50}=0.04472.$

17. $\frac{1}{\sqrt{243}}=\frac{1}{9\sqrt{3}}=\frac{\sqrt{3}}{27}=\frac{1.73205}{27}=0.06415$

18. $\frac{1}{2\sqrt{3}}=\frac{\sqrt{3}}{6}=\frac{1.73205}{6}=0.28867$

19. $\frac{1}{3\sqrt{2}}=\frac{\sqrt{2}}{6}=\frac{1.41421}{6}=0.23570.$

20. $\frac{1}{\sqrt{125}}=\frac{1}{5\sqrt{5}}=\frac{\sqrt{5}}{25}=\frac{2.23607}{25}=0.08944.$

21. $\frac{1}{4\sqrt{5}} = \frac{\sqrt{5}}{20} = \frac{2.23607}{20} = 0.11180.$

22. $\frac{7 - 3\sqrt{5}}{5 + 4\sqrt{5}} = \frac{(7 - 3\sqrt{5})(5 - 4\sqrt{5})}{(5 + 4\sqrt{5})(5 - 4\sqrt{5})} = \frac{35 - 43\sqrt{5} + 60}{25 - 80} = \frac{95 - 43\sqrt{5}}{-55}$

$= \frac{43\sqrt{5} - 95}{55} = \frac{43 \times 2.23607 - 95}{55} = 0.02093.$

23. $\frac{3 + \sqrt{5}}{\sqrt{5} - 2} = \frac{(3 + \sqrt{5})(\sqrt{5} + 2)}{(\sqrt{5} - 2)(\sqrt{5} + 2)} = \frac{5\sqrt{5} + 5 + 6}{5 - 4} = 11 + 5\sqrt{5}$

$= 11 + 5 \times 2.23607 = 22.18035.$

24. $\frac{3\sqrt{2} - 1}{3\sqrt{2} + 1} = \frac{(3\sqrt{2} - 1)^2}{(3\sqrt{2} + 1)(3\sqrt{2} - 1)} = \frac{19 - 6\sqrt{2}}{18 - 1} = \frac{19 - 6\sqrt{2}}{17}$

$= \frac{19 - 6 \times 1.41421}{17} = 0.61851.$

Exercise 88. Page 237.

Perform the operations indicated:

1. $(\sqrt[3]{m^2})^3 = (m^{\frac{2}{3}})^3 = m^2.$

2. $(\sqrt{m^3})^5 = (m^{\frac{3}{2}})^5 = m^{\frac{15}{2}} = \sqrt{m^{15}}.$

3. $(\sqrt[5]{x^4})^{15} = (x^{\frac{4}{5}})^{15} = x^{12}.$

4. $(\sqrt[3]{y^5})^{12} = (y^{\frac{5}{3}})^{12} = y^{20}.$

5. $\sqrt[4]{\sqrt{(x - y)^8}} = [(x - y)^{\frac{8}{2}}]^{\frac{1}{4}} = [(x - y)^4]^{\frac{1}{4}} = x - y.$

6. $\sqrt[3]{\sqrt[6]{(a - b)^{36}}} = [(a - b)^{\frac{36}{6}}]^{\frac{1}{3}} = [(a - b)^6]^{\frac{1}{3}} = (a - b)^2.$

7. $(\sqrt{2a^3b})^4 = (2^{\frac{1}{2}}a^{\frac{3}{2}}b^{\frac{1}{2}})^4 = 2^2a^6b^2 = 4a^6b^2.$

8. $\sqrt[3]{(x^2 - y^2)^6} = (x^2 - y^2)^{\frac{6}{3}} = (x^2 - y^2)^2.$

9. $(\sqrt[8]{x})^4 = (x^{\frac{1}{8}})^4 = x^{\frac{4}{8}} = x^{\frac{1}{2}} = \sqrt{x}.$

10. $\sqrt{\sqrt[3]{a^2}} = (a^{\frac{2}{3}})^{\frac{1}{2}} = a^{\frac{1}{3}} = \sqrt[3]{a}.$

11. $\sqrt[4]{\sqrt[3]{729}} = [(729)^{\frac{1}{3}}]^{\frac{1}{4}} = (729)^{\frac{1}{12}} = 3^{\frac{6}{12}} = \sqrt{3}.$

12. $\sqrt[3]{\sqrt{125}} = [(125)^{\frac{1}{2}}]^{\frac{1}{3}} = (125)^{\frac{1}{6}} = 5^{\frac{3}{6}} = \sqrt{5}.$

13. $\sqrt[7]{\sqrt[3]{(3a-2b)^{14}}}=[(3a-2b)^{\frac{14}{3}}]^{\frac{1}{7}}=(3a-2b)^{\frac{2}{3}}=\sqrt[3]{(3a-2b)^2}$

14. $\sqrt[3]{\sqrt[5]{32a^{45}}}=[(32a^{45})^{\frac{1}{5}}]^{\frac{1}{3}}=(2a^9)^{\frac{1}{3}}=a^3\sqrt[3]{2}.$

15. $\sqrt[7]{128\sqrt[5]{243a^{70}}}=[128(243a^{70})^{\frac{1}{5}}]^{\frac{1}{7}}=(128\times 3a^{14})^{\frac{1}{7}}=2a^2\sqrt[7]{3}.$

Exercise 89. Page 241.

1. $\sqrt{7-4\sqrt{3}}=\sqrt{4-4\sqrt{3}+3}=2-\sqrt{3}.$

2. $\sqrt{11+\sqrt{72}}=\sqrt{9+\sqrt{72}+2}=\sqrt{9+6\sqrt{2}+2}=3+\sqrt{2}.$

3. $\sqrt{7+2\sqrt{10}}=\sqrt{2+2\sqrt{10}+5}=\sqrt{2}+\sqrt{5}.$

4. $\sqrt{18+8\sqrt{5}}=\sqrt{10+8\sqrt{5}+8}=\sqrt{10+2\sqrt{80}+8}=\sqrt{10}+\sqrt{8}$

5. $\sqrt{8+2\sqrt{15}}=\sqrt{3+2\sqrt{15}+5}=\sqrt{3}+\sqrt{5}.$

6. $\sqrt{15-4\sqrt{14}}=\sqrt{7-4\sqrt{14}+8}=\sqrt{7-2\sqrt{56}+8}=\sqrt{7}-\sqrt{8}.$

7. $\sqrt{16+5\sqrt{7}}=\sqrt{\frac{25}{2}+5\sqrt{7}+\frac{7}{2}}=\sqrt{\frac{25}{2}}+\sqrt{\frac{7}{2}}$
$$=\frac{5+\sqrt{7}}{\sqrt{2}}=\frac{5\sqrt{2}+\sqrt{14}}{2}.$$

8. $\sqrt{75+12\sqrt{21}}=\sqrt{63+12\sqrt{21}+12}=\sqrt{63+2\sqrt{63\times 12}+12}$
$$=\sqrt{63}+\sqrt{12}=3\sqrt{7}+2\sqrt{3}.$$

9. $\sqrt{19+8\sqrt{3}}=\sqrt{16+8\sqrt{3}+3}=4+\sqrt{3}.$

10. $\sqrt{8\sqrt{6}+20}=\sqrt{8+8\sqrt{6}+12}=\sqrt{8}+\sqrt{12}=2\sqrt{2}+2\sqrt{3}.$

11. $\sqrt{28-16\sqrt{3}}=\sqrt{16-16\sqrt{3}+12}=\sqrt{16-2\sqrt{192}+12}$
$$=\sqrt{16}-\sqrt{12}=4-2\sqrt{3}.$$

12. $\sqrt{51+36\sqrt{2}}=\sqrt{27+36\sqrt{2}+24}=\sqrt{27+2\sqrt{648}+24}$
$$=\sqrt{27}+\sqrt{24}=3\sqrt{3}+2\sqrt{6}.$$

13. $\sqrt{94+42\sqrt{5}}=\sqrt{49+42\sqrt{5}+45}=7+\sqrt{45}=7+3\sqrt{5}.$

14. $\sqrt{11-2\sqrt{30}}=\sqrt{6-2\sqrt{30}+5}=\sqrt{6}-\sqrt{5}.$

15. $\sqrt{47-4\sqrt{33}}=\sqrt{44-4\sqrt{33}+3}=\sqrt{44}-\sqrt{3}=2\sqrt{11}-\sqrt{3}.$

16. $\sqrt{29+6\sqrt{22}}=\sqrt{18+6\sqrt{22}+11}=\sqrt{18}+\sqrt{11}=3\sqrt{2}+\sqrt{11}.$

17. $\sqrt{83+12\sqrt{35}}=\sqrt{63+12\sqrt{35}+20}=\sqrt{63}+\sqrt{20}=3\sqrt{7}+2\sqrt{5}.$

18. $\sqrt{55-12\sqrt{21}}=\sqrt{28-12\sqrt{21}+27}=\sqrt{28}-\sqrt{27}=2\sqrt{7}-3\sqrt{3}.$

Exercise 90. Page 242.

Solve:

1.
$$\begin{aligned} 2\sqrt{x+5}&=\sqrt{28}\\ 4(x+5)&=28\\ 4x&=8\\ \therefore\ x&=2 \end{aligned}$$

2.
$$\begin{aligned} 3\sqrt{4x-8}&=\sqrt{13x-3}\\ 9(4x-8)&=13x-3\\ 23x&=69\\ \therefore\ x&=3 \end{aligned}$$

3.
$$\begin{aligned} \sqrt{x+9}&=5\sqrt{x-3}\\ x+9&=25(x-3)\\ 24x&=84\\ \therefore\ x&=\tfrac{7}{2} \end{aligned}$$

4.
$$\begin{aligned} 4&=2\sqrt{x}-3\\ 7&=2\sqrt{x}\\ 49&=4x\\ \therefore\ x&=\tfrac{49}{4} \end{aligned}$$

5.
$$\begin{aligned} 5-\sqrt{3y}&=4\\ \sqrt{3y}&=1\\ 3y&=1\\ \therefore\ y&=\tfrac{1}{3} \end{aligned}$$

6.
$$\begin{aligned} 7+2\sqrt[3]{3x}&=5\\ \sqrt[3]{3x}&=-1\\ 3x&=-1\\ \therefore\ x&=-\tfrac{1}{3} \end{aligned}$$

7.
$$\begin{aligned} \sqrt[3]{2x-3}&=-3\\ 2x-3&=-27\\ 2x&=-24\\ \therefore\ x&=-12 \end{aligned}$$

8.
$$\begin{aligned} \sqrt[3]{3x+7}&=3\\ 3x+7&=27\\ 3x&=20\\ \therefore\ x&=\tfrac{20}{3} \end{aligned}$$

9.
$$\begin{aligned} 14+\sqrt[3]{4x-40}&=10\\ \sqrt[3]{4x-40}&=-4\\ 4x-40&=-64\\ 4x&=-24\\ \therefore\ x&=-6 \end{aligned}$$

10.
$$\begin{aligned} \sqrt[3]{10y-4}&=\sqrt[3]{7y+11}\\ 10y-4&=7y+11\\ 3y&=15\\ \therefore\ y&=5 \end{aligned}$$

11.
$$
\begin{aligned}
2\sqrt{x-2} &= \sqrt[4]{32(x-2)^3}\\
(2\sqrt{x-2})^4 &= 32(x-2)^3\\
16(x-2)^2 &= 32(x-2)^3\\
1 &= 2(x-2)\\
2x &= 5\\
\therefore\ x &= \tfrac{5}{2}
\end{aligned}
$$

12.
$$
\begin{aligned}
\sqrt{\tfrac{15}{4}+x} &= \tfrac{3}{2}+\sqrt{x}\\
\tfrac{15}{4}+x &= \tfrac{9}{4}+3\sqrt{x}+x\\
3\sqrt{x} &= \tfrac{3}{2}\\
\sqrt{x} &= \tfrac{1}{2}\\
\therefore\ x &= \tfrac{1}{4}
\end{aligned}
$$

13.
$$
\begin{aligned}
\sqrt{32+x} &= 16-\sqrt{x}\\
32+x &= 256-32\sqrt{x}+x\\
32\sqrt{x} &= 224\\
\sqrt{x} &= 7\\
\therefore\ x &= 49
\end{aligned}
$$

14.
$$
\begin{aligned}
\sqrt{x}-\sqrt{x-5} &= \sqrt{5}\\
\sqrt{x} &= \sqrt{5}+\sqrt{x-5}\\
x &= 5+2\sqrt{5(x-5)}+x-5\\
2\sqrt{5(x-5)} &= 0\\
5(x-5) &= 0\\
\therefore\ x &= 5
\end{aligned}
$$

15.
$$
\begin{aligned}
\sqrt{x+20}-\sqrt{x-1}-3 &= 0\\
\sqrt{x+20} &= \sqrt{x-1}+3\\
x+20 &= x-1+6\sqrt{x-1}+9\\
6\sqrt{x-1} &= 12\\
x-1 &= 4\\
\therefore\ x &= 5
\end{aligned}
$$

16.
$$
\begin{aligned}
\sqrt{x+15}-7 &= 7-\sqrt{x-13}\\
\sqrt{x+15} &= 14-\sqrt{x-13}\\
x+15 &= 196-28\sqrt{x-13}+x-13\\
28\sqrt{x-13} &= 168\\
\sqrt{x-13} &= 6\\
x-13 &= 36\\
\therefore\ x &= 49
\end{aligned}
$$

17.
$$
\begin{aligned}
x &= 7-\sqrt{x^2-7}\\
x-7 &= -\sqrt{x^2-7}\\
(x-7)^2 &= x^2-7\\
x^2-14x+49 &= x^2-7\\
14x &= 56\\
\therefore\ x &= 4
\end{aligned}
$$

18.
$$
\begin{aligned}
\sqrt{x-7} &= \sqrt{x+1}-2\\
x-7 &= x+1-4\sqrt{x+1}+4\\
4\sqrt{x+1} &= 12\\
\sqrt{x+1} &= 3\\
x+1 &= 9\\
\therefore\ x &= 8
\end{aligned}
$$

19.
$$
\begin{aligned}
\frac{\sqrt{x}-3}{\sqrt{x}+3} &= \frac{\sqrt{x}+1}{\sqrt{x}-2}\\
(\sqrt{x}-3)(\sqrt{x}-2) &= (\sqrt{x}+1)(\sqrt{x}+3)\\
x-5\sqrt{x}+6 &= x+4\sqrt{x}+3\\
9\sqrt{x} &= 3\\
\sqrt{x} &= \tfrac{1}{3}\\
\therefore\ x &= \tfrac{1}{9}
\end{aligned}
$$

20.

$$\frac{1}{2}-\frac{3}{x}=\sqrt{\frac{1}{4}-\frac{1}{x}\sqrt{9-\frac{36}{x}}}$$

$$\frac{1}{4}-\frac{3}{x}+\frac{9}{x^2}=\frac{1}{4}-\frac{1}{x}\sqrt{9-\frac{36}{x}}$$

$$\frac{9}{x^2}-\frac{3}{x}=-\frac{1}{x}\sqrt{9-\frac{36}{x}}$$

$$9-3x=-x\sqrt{9-\frac{36}{x}}$$

$$81-54x+9x^2=9x^2-36x$$

$$18x=81$$

$$\therefore\ x=\frac{9}{2}$$

21.

$$\frac{1+(1-x)^{\frac{1}{2}}}{1-(1-x)^{\frac{1}{2}}}=3$$

$$1+(1-x)^{\frac{1}{2}}=3[1-(1-x)^{\frac{1}{2}}]$$

$$4(1-x)^{\frac{1}{2}}=2$$

$$1-x=\frac{1}{4}$$

$$\therefore\ x=\frac{3}{4}$$

22.

$$(x-3)^{\frac{1}{2}}+x^{\frac{1}{2}}=\frac{3}{(x-3)^{\frac{1}{2}}}$$

$$(x-3)+[x(x-3)]^{\frac{1}{2}}=3$$

$$[x(x-3)]^{\frac{1}{2}}=6-x$$

$$x^2-3x=36-12x+x^2$$

$$9x=36$$

$$\therefore\ x=4$$

23.

$$\sqrt{a+\sqrt{x}}+\sqrt{a-\sqrt{x}}=\sqrt{x}$$

$$a+\sqrt{x}+2\sqrt{a^2-x}+a-\sqrt{x}=x$$

$$2\sqrt{a^2-x}=x-2a$$

$$4a^2-4x=x^2-4ax+4a^2$$

$$x^2+(4-4a)x=0$$

$$x+4-4a=0$$

$$\therefore\ x=4a-4$$

24.

$$\sqrt{ax}-1=4+\tfrac{1}{2}\sqrt{ax}-\tfrac{1}{2}$$

$$\tfrac{1}{2}\sqrt{ax}=4\tfrac{1}{2}$$

$$\sqrt{ax}=9$$

$$ax=81$$

$$\therefore\ x=\frac{81}{a}$$

25.

$$3\sqrt{x}-3\sqrt{a}=\sqrt{x}-\sqrt{a}+2\sqrt{a}$$

$$2\sqrt{x}=4\sqrt{a}$$

$$\sqrt{x}=2\sqrt{a}$$

$$\therefore\ x=4a$$

26.

$$\sqrt{9+2x}-\sqrt{2x}=\frac{5}{\sqrt{9+2x}}$$

$$9+2x-\sqrt{2x(9+2x)}=5$$

$$\sqrt{2x(9+2x)}=4+2x$$

$$2x(9+2x)=16+16x+4x^2$$

$$2x=16$$

$$\therefore\ x=8$$

Exercise 91. Page 247.

Reduce to the form $b\sqrt{-1}$:

1. $\sqrt{-9} = 3\sqrt{-1}.$

2. $\sqrt{-16} = 4\sqrt{-1}.$

3. $\sqrt{-25} = 5\sqrt{-1}.$

4. $\sqrt{-144} = 12\sqrt{-1}.$

5. $\sqrt{-169} = 13\sqrt{-1}.$

6. $\sqrt{-x^2} = x\sqrt{-1}.$

7. $\sqrt{-81} = 9\sqrt{-1}.$

8. $\sqrt{-256} = 16\sqrt{-1}.$

9. $\sqrt{-625} = 25\sqrt{-1}.$

10. $\sqrt{-36} = 6\sqrt{-1}.$

11. $\sqrt[6]{-64} = \sqrt{\sqrt[3]{-64}} = \sqrt{-4} = 2\sqrt{-1}.$

12. $\sqrt[6]{-729} = \sqrt{\sqrt[3]{-729}} = \sqrt{-9} = 3\sqrt{-1}.$

13. $\sqrt{-289} = 17\sqrt{-1}.$

14. $\sqrt[10]{-1024} = \sqrt{\sqrt[5]{-1024}} = \sqrt{-4} = 2\sqrt{-1}.$

15. $\sqrt{-x^8} = x^4\sqrt{-1}.$

16. $\sqrt{-x^9} = x^{\frac{9}{2}}\sqrt{-1}.$

17. $\sqrt[6]{-x^{18}} = \sqrt{\sqrt[3]{-x^{18}}} = \sqrt{-x^6} = x^3\sqrt{-1}.$

18. $\sqrt{-\frac{1}{4}} = \frac{1}{2}\sqrt{-1}.$

19. $\sqrt{-a^4b^{-2}} = a^2b^{-1}\sqrt{-1}.$

20. $\sqrt{-9x^4} = 3x^2\sqrt{-1}.$

21. $\sqrt[6]{-(2x-3y)^{12}} = \sqrt{\sqrt[3]{-(2x-3y)^{12}}} = \sqrt{-(2x-3y)^4}$
$= (2x-3y)^2\sqrt{-1}.$

22. $\sqrt[10]{-(x-2y)^{20}} = \sqrt{\sqrt[5]{-(x-2y)^{20}}} = \sqrt{-(x-2y)^4}$
$= (x-2y)^2\sqrt{-1}.$

23. $\sqrt{-(x^2+y^2)} = (x^2+y^2)^{\frac{1}{2}}\sqrt{-1}.$

24. $\sqrt{-(x^2-y^2)} = (x^2-y^2)^{\frac{1}{2}}\sqrt{-1}.$

Add:

25. $\sqrt{-25} + \sqrt{-49} - \sqrt{-121} = 5\sqrt{-1} + 7\sqrt{-1} - 11\sqrt{-1} = \sqrt{-1}.$

26. $\sqrt{-64} + \sqrt{-1} - \sqrt{-36} = 8\sqrt{-1} + \sqrt{-1} - 6\sqrt{-1} = 3\sqrt{-1}.$

27. $\sqrt{-a^4}+\sqrt{-4a^4}+\sqrt{-16a^4}=a^2\sqrt{-1}+2a^2\sqrt{-1}+4a^2\sqrt{-1}$
$=7a^2\sqrt{-1}.$

28. $\sqrt{-a^2}+\sqrt{-81a^2}-\sqrt{-a^2}=a\sqrt{-1}+9a\sqrt{-1}-a\sqrt{-1}=9a\sqrt{-1}.$

29. $a-b\sqrt{-1}+a+b\sqrt{-1}=2a.$

30. $2+3\sqrt{-1}-2+3\sqrt{-1}=6\sqrt{-1}.$

31. $a+b\sqrt{-1}+c-d\sqrt{-1}=a+c+(b-d)\sqrt{-1}.$

32. $3a\sqrt{-1}-(2a-b)\sqrt{-1}=(3a-2a+b)\sqrt{-1}=(a+b)\sqrt{-1}.$

Multiply:

33. $\sqrt{-3}\times\sqrt{-5}=-\sqrt{15}.$

34. $(-\sqrt{-5})\times\sqrt{-5}=-(-5)=5.$

35. $\sqrt{-16}\times\sqrt{-9}=-\sqrt{144}=-12.$

36. $\sqrt{-x^2}\times\sqrt{-x}=-\sqrt{x^3}=-x^{\frac{3}{2}}.$

37. $\sqrt{-x^2}\times\sqrt{-y^2}=-\sqrt{x^2y^2}=-xy.$

38. $\sqrt{-8}\times\sqrt{-16}=-\sqrt{128}=-8\sqrt{2}.$

39. $\sqrt{-25}\times\sqrt{-64}=-\sqrt{1600}=-40.$

40. $\sqrt{-(a+b)}\times\sqrt{-(a-b)}=-\sqrt{a^2-b^2}.$

41. $(3\sqrt{-3})\times(2\sqrt{-2})=-6\sqrt{6}.$

42. $(-5\sqrt{-2})\times(2\sqrt{-5})=10\sqrt{10}.$

43. $(\sqrt{-2}+\sqrt{-3})(\sqrt{-4}-\sqrt{-5})=-\sqrt{8}+\sqrt{10}-\sqrt{12}+\sqrt{15}$
$=-2\sqrt{2}+\sqrt{10}-2\sqrt{3}+\sqrt{15}.$

44. $$\left(x-\frac{1+\sqrt{-3}}{2}\right)\left(x-\frac{1-\sqrt{-3}}{2}\right)$$
$$=x^2-\left(\frac{1+\sqrt{-3}}{2}+\frac{1-\sqrt{-3}}{2}\right)x+\frac{(1+\sqrt{-3})(1-\sqrt{-3})}{4}$$
$$=x^2-x+\frac{1-(-3)}{4}=x^2-x+1.$$

45. $$(a\sqrt{-a}+b\sqrt{-b})(a\sqrt{-a}-b\sqrt{-b})=(a\sqrt{-a})^2-(b\sqrt{-b})^2$$
$$=-a^3-(-b^3)=b^3-a^3.$$

46. $$(2\sqrt{-2}+3\sqrt{-3})(3\sqrt{-4}-2\sqrt{-5})$$
$$=-6\sqrt{8}+4\sqrt{10}-9\sqrt{12}+6\sqrt{15}$$
$$=-12\sqrt{2}+4\sqrt{10}-18\sqrt{3}+6\sqrt{15}.$$

47. $$(\sqrt{3}+2\sqrt{-3})(\sqrt{3}-2\sqrt{-3})=(\sqrt{3})^2-(2\sqrt{-3})^2$$
$$=3-(-12)=15.$$

48. $$(m-3\sqrt{-b})(n+4\sqrt{-c})=mn+4m\sqrt{-c}-3n\sqrt{-b}+12\sqrt{bc}.$$

Perform the divisions indicated:

49. $$\frac{a}{\sqrt{-1}}=\frac{a\sqrt{-1}}{(\sqrt{-1})^2}=\frac{a\sqrt{-1}}{-1}=-a\sqrt{-1}.$$

50. $$\frac{b}{\sqrt{-b^2}}=\frac{b}{b\sqrt{-1}}=\frac{1}{\sqrt{-1}}=\frac{\sqrt{-1}}{(\sqrt{-1})^2}=-\sqrt{-1}.$$

51. $$\frac{c}{\sqrt{-4}}=\frac{c}{2\sqrt{-1}}=\frac{c\sqrt{-1}}{2(\sqrt{-1})^2}=-\frac{c\sqrt{-1}}{2}.$$

52. $$\frac{\sqrt{-9}}{\sqrt{-81}}=\frac{3\sqrt{-1}}{9\sqrt{-1}}=\frac{1}{3}.$$

53. $$\frac{\sqrt{-a}}{\sqrt{-b}}=\frac{\sqrt{a}\,\sqrt{-1}}{\sqrt{b}\,\sqrt{-1}}=\frac{\sqrt{a}}{\sqrt{b}}.$$

54. $$\frac{\sqrt{-ax}}{\sqrt{-x}}=\frac{\sqrt{ax}\,\sqrt{-1}}{\sqrt{x}\,\sqrt{-1}}=\sqrt{a}.$$

55. $$\frac{x}{\sqrt{-x}}=\frac{x}{\sqrt{x}\,\sqrt{-1}}=\frac{\sqrt{x}}{\sqrt{-1}}=-\sqrt{x}\,\sqrt{-1}=-\sqrt{-x}.$$

56. $$\frac{x}{\sqrt{-x^2}}=\frac{x}{x\sqrt{-1}}=\frac{1}{\sqrt{-1}}=-\sqrt{-1}.$$

57. $$\frac{\sqrt{-x^2}}{-\sqrt{-x}}=\frac{x\sqrt{-1}}{-\sqrt{x}\,\sqrt{-1}}=-\sqrt{x}$$

58. $\dfrac{\sqrt{-8}}{\sqrt{-2}}=\dfrac{2\sqrt{-2}}{\sqrt{-2}}=2.$

59. $\dfrac{\sqrt{-10x^3}}{\sqrt{-5x}}=\dfrac{\sqrt{10x^3}\,\sqrt{-1}}{\sqrt{5x}\,\sqrt{-1}}=\sqrt{2x^2}=x\sqrt{2}.$

60. $\dfrac{8\sqrt{-x^2}}{2\sqrt{x}}=\dfrac{8x\sqrt{-1}}{2\sqrt{x}}=4\sqrt{x}\,\sqrt{-1}=4\sqrt{-x}.$

61. $\dfrac{1}{3-\sqrt{-2}}=\dfrac{3+\sqrt{-2}}{(3-\sqrt{-2})(3+\sqrt{-2})}=\dfrac{3+\sqrt{-2}}{3^2-(\sqrt{-2})^2}=\dfrac{3+\sqrt{-2}}{9+2}$

$$=\frac{3+\sqrt{-2}}{11}.$$

62. $\dfrac{2+\sqrt{-2}}{1-\sqrt{-1}}=\dfrac{(2+\sqrt{-2})(1+\sqrt{-1})}{(1-\sqrt{-1})(1+\sqrt{-1})}=\dfrac{2+2\sqrt{-1}+\sqrt{-2}-\sqrt{2}}{1^2-(\sqrt{-1})^2}$

$$=\frac{2+2\sqrt{-1}+\sqrt{-2}-\sqrt{2}}{2}.$$

63. $\dfrac{a+x\sqrt{-1}}{a-x\sqrt{-1}}=\dfrac{(a+x\sqrt{-1})^2}{(a-x\sqrt{-1})(a+x\sqrt{-1})}=\dfrac{a^2+2ax\sqrt{-1}-x^2}{a^2-(x\sqrt{-1})^2}$

$$=\frac{a^2-x^2+2ax\sqrt{-1}}{a^2+x^2}.$$

64. $\dfrac{\sqrt{5}+\sqrt{-6}}{\sqrt{6}-\sqrt{-8}}=\dfrac{(\sqrt{5}+\sqrt{-6})(\sqrt{6}+\sqrt{-8})}{(\sqrt{6}-\sqrt{-8})(\sqrt{6}+\sqrt{-8})}$

$$=\frac{\sqrt{30}+\sqrt{-40}+\sqrt{-36}-\sqrt{48}}{(\sqrt{6})^2-(\sqrt{-8})^2}=\frac{\sqrt{30}+2\sqrt{-10}+6\sqrt{-1}-4\sqrt{3}}{14}.$$

65. $\dfrac{2a+3b\sqrt{-1}}{2a-3b\sqrt{-1}}=\dfrac{(2a+3b\sqrt{-1})^2}{(2a+3b\sqrt{-1})(2a-3b\sqrt{-1})}$

$$=\frac{4a^2+12ab\sqrt{-1}-9b^2}{(2a)^2-(3b\sqrt{-1})^2}=\frac{4a^2-9b^2+12ab\sqrt{-1}}{4a^2+9b^2}.$$

66. $\dfrac{\frac{1}{2}a-4b\sqrt{-1}}{4a-\frac{1}{2}b\sqrt{-1}}=\dfrac{(\frac{1}{2}a-4b\sqrt{-1})(4a+\frac{1}{2}b\sqrt{-1})}{(4a-\frac{1}{2}b\sqrt{-1})(4a+\frac{1}{2}b\sqrt{-1})}$

$$=\frac{2a^2+\frac{1}{4}ab\sqrt{-1}-16ab\sqrt{-1}+2b^2}{(4a)^2-(\frac{1}{2}b\sqrt{-1})^2}=\frac{2a^2+2b^2-15\frac{3}{4}ab\sqrt{-1}}{16a^2+\dfrac{b^2}{4}}.$$

Exercise 92. Page 251.

Solve:

1.
$$\begin{aligned} 3x^2-2&=x^2+6\\ 2x^2&=8\\ x^2&=4\\ \therefore\ x&=\pm 2 \end{aligned}$$

2.
$$\begin{aligned} 5x^2+10&=6x^2+1\\ -x^2&=-9\\ x^2&=9\\ \therefore\ x&=\pm 3 \end{aligned}$$

3.
$$\begin{aligned} 7x^2-50&=4x^2+25\\ 3x^2&=75\\ x^2&=25\\ \therefore\ x&=\pm 5 \end{aligned}$$

4.
$$\begin{aligned} 6x^2-\tfrac{1}{6}&=4x^2+\tfrac{11}{9}\\ 2x^2&=\tfrac{11}{9}+\tfrac{1}{6}=\tfrac{25}{18}\\ x^2&=\tfrac{25}{36}\\ \therefore\ x&=\pm\tfrac{5}{6} \end{aligned}$$

5.
$$\begin{aligned} \frac{x^2+1}{5}&=10\\ x^2+1&=50\\ x^2&=49\\ \therefore\ x&=\pm 7 \end{aligned}$$

6.
$$\begin{aligned} \frac{3x^2-8}{10}&=4\\ 3x^2-8&=40\\ 3x^2&=48\\ x^2&=16\\ \therefore\ x&=\pm 4 \end{aligned}$$

7.
$$\begin{aligned} \frac{x^2-9}{4}&=\frac{x^2+1}{5}.\\ 5(x^2-9)&=4(x^2+1)\\ 5x^2-45&=4x^2+4\\ x^2&=49\\ \therefore\ x&=\pm 7 \end{aligned}$$

8.
$$\begin{aligned} \frac{2x^2-4}{7}+\frac{x^2+4}{5}&=8\\ 5(2x^2-4)+7(x^2+4)&=280\\ 10x^2-20+7x^2+28&=280\\ 17x^2&=272\\ x^2&=16\\ \therefore\ x&=\pm 4 \end{aligned}$$

9.
$$\begin{aligned} \frac{3-x^2}{11}+\frac{x^2+5}{6}&=3\\ 6(3-x^2)+11(x^2+5)&=198\\ 18-6x^2+11x^2+55&=198\\ 5x^2&=125\\ x^2&=25\\ \therefore\ x&=\pm 5 \end{aligned}$$

10.
$$\begin{aligned} \frac{5x^2+3}{8}-\frac{17-x^2}{4}&=4\\ 4(5x^2+3)-8(17-x^2)&=128\\ 20x^2+12-136+8x^2&=128\\ 28x^2&=252\\ x^2&=9\\ \therefore\ x&=\pm 3 \end{aligned}$$

11.
$$\begin{aligned} \frac{3}{4x^2}-\frac{1}{6x^2}&=\frac{7}{3}\\ 9-2&=28x^2\\ 28x^2&=7\\ x^2&=\tfrac{1}{4}\\ \therefore\ x&=\pm\tfrac{1}{2}. \end{aligned}$$

12.
$$\begin{aligned} \frac{5}{3x^2}-\frac{3}{5x^2}&=\frac{4}{15}\\ 25-9&=4x^2\\ 4x^2&=16\\ x^2&=4\\ \therefore\ x&=\pm 2 \end{aligned}$$

13.
$$\begin{aligned}\frac{1}{x-1}-\frac{1}{x+1}&=\frac{1}{4}\\4(x+1)-4(x-1)&=(x-1)(x+1)\\4x+4-4x+4&=x^2-1\\x^2&=9\\\therefore\ x&=\pm 3\end{aligned}$$

14.
$$\begin{aligned}\frac{15}{8-x}+\frac{7}{2-3x}&=2\\15(2-3x)+7(8-x)&=2(8-x)(2-3x)\\30-45x+56-7x&=32-52x+6x^2\\6x^2&=54\\x^2&=9\\\therefore\ x&=\pm 3\end{aligned}$$

15.
$$\begin{aligned}3x^2+11x&=10x+8+x^2+x\\2x^2&=8\\x^2&=4\\\therefore\ x&=\pm 2\end{aligned}$$

16.
$$\begin{aligned}(x+4)(x+5)&=3(x+1)(x+2)-4\\x^2+9x+20&=3x^2+9x+6-4\\-2x^2&=-18\\x^2&=9\\\therefore\ x&=\pm 3\end{aligned}$$

17.
$$\begin{aligned}3(x-2)(x+3)&=(x+1)(x+2)+x^2+5\\3x^2+3x-18&=x^2+3x+2+x^2+5\\x^2&=25\\\therefore\ x&=\pm 5\end{aligned}$$

18.
$$\begin{aligned}(2x+1)(3x-2)+(1-x)(3+4x)&=3x^2-15\\6x^2-x-2+3+x-4x^2&=3x^2-15\\x^2&=16\\\therefore\ x&=\pm 4\end{aligned}$$

19.
$$\begin{aligned}\frac{x^2+9}{17}-\frac{2x^2-5}{9}+\frac{3x^2+10}{5}&=14\\45(x^2+9)-85(2x^2-5)+153(3x^2+10)&=10710\\45x^2+405-170x^2+425+459x^2+1530&=10710\\334x^2&=8350\\x^2&=25\\\therefore\ x&=\pm 5\end{aligned}$$

20.
$$\begin{aligned}
\frac{3x^2-5}{7}+\frac{2x^2+4}{9}-\frac{x^2-3}{2}&=5\\
18(3x^2-5)+14(2x^2+4)-63(x^2-3)&=630\\
54x^2-90+28x^2+56-63x^2+189&=630\\
19x^2&=475\\
x^2&=25\\
\therefore\ x&=\pm 5
\end{aligned}$$

21.
$$\begin{aligned}
\frac{10x^2+7}{18}-\frac{12x^2+2}{11x^2-8}&=\frac{5x^2-9}{9}\\
\frac{10x^2+7}{18}-\frac{5x^2-9}{9}&=\frac{12x^2+2}{11x^2-8}\\
\frac{10x^2+7-(10x^2-18)}{18}&=\frac{12x^2+2}{11x^2-8}\\
\frac{25}{18}&=\frac{12x^2+2}{11x^2-8}\\
25(11x^2-8)&=18(12x^2+2)\\
275x^2-200&=216x^2+36\\
59x^2&=236\\
x^2&=4\\
\therefore\ x&=\pm 2
\end{aligned}$$

22.
$$\begin{aligned}
\frac{x-1}{x+1}+\frac{x+1}{x-1}&=\frac{5}{2}\\
2(x-1)^2+2(x+1)^2&=5(x^2-1)\\
2x^2-4x+2+2x^2+4x+2&=5x^2-5\\
x^2&=9\\
\therefore\ x&=\pm 3
\end{aligned}$$

23.
$$\begin{aligned}
ax^2+b&=c\\
ax^2&=c-b\\
x^2&=\frac{c-b}{a}\\
\therefore\ x&=\pm\sqrt{\frac{c-b}{a}}
\end{aligned}$$

24.
$$\begin{aligned}
ax^2+b&=bx^2+a\\
(a-b)x^2&=a-b\\
x^2&=1\\
\therefore\ x&=\pm 1
\end{aligned}$$

25.
$$\begin{aligned}
x^2+2bx+c&=b(2x+1)\\
x^2+2bx+c&=2bx+b\\
x^2&=b-c\\
\therefore\ x&=\pm\sqrt{b-c}
\end{aligned}$$

26.
$$\begin{aligned}
\frac{a}{x}+\frac{x}{a}&=\frac{9a^2-x^2}{ax}\\
a^2+x^2&=9a^2-x^2\\
2x^2&=8a^2\\
x^2&=4a^2\\
\therefore\ x&=\pm 2a
\end{aligned}$$

27.
$$\frac{x+a}{x-a}+\frac{x-a}{x+a}=\frac{5}{2}$$
$$2(x+a)^2+2(x-a)^2=5(x^2-a^2)$$
$$2x^2+4ax+2a^2+2x^2-4ax+2a^2=5x^2-5a^2$$
$$x^2=9a^2$$
$$\therefore\ x=\pm 3a$$

28.
$$\frac{2b}{x-b}+\frac{5x+2b}{3x}=-1$$
$$6bx+(x-b)(5x+2b)=-3x(x-b)$$
$$6bx+5x^2-3bx-2b^2=-3x^2+3bx$$
$$8x^2=2b^2$$
$$x^2=\frac{b^2}{4}$$
$$\therefore\ x=\pm\frac{b}{2}$$

29.
$$2[(x+a)(x+b)+(x-a)(x-b)]=a^2+4b^2$$
$$2(x^2+ax+bx+ab+x^2-ax-bx+ab)=a^2+4b^2$$
$$4x^2+4ab=a^2+4b^2$$
$$4x^2=a^2-4ab+4b^2$$
$$x^2=\frac{a^2-4ab+4b^2}{4}$$
$$\therefore\ x=\pm\frac{a-2b}{2}$$

30.
$$2[(x-a)(x+b)+(x+a)(x-b)]=9a^2+2ab+b^2$$
$$2(x^2-ax+bx-ab+x^2+ax-bx-ab)=9a^2+2ab+b^2$$
$$4x^2-4ab=9a^2+2ab+b^2$$
$$4x^2=9a^2+6ab+b^2$$
$$x^2=\frac{9a^2+6ab+b^2}{4}$$
$$\therefore\ x=\pm\frac{3a+b}{2}$$

Exercise 93. Page 254.

Solve:

1.
$$x^2+2x=8$$
$$x^2+2x+1=9$$
$$x+1=\pm 3$$
$$\therefore\ x=2,\text{ or }-4$$

2.
$$x^2-6x=7$$
$$x^2-6x+9=16$$
$$x-3=\pm 4$$
$$\therefore\ x=7,\text{ or }-1$$

3.
$$\begin{aligned} x^2-4x&=12\\ x^2-4x+4&=16\\ x-2&=\pm 4\\ \therefore\ x&=6,\text{ or }-2 \end{aligned}$$

4.
$$\begin{aligned} x^2+4x&=5\\ x^2+4x+4&=9\\ x+2&=\pm 3\\ \therefore\ x&=1,\text{ or }-5 \end{aligned}$$

5.
$$\begin{aligned} x^2+5x&=14\\ x^2+5x+\tfrac{25}{4}&=\tfrac{81}{4}\\ x+\tfrac{5}{2}&=\pm\tfrac{9}{2}\\ \therefore\ x&=2,\text{ or }-7 \end{aligned}$$

6.
$$\begin{aligned} x^2-3x&=28\\ x^2-3x+\tfrac{9}{4}&=\tfrac{121}{4}\\ x-\tfrac{3}{2}&=\pm\tfrac{11}{2}\\ \therefore\ x&=7,\text{ or }-4 \end{aligned}$$

7
$$\begin{aligned} 2x^2+x&=15\\ x^2+\tfrac{1}{2}x&=\tfrac{15}{2}\\ x^2+\tfrac{1}{2}x+\tfrac{1}{16}&=\tfrac{121}{16}\\ x+\tfrac{1}{4}&=\pm\tfrac{11}{4}\\ \therefore\ x&=\tfrac{5}{2},\text{ or }-3 \end{aligned}$$

8.
$$\begin{aligned} 5x^2+3x&=2\\ x^2+\tfrac{3}{5}x&=\tfrac{2}{5}\\ x^2+\tfrac{3}{5}x+\tfrac{9}{100}&=\tfrac{49}{100}\\ x+\tfrac{3}{10}&=\pm\tfrac{7}{10}\\ \therefore\ x&=\tfrac{2}{5},\text{ or }-1 \end{aligned}$$

9.
$$\begin{aligned} x^2+\tfrac{2}{3}x&=40\\ x^2+\tfrac{2}{3}x+\tfrac{1}{9}&=\tfrac{361}{9}\\ x+\tfrac{1}{3}&=\pm\tfrac{19}{3}\\ \therefore\ x&=6,\text{ or }-\tfrac{20}{3} \end{aligned}$$

10.
$$\begin{aligned} 3x^2-4x&=4\\ x^2-\tfrac{4}{3}x&=\tfrac{4}{3}\\ x^2-\tfrac{4}{3}x+\tfrac{4}{9}&=\tfrac{16}{9}\\ x-\tfrac{2}{3}&=\pm\tfrac{4}{3}\\ \therefore\ x&=2,\text{ or }-\tfrac{2}{3} \end{aligned}$$

11.
$$\begin{aligned} 6x^2+x&=1\\ x^2+\tfrac{1}{6}x&=\tfrac{1}{6}\\ x^2+\tfrac{1}{6}x+\tfrac{1}{144}&=\tfrac{25}{144}\\ x+\tfrac{1}{12}&=\pm\tfrac{5}{12}\\ \therefore\ x&=\tfrac{1}{3},\text{ or }-\tfrac{1}{2} \end{aligned}$$

12.
$$\begin{aligned} 6x^2-x&=2\\ x^2-\tfrac{1}{6}x&=\tfrac{1}{3}\\ x^2-\tfrac{1}{6}x+\tfrac{1}{144}&=\tfrac{49}{144}\\ x-\tfrac{1}{12}&=\pm\tfrac{7}{12}\\ \therefore\ x&=\tfrac{2}{3},\text{ or }-\tfrac{1}{2} \end{aligned}$$

13.
$$\begin{aligned} 12x^2-11x+2&=0\\ x^2-\tfrac{11}{12}x&=-\tfrac{1}{6}\\ x^2-\tfrac{11}{12}x+\tfrac{121}{576}&=\tfrac{25}{576}\\ x-\tfrac{11}{24}&=\pm\tfrac{5}{24}\\ \therefore\ x&=\tfrac{2}{3},\text{ or }\tfrac{1}{4} \end{aligned}$$

14.
$$\begin{aligned} 15x^2-2x-1&=0\\ x^2-\tfrac{2}{15}x&=\tfrac{1}{15}\\ x^2-\tfrac{2}{15}x+\tfrac{1}{225}&=\tfrac{16}{225}\\ x-\tfrac{1}{15}&=\pm\tfrac{4}{15}\\ \therefore\ x&=\tfrac{1}{3},\text{ or }-\tfrac{1}{5} \end{aligned}$$

15.
$$\begin{aligned} \frac{(x+1)(x+2)}{5}-\frac{(x-1)(x-2)}{2}&=3\\ 2(x+1)(x+2)-5(x-1)(x-2)&=30\\ 2x^2+6x+4-5x^2+15x-10&=30\\ 3x^2-21x&=-36\\ x^2-7x&=-12\\ x^2-7x+\tfrac{49}{4}&=\tfrac{1}{4}\\ x-\tfrac{7}{2}&=\pm\tfrac{1}{2}\\ \therefore\ x&=4,\text{ or }3 \end{aligned}$$

16.
$$\frac{(2x-3)x}{4}-\frac{(x+4)(x-1)}{6}=1$$
$$6x(2x-3)-4(x+4)(x-1)=24$$
$$12x^2-18x-4x^2-12x+16=24$$
$$8x^2-30x=8$$
$$x^2-\tfrac{15}{4}x=1$$
$$x^2-\tfrac{15}{4}x+\tfrac{225}{64}=\tfrac{289}{64}$$
$$x-\tfrac{15}{8}=\pm\tfrac{17}{8}$$
$$\therefore\ x=4,\ \text{or}\ -\tfrac{1}{4}$$

17.
$$\frac{3x+5}{x+4}+\frac{2x-5}{x-2}=3$$
$$(3x+5)(x-2)+(2x-5)(x+4)=3(x+4)(x-2)$$
$$3x^2-x-10+2x^2+3x-20=3x^2+6x-24$$
$$2x^2-4x=6$$
$$x^2-2x=3$$
$$x^2-2x+1=4$$
$$x-1=\pm 2$$
$$\therefore\ x=3,\ \text{or}\ -1$$

18.
$$\frac{x-6}{x-2}+\frac{x+5}{2x+1}=1$$
$$(x-6)(2x+1)+(x-2)(x+5)=(x-2)(2x+1)$$
$$2x^2-11x-6+x^2+3x-10=2x^2-3x-2$$
$$x^2-5x=14$$
$$x^2-5x+\tfrac{25}{4}=\tfrac{81}{4}$$
$$x-\tfrac{5}{2}=\pm\tfrac{9}{2}$$
$$\therefore\ x=7,\ \text{or}\ -2$$

19.
$$\frac{4-3x}{2+x}-\frac{1+2x}{1-x}=\frac{9}{2}$$
$$2(4-3x)(1-x)-2(1+2x)(2+x)=9(2+x)(1-x)$$
$$8-14x+6x^2-4-10x-4x^2=18-9x-9x^2$$
$$11x^2-15x=14$$
$$x^2-\tfrac{15}{11}x=\tfrac{14}{11}$$
$$x^2-\tfrac{15}{11}x+\tfrac{225}{484}=\tfrac{841}{484}$$
$$x-\tfrac{15}{22}=\pm\tfrac{29}{22}$$
$$\therefore\ x=2,\ \text{or}\ -\tfrac{7}{11}$$

20.
$$\begin{aligned}\frac{x}{x+1}+\frac{x+1}{x}&=\frac{13}{6}\\6x^2+6(x+1)^2&=13x(x+1)\\6x^2+6x^2+12x+6&=13x^2+13x\\x^2+x&=6\\x^2+x+\tfrac{1}{4}&=\tfrac{25}{4}\\x+\tfrac{1}{2}&=\pm\tfrac{5}{2}\\\therefore\ x&=2,\text{ or }-3\end{aligned}$$

21.
$$\begin{aligned}5x^2-4x&=1\\x^2-\frac{4x}{5}&=\frac{1}{5}\\x^2-\frac{4x}{5}+\frac{4}{25}&=\frac{9}{25}\\x-\tfrac{2}{5}&=\pm\tfrac{3}{5}\\\therefore\ x&=1,\text{ or }-\tfrac{1}{5}\end{aligned}$$

22.
$$\begin{aligned}\frac{x+2}{x-1}-\frac{4-x}{2x}&=\frac{7}{3}.\\6x(x+2)-3(4-x)(x-1)&=14x(x-1)\\6x^2+12x+3x^2-15x+12&=14x^2-14x\\5x^2-11x&=12\\x^2-\tfrac{11}{5}x&=\tfrac{12}{5}\\x^2-\tfrac{11}{5}x+\tfrac{121}{100}&=\tfrac{361}{100}\\x-\tfrac{11}{10}&=\pm\tfrac{19}{10}\\\therefore\ x&=3,\text{ or }-\tfrac{4}{5}\end{aligned}$$

23.
$$\begin{aligned}\frac{x+3}{x-2}&=\frac{5x+8}{x+4}\\(x+3)(x+4)&=(5x+8)(x-2)\\x^2+7x+12&=5x^2-2x-16\\4x^2-9x&=28\\x^2-\tfrac{9}{4}x&=7\\x^2-\tfrac{9}{4}x+\tfrac{81}{64}&=\tfrac{529}{64}\\x-\tfrac{9}{8}&=\pm\tfrac{23}{8}\\x&=4,\text{ or }-\tfrac{7}{4}\end{aligned}$$

24.
$$\begin{aligned}\frac{2x-1}{3}+\frac{3}{2x-1}&=2\\(2x-1)^2+9&=6(2x-1)\\4x^2-4x+1+9&=12x-6\\4x^2-16x&=-16\\x^2-4x&=-4\\x^2-4x+4&=0\\x-2&=0\\\therefore\ x&=2\end{aligned}$$

25.
$$\begin{aligned}\frac{x+1}{x+2}+\frac{x+2}{x+1}&=\frac{13}{6}\\6(x+1)^2+6(x+2)^2&=13(x+2)(x+1)\\6x^2+12x+6+6x^2+24x+24&=13x^2+39x+26\\x^2+3x&=4\\x^2+3x+\tfrac{9}{4}&=\tfrac{25}{4}\\x+\tfrac{3}{2}&=\pm\tfrac{5}{2}\\\therefore\ x&=1,\text{ or }-4\end{aligned}$$

26.
$$\begin{aligned}7x^2-8x&=-1\\x^2-\tfrac{8}{7}x&=-\tfrac{1}{7}\\{}^2-\tfrac{8}{7}x+\tfrac{16}{49}&=\tfrac{9}{49}\\x-\tfrac{4}{7}&=\pm\tfrac{3}{7}\\\therefore\ x&=1,\text{ or }\tfrac{1}{7}\end{aligned}$$

Exercise 94. Page 259.

Solve:

1.
$$\begin{aligned}3x^2-2x&=8\\9x^2-6x&=24\\9x^2-6x+1&=25\\3x-1&=\pm 5\\3x&=6,\text{ or }-4\\\therefore\ x&=2,\text{ or }-\tfrac{4}{3}\end{aligned}$$

2.
$$\begin{aligned}5x^2-6x&=27\\25x^2-30x&=135\\25x^2-30x+9&=144\\5x-3&=\pm 12\\5x&=15,\text{ or }-9\\\therefore\ x&=3,\text{ or }-\tfrac{9}{5}\end{aligned}$$

3.
$$\begin{aligned}2x^2+3x&=5\\16x^2+24x&=40\\16x^2+24x+9&=49\\4x+3&=\pm 7\\4x&=4,\text{ or }-10\\\therefore\ x&=1,\text{ or }-\tfrac{5}{2}\end{aligned}$$

4.
$$\begin{aligned}2x^2-5x&=7\\16x^2-40x&=56\\16x^2-40x+25&=81\\4x-5&=\pm 9\\4x&=14,\text{ or }-4\\\therefore\ x&=\tfrac{7}{2},\text{ or }-1\end{aligned}$$

5.
$$\begin{aligned}3x^2+7x&=6\\36x^2+84x&=72\\36x^2+84x+49&=121\\6x+7&=\pm 11\\6x&=4,\text{ or }-18\\\therefore\ x&=\tfrac{2}{3},\text{ or }-3\end{aligned}$$

6.
$$\begin{aligned}5x^2-7x&=24\\100x^2-140x&=480\\100x^2-140x+49&=529\\10x-7&=\pm 23\\10x&=30,\text{ or }-16\\\therefore\ x&=3,\text{ or }-\tfrac{8}{5}\end{aligned}$$

7.
$$\begin{aligned}8x^2+3x&=26\\256x^2+96x&=832\\256x^2+96x+9&=841\\16x+3&=\pm 29\\16x&=26,\text{ or }-32\\\therefore\ x&=\tfrac{13}{8},\text{ or }-2\end{aligned}$$

8.
$$\begin{aligned}7x^2+5x&=150\\196x^2+140x&=4200\\196x^2+140x+25&=4225\\14x+5&=\pm 65\\14x&=60,\text{ or }-70\\\therefore\ x&=\tfrac{30}{7},\text{ or }-5\end{aligned}$$

9.
$$\begin{aligned}6x^2+5x&=14\\144x^2+120x&=336\\144x^2+120x+25&=361\\12x+5&=\pm 19\\12x&=14,\text{ or }-24\\\therefore\ x&=\tfrac{7}{6},\text{ or }-2\end{aligned}$$

10.
$$\begin{aligned}7x^2-2x&=\tfrac{3}{4}\\196x^2-56x&=21\\196x^2-56x+4&=25\\14x-2&=\pm 5\\14x&=7,\text{ or }-3\\\therefore\ x&=\tfrac{1}{2},\text{ or }-\tfrac{3}{14}\end{aligned}$$

11.
$$\begin{aligned}8x^2+7x&=51\\256x^2+224x&=1632\\256x^2+224x+49&=1681\\16x+7&=\pm 41\\16x&=34,\text{ or }-48\\\therefore\ x&=\tfrac{17}{8},\text{ or }-3\end{aligned}$$

12.
$$\begin{aligned}7x^2-20x&=75\\196x^2-560x&=2100\\196x^2-560x+400&=2500\\14x-20&=\pm 50\\14x&=70,\text{ or }-30\\\therefore\ x&=5,\text{ or }-\tfrac{15}{7}\end{aligned}$$

13.

$$\begin{aligned} 11x^2 - 10x &= 24 \\ 121x^2 - 110x &= 264 \\ 121x^2 - 110x + 25 &= 289 \\ 11x - 5 &= \pm 17 \\ 11x &= 22, \text{ or } -12 \\ \therefore\ x &= 2, \text{ or } -\tfrac{12}{11} \end{aligned}$$

14.

$$\begin{aligned} 3x^2 + \frac{2x}{3} &= 25 \\ 81x^2 + 18x &= 675 \\ 81x^2 + 18x + 1 &= 676 \\ 9x + 1 &= \pm 26 \\ 9x &= 25, \text{ or } -27 \\ \therefore\ x &= \tfrac{25}{9}, \text{ or } -3 \end{aligned}$$

15.

$$\begin{aligned} x^2 - \frac{3x}{4} &= 3x + 1 \\ x^2 - \frac{15x}{4} &= 1 \\ 64x^2 - 240x &= 64 \\ 64x^2 - 240x + 225 &= 289 \\ 8x - 15 &= \pm 17 \\ 8x &= 32, \text{ or } -2 \\ \therefore\ x &= 4, \text{ or } -\tfrac{1}{4} \end{aligned}$$

16.

$$\begin{aligned} \frac{x^2}{2} - \frac{x}{3} &= 2(x - 2) \\ 3x^2 - 14x &= -24 \\ 36x^2 - 168x + 196 &= -92 \\ 6x - 14 &= \pm 2\sqrt{-23} \\ \therefore\ x &= \tfrac{1}{3}(7 \pm \sqrt{-23}) \end{aligned}$$

17.

$$\begin{aligned} \frac{2x^2}{3} + \frac{3x}{2} &= 15 \\ 4x^2 + 9x &= 90 \\ 64x^2 + 144x &= 1440 \\ 64x^2 + 144x + 81 &= 1521 \\ 8x + 9 &= \pm 39 \\ 8x &= 30, \text{ or } -48 \\ \therefore\ x &= \tfrac{15}{4}, \text{ or } -6 \end{aligned}$$

18.

$$\begin{aligned} \frac{3x}{4} - 2x^2 &= \frac{1}{16} \\ 2x^2 - \frac{3x}{4} &= -\frac{1}{16} \\ 32x^2 - 12x &= -1 \\ 256x^2 - 96x &= -8 \\ 256x^2 - 96x + 9 &= 1 \\ 16x - 3 &= \pm 1 \\ 16x &= 4, \text{ or } 2 \\ \therefore\ x &= \tfrac{1}{4}, \text{ or } \tfrac{1}{8} \end{aligned}$$

19.

$$\begin{aligned} \tfrac{3}{5}x^2 + \tfrac{5}{3}x &= \tfrac{20}{3} \\ 9x^2 + 25x &= 100 \\ 324x^2 + 900x &= 3600 \\ 324x^2 + 900x + 625 &= 4225 \\ 18x + 25 &= \pm 65 \\ 18x &= 40, \text{ or } -90 \\ \therefore\ x &= \tfrac{20}{9}, \text{ or } -5 \end{aligned}$$

20.

$$\begin{aligned} 2x - 3 &= \frac{2}{x} \\ 2x^2 - 3x &= 2 \\ 16x^2 - 24x &= 16 \\ 16x^2 - 24x + 9 &= 25 \\ 4x - 3 &= \pm 5 \\ 4x &= 8, \text{ or } -2 \\ \therefore\ x &= 2, \text{ or } -\tfrac{1}{2} \end{aligned}$$

21.

$$\begin{aligned} \frac{7x}{5} - \frac{5}{3x} &= \frac{20}{3} \\ 21x^2 - 25 &= 100x \\ 21x^2 - 100x &= 25 \\ 1764x^2 - 8400x &= 2100 \\ 1764x^2 - 8400x + 10000 &= 12100 \\ 42x - 100 &= \pm 110 \\ 42x &= 210, \text{ or } -10 \\ \therefore\ x &= 5, \text{ or } -\tfrac{5}{21} \end{aligned}$$

22.

$$\begin{aligned}
\frac{2x}{3}+\frac{3}{2x}&=\frac{10}{3}\\
4x^2+9&=20x\\
4x^2-20x&=-9\\
4x^2-20x+25&=16\\
2x-5&=\pm 4\\
2x&=9,\text{ or }1\\
\therefore\ x&=\tfrac{9}{2},\text{ or }\tfrac{1}{2}
\end{aligned}$$

23.

$$\begin{aligned}
(x+2)(2x+1)+(x-1)(3x+2)&=57\\
2x^2+5x+2+3x^2-x-2&=57\\
5x^2+4x&=57\\
25x^2+20x&=285\\
25x^2+20x+4&=289\\
5x+2&=\pm 17\\
5x&=15,\text{ or }-19\\
\therefore\ x&=3,\text{ or }-\tfrac{19}{5}
\end{aligned}$$

24.

$$\begin{aligned}
3x(2x+5)-(x+3)(3x-1)&=1\\
6x^2+15x-3x^2-8x+3&=1\\
3x^2+7x&=-2\\
36x^2+84x&=-24\\
36x^2+84x+49&=25\\
6x+7&=\pm 5\\
6x&=-2,\text{ or }-12\\
\therefore\ x&=-\tfrac{1}{3},\text{ or }-2
\end{aligned}$$

25.

$$\begin{aligned}
\frac{(2x+5)(x-3)}{3}+\frac{x(3x+4)}{5}&=5\\
5(2x+5)(x-3)+3x(3x+4)&=75\\
10x^2-5x-75+9x^2+12x&=75\\
19x^2+7x&=150\\
1444x^2+532x&=11400\\
1444x^2+532x+49&=11449\\
38x+7&=\pm 107\\
38x&=100,\text{ or }-144\\
x&=\tfrac{50}{19},\text{ or }-3
\end{aligned}$$

26.

$$\begin{aligned}
\tfrac{2}{3}(5x^2-8x-6)-\tfrac{1}{2}(x^2-3)&=2x+1\\
4(5x^2-8x-6)-3(x^2-3)&=6(2x+1)\\
20x^2-32x-24-3x^2+9&=12x+6\\
17x^2-44x&=21\\
1156x^2-2992x&=1428\\
1156x^2-2992x+1936&=3364\\
34x-44&=\pm 58\\
34x&=102,\text{ or }-14\\
\therefore\ x&=3,\text{ or }-\tfrac{7}{17}
\end{aligned}$$

27.

$$\begin{aligned}
\frac{2}{x+3}+\frac{5}{x}&=2\\
2x+5(x+3)&=2x(x+3)\\
2x+5x+15&=2x^2+6x\\
2x^2-x&=15\\
16x^2-8x&=120\\
16x^2-8x+1&=121\\
4x-1&=\pm 11\\
4x&=12,\text{ or }-10\\
\therefore\ x&=3,\text{ or }-\tfrac{5}{2}
\end{aligned}$$

28.

$$\begin{aligned}
\frac{5}{x-1}-\frac{3}{2x+1}&=\frac{4}{3}\\
15(2x+1)-9(x-1)&=4(x-1)(2x+1)\\
30x+15-9x+9&=8x^2-4x-4\\
8x^2-25x&=28\\
256x^2-800x&=896\\
256x^2-800x+625&=1521\\
16x-25&=\pm 39\\
16x&=64,\text{ or }-14\\
\therefore\ x&=4,\text{ or }-\tfrac{7}{8}
\end{aligned}$$

29.

$$\begin{aligned}
\frac{7}{3x-2}+\frac{4}{2x-5}&=5\\
7(2x-5)+4(3x-2)&=5(3x-2)(2x-5)\\
14x-35+12x-8&=30x^2-95x+50\\
30x^2-121x&=-93\\
4\times 30^2x^2-4\times 30\times 121x&=-11160\\
4\times 30^2x^2-4\times 30\times 121x+121^2&=3481\\
60x-121&=\pm 59\\
60x&=180,\text{ or }62\\
\therefore\ x&=3,\text{ or }\tfrac{31}{30}
\end{aligned}$$

30.

$$\frac{2}{x-1}=\frac{3}{x-2}+\frac{2}{x-4}$$

$$2(x-2)(x-4)=3(x-1)(x-4)+2(x-1)(x-2)$$

$$2x^2-12x+16=3x^2-15x+12+2x^2-6x+4$$

$$3x^2-9x=0$$

$$x(x-3)=0$$

$$\therefore\ x=0, \text{ or } 3$$

31.

$$\frac{x+2}{x-4}+\frac{x+3}{x-2}=-5$$

$$(x+2)(x-2)+(x+3)(x-4)+5(x-4)(x-2)=0$$

$$x^2-4+x^2-x-12+5x^2-30x+40=0$$

$$7x^2-31x=-24$$

$$196x^2-868x=-672$$

$$196x^2-868x+961=289$$

$$14x-31=\pm 17$$

$$14x=48, \text{ or } 14$$

$$\therefore\ x=\tfrac{24}{7}, \text{ or } 1$$

32.

$$\frac{2x-3}{x-4}+\frac{5-3x}{x+2}=1$$

$$(2x-3)(x+2)+(5-3x)(x-4)=(x-4)(x+2)$$

$$2x^2+x-6-3x^2+17x-20=x^2-2x-8$$

$$2x^2-20x=-18$$

$$x^2-10x=-9$$

$$x^2-10x+25=16$$

$$x-5=\pm 4$$

$$\therefore\ x=9, \text{ or } 1$$

33.

$$\frac{11-3x}{1-x}+\frac{2(7-4x)}{1-2x}=1$$

$$(11-3x)(1-2x)+2(7-4x)(1-x)=(1-x)(1-2x)$$

$$6x^2-25x+11+8x^2-22x+14=2x^2-3x+1$$

$$12x^2-44x=-24$$

$$36x^2-132x=-72$$

$$36x^2-132x+121=49$$

$$6x-11=\pm 7$$

$$6x=18, \text{ or } 4$$

$$\therefore\ x=3, \text{ or } \tfrac{2}{3}$$

34.
$$\frac{x+1}{x^2-4}+\frac{1-x}{x+2}=\frac{2}{5(x-2)}$$
$$5x+5+5(x-2)(1-x)=2(x+2)$$
$$5x+5-5x^2-10+15x=2x+4$$
$$5x^2-18x=-9$$
$$25x^2-90x=-45$$
$$25x^2-90x+81=36$$
$$5x-9=\pm 6$$
$$5x=15, \text{ or } 3$$
$$\therefore\ x=3, \text{ or } \tfrac{3}{5}$$

35.
$$\frac{2x+7}{2x-3}+\frac{3x-2}{x+1}=5$$
$$(2x+7)(x+1)+(3x-2)(2x-3)=5(2x-3)(x+1)$$
$$2x^2+9x+7+6x^2-13x+6=10x^2-5x-15$$
$$2x^2-x=28$$
$$16x^2-8x=224$$
$$16x^2-8x+1=225$$
$$4x-1=\pm 15$$
$$4x=16, \text{ or } -14$$
$$\therefore\ x=4, \text{ or } -\tfrac{7}{2}$$

36.
$$\frac{2x+3}{2(2x-1)}-\frac{7-x}{2(x+1)}=\frac{7-3x}{4-3x}$$
$$2(2x+3)(x+1)(4-3x)-2(7-x)(2x-1)(4-3x)$$
$$=4(7-3x)(2x-1)(x+1)$$
$$-12x^3-14x^2+22x+24-12x^3+106x^2-162x+56$$
$$=-24x^3+44x^2+40x-28$$
$$48x^2-180x=-108$$
$$16x^2-60x=-36$$
$$64x^2-240x=-144$$
$$64x^2-240x+225=81$$
$$8x-15=\pm 9$$
$$8x=24, \text{ or } 6$$
$$\therefore\ x=3, \text{ or } \tfrac{3}{4}$$

37.
$$\frac{2x-1}{3}-\frac{3}{x-8}=\frac{x-2}{x-8}+5$$
$$\frac{2x-1}{3}-\frac{3+(x-2)}{x-8}=5$$
$$\frac{2x-1}{3}-\frac{x+1}{x-8}=5$$

$$\begin{aligned}
(2x-1)(x-8)-3(x+1) &= 15(x-8)\\
2x^2-17x+8-3x-3 &= 15x-120\\
2x^2-35x &= -125\\
16x^2-280x &= -1000\\
16x^2-280x+1225 &= 225\\
4x-35 &= \pm 15\\
4x &= 50, \text{ or } 20\\
\therefore\ x &= \tfrac{25}{2}, \text{ or } 5
\end{aligned}$$

38. $$\frac{3x+2}{2x-1}+\frac{7-x}{2x+1}=\frac{7x-1}{4x^2-1}+5$$

$$\begin{aligned}
(3x+2)(2x+1)+(7-x)(2x-1) &= 7x-1+5(4x^2-1)\\
6x^2+7x+2-2x^2+15x-7 &= 7x-1+20x^2-5\\
16x^2-15x &= 1\\
4\times 16^2x^2-4\times 16\times 15x &= 4\times 16\\
4\times 16^2x^2-4\times 16\times 15x+15^2 &= 289\\
32x-15 &= \pm 17\\
32x &= 32, \text{ or } -2\\
\therefore\ x &= 1, \text{ or } -\tfrac{1}{16}
\end{aligned}$$

39. $$\frac{x-5}{x+3}+\frac{x-8}{x-3}=\frac{80}{x^2-9}+\frac{1}{2}$$

$$\begin{aligned}
2(x-5)(x-3)+2(x-8)(x+3) &= 160+x^2-9\\
2x^2-16x+30+2x^2-10x-48 &= 160+x^2-9\\
3x^2-26x &= 169\\
36x^2-312x &= 2028\\
36x^2-312x+676 &= 2704\\
6x-26 &= \pm 52\\
6x &= 78, \text{ or } -26\\
\therefore\ x &= 13, \text{ or } -\tfrac{13}{3}
\end{aligned}$$

40. $$\frac{2x+1}{7-x}+\frac{4x+1}{7+x}=\frac{45}{49-x^2}+1$$

$$\begin{aligned}
(2x+1)(7+x)+(4x+1)(7-x) &= 45+49-x^2\\
2x^2+15x+7-4x^2+27x+7 &= 45+49-x^2\\
x^2-42x &= -80\\
x^2-42x+441 &= 361\\
x-21 &= \pm 19\\
\therefore\ x &= 40, \text{ or } 2
\end{aligned}$$

Exercise 95. Page 262.

Solve:

1. $x^2 + 2ax = 3a^2$
$x^2 + 2ax + a^2 = 4a^2$
$x + a = \pm 2a$
$\therefore\ x = a$, or $-3a$

2. $x^2 - 4ax = 12a^2$
$x^2 - 4ax + 4a^2 = 16a^2$
$x - 2a = \pm 4a$
$\therefore\ x = 6a$, or $-2a$

3. $x^2 + 8bx = 9b^2$
$x^2 + 8bx + 16b^2 = 25b^2$
$x + 4b = \pm 5b$
$\therefore\ x = b$, or $-9b$

4. $x^2 + 3bx = 10b^2$
$x^2 + 3bx + \frac{9b^2}{4} = \frac{49b^2}{4}$
$x + \frac{3b}{2} = \pm \frac{7b}{2}$
$\therefore\ x = 2b$, or $-5b$

5. $x^2 + 5ax = 14a^2$
$x^2 + 5ax + \frac{25}{4}a^2 = \frac{81}{4}a^2$
$x + \frac{5a}{2} = \pm \frac{9a}{2}$
$\therefore\ x = 2a$, or $-7a$

6. $3x^2 + 4cx = 4c^2$
$9x^2 + 12cx + 4c^2 = 16c^2$
$3x + 2c = \pm 4c$
$3x = 2c$, or $-6c$
$\therefore\ x = \frac{2c}{3}$, or $-2c$

7. $5ax - 2x^2 = 2a^2$
$2x^2 - 5ax = -2a^2$
$16x^2 - 40ax + 25a^2 = 9a^2$
$4x - 5a = \pm 3a$
$4x = 8a$, or $2a$
$\therefore\ x = 2a$, or $\frac{a}{2}$

8. $6x^2 - ax - a^2 = 0$
$36x^2 - 6ax + \frac{a^2}{4} = \frac{25a^2}{4}$
$6x - \frac{a}{2} = \pm \frac{5a}{2}$
$6x = 3a$, or $-2a$
$\therefore\ x = \frac{a}{2}$, or $-\frac{a}{3}$

9. $2a^2x^2 + ax - 1 = 0$
$16a^2x^2 + 8ax + 1 = 9$
$4ax + 1 = \pm 3$
$4ax = 2$, or -4
$\therefore\ x = \frac{1}{2a}$, or $-\frac{1}{a}$

10. $12b^2x^2 - 5bx = 3$
$144b^2x^2 - 60bx + \frac{25}{4} = \frac{169}{4}$
$12bx - \frac{5}{2} = \pm \frac{13}{2}$
$12bx = 9$, or -4
$\therefore\ x = \frac{3}{4b}$, or $-\frac{1}{3b}$

11. $\frac{2x^2}{3} + \frac{ax}{4} = 11a(x - 3a)$
$8x^2 + 3ax = 132ax - 396a^2$
$8x^2 - 129ax = -396a^2$
$16x^2 - 258ax + \frac{16641a^2}{16} = \frac{3969}{16}a^2$
$4x - \frac{129}{4}a = \pm \frac{63}{4}a$
$4x = 48a$, or $\frac{33}{2}a$
$\therefore\ x = 12a$, or $\frac{33}{8}a$

12. $\frac{3x^2}{4} + \frac{x}{2a} = \frac{3}{2a^2}$
$3a^2x^2 + 2ax = 6$
$9a^2x^2 + 6ax + 1 = 19$
$3ax + 1 = \pm\sqrt{19}$
$3ax = -1 \pm \sqrt{19}$
$\therefore\ x = \frac{-1 \pm \sqrt{19}}{3a}$

13.
$$x^2 - \frac{x}{a} = \frac{3}{4a^2}$$
$$x^2 - \frac{x}{a} + \frac{1}{4a^2} = \frac{1}{a^2}$$
$$x - \frac{1}{2a} = \pm\frac{1}{a}$$
$$\therefore\ x = \frac{3}{2a},\ \text{or}\ -\frac{1}{2a}$$

14.
$$\frac{3ax^2}{4} + \frac{2x}{3} = \frac{13}{3a}$$
$$9a^2x^2 + 8ax = 52$$
$$9a^2x^2 + 8ax + \tfrac{16}{9} = \tfrac{484}{9}$$
$$3ax + \tfrac{4}{3} = \pm\tfrac{22}{3}$$
$$3ax = 6,\ \text{or}\ -\tfrac{26}{3}$$
$$\therefore\ x = \frac{2}{a},\ \text{or}\ -\frac{26}{9a}$$

15.
$$\frac{x^2 + ax + a^2}{3} + \frac{a^2 - x^2}{4} = ax$$
$$4(x^2 + ax + a^2) + 3(a^2 - x^2) = 12ax$$
$$4x^2 + 4ax + 4a^2 + 3a^2 - 3x^2 = 12ax$$
$$x^2 - 8ax = -7a^2$$
$$x^2 - 8ax + 16a^2 = 9a^2$$
$$x - 4x = \pm 3a$$
$$\therefore\ x = 7a,\ \text{or}\ a$$

16.
$$2x^2 - x + a = 2a^2$$
$$2x^2 - x = 2a^2 - a$$
$$16x^2 - 8x + 1 = 16a^2 - 8a + 1$$
$$4x - 1 = \pm(4a - 1)$$
$$4x = 4a,\ \text{or}\ 2 - 4a$$
$$\therefore\ x = a,\ \text{or}\ \frac{1 - 2a}{2}$$

17.
$$\frac{x(a - x)}{a + x} + \frac{x}{3} = 5a$$
$$3x(a - x) + x(a + x) = 15a(a + x)$$
$$3ax - 3x^2 + ax + x^2 = 15a^2 + 15ax$$
$$2x^2 + 11ax = -15a^2$$
$$16x^2 + 88ax + 121a^2 = a^2$$
$$4x + 11a = \pm a$$
$$4x = -10a,\ \text{or}\ -12a$$
$$\therefore\ x = -\frac{5a}{2},\ \text{or}\ -3a$$

18.
$$\frac{x(3x - a)}{4x + a} = \frac{a}{12}$$
$$12x(3x - a) = a(4x + a)$$
$$36x^2 - 12ax = 4ax + a^2$$

$$36x^2 - 16ax + \tfrac{16}{9}a^2 = \tfrac{25}{9}a^2$$

$$6x - \tfrac{4}{3}a = \pm\tfrac{5}{3}a$$

$$6x = 3a, \text{ or } -\frac{a}{3}$$

$$\therefore\ x = \frac{a}{2}, \text{ or } -\frac{a}{18}$$

19.

$$x^2 + \frac{m^2 - n^2}{mn}x = 1$$

$$x^2 + \frac{m^2 - n^2}{mn}x + \frac{(m^2 - n^2)^2}{4m^2n^2} = \frac{(m^2 + n^2)^2}{4m^2n^2}$$

$$x + \frac{m^2 - n^2}{2mn} = \pm\frac{m^2 + n^2}{2mn}$$

$$\therefore\ x = \frac{2n^2}{2mn}, \text{ or } -\frac{2m^2}{2mn} = \frac{n}{m}, \text{ or } -\frac{m}{n}$$

20.

$$\frac{x + a}{b - a} + \frac{b - a}{x + a} = 2$$

$$(x + a)^2 + (b - a)^2 = 2(b - a)(x + a)$$

$$x^2 + 2ax + a^2 + b^2 - 2ab + a^2 = 2bx - 2ax + 2ab - 2a^2$$

$$x^2 + (4a - 2b)x = -4a^2 + 4ab - b^2$$

$$x^2 + (4a - 2b)x + (2a - b)^2 = 0$$

$$x + 2a - b = 0$$

$$\therefore\ x = b - 2a$$

21.

$$\frac{a + 2b}{3b - x} + \frac{a + b}{x + 2a} = 2$$

$$(a + 2b)(x + 2a) + (a + b)(3b - x) = 2(3b - x)(x + 2a)$$

$$ax + 2bx + 2a^2 + 4ab + 3ab + 3b^2 - ax - bx = -2x^2 + 6bx - 4ax + 12ab$$

$$2x^2 + (4a - 5b)x = -2a^2 + 5ab - 3b^2$$

$$16x^2 + 8(4a - 5b)x = -16a^2 + 40ab - 24b^2$$

$$16x^2 + 8(4a - 5b)x + (4a - 5b)^2 = b^2$$

$$4x + 4a - 5b = \pm b$$

$$4x = 6b - 4a, \text{ or } 4b - 4a$$

$$\therefore\ x = \frac{3b - 2a}{2}, \text{ or } b - a$$

22.

$$x^2 + (a - b)x = ab$$

$$4x^2 + 4(a - b)x + (a - b)^2 = a^2 + 2ab + b^2$$

$$2x + a - b = \pm(a + b)$$

$$2x = 2b, \text{ or } -2a$$

$$\therefore\ x = b, \text{ or } -a$$

23. $$\frac{2a+x}{2a-x}+\frac{a-2x}{a+2x}=\frac{8}{3}$$

$$3(2a+x)(a+2x)+3(a-2x)(2a-x)=8(2a-x)(a+2x)$$

$$6x^2+15ax+6a^2+6x^2-15ax+6a^2=16a^2+24ax-16x^2$$

$$28x^2-24ax=4a^2$$

$$196x^2-168ax+36a^2=64a^2$$

$$14x-6a=\pm 8a$$

$$14x=14a, \text{ or } -2a$$

$$\therefore\ x=a, \text{ or } -\frac{a}{7}$$

24. $$\frac{2x-3a}{x+4a}+\frac{3x+2a}{4x-a}=\frac{10}{7}$$

$$7(2x-3a)(4x-a)+7(3x+2a)(x+4a)=10(x+4a)(4x-a)$$

$$56x^2-98ax+21a^2+21x^2+98ax+56a^2=40x^2+150ax-40a^2$$

$$37x^2-150ax=-117a^2$$

$$4\times 37^2x^2-4\times 37\times 150ax+150^2a^2=5184a^2$$

$$74x-150a=\pm 72a$$

$$74x=222a, \text{ or } 78a$$

$$\therefore\ x=3a, \text{ or } \tfrac{39}{37}a$$

25. $$\frac{x^2}{6b^2}-\frac{5x}{6ab}+\frac{1}{a^2}=0$$

$$a^2x^2-5abx+6b^2=0$$

$$a^2x^2-5abx+\tfrac{25}{4}b^2=\tfrac{1}{4}b^2$$

$$ax-\tfrac{5}{2}b=\pm\tfrac{1}{2}b$$

$$ax=3b, \text{ or } 2b$$

$$\therefore\ x=\frac{3b}{a}, \text{ or } \frac{2b}{a}$$

26. $$\frac{a-b+x}{a+b+x}+\frac{a+b}{x+b}=2$$

$$(a-b+x)(x+b)+(a+b+x)(a+b)=2(a+b+x)(x+b)$$

$$x^2+ax+ab-b^2+a^2+2ab+b^2+ax+bx=2x^2+2ax+4bx+2ab+2b^2$$

$$x^2+3bx=a^2+ab-2b^2$$

$$x^2+3bx+\frac{9b^2}{4}=a^2+ab+\frac{b^2}{4}$$

$$x+\frac{3b}{2}=\pm\left(a+\frac{b}{2}\right)$$

$$\therefore\ x=a-b, \text{ or } -a-2b$$

27.
$$\frac{1}{a+b+x}=\frac{1}{a}+\frac{1}{b}+\frac{1}{x}$$
$$\begin{aligned}
abx &= bx(a+b+x)+ax(a+b+x)+ab(a+b+x)\\
abx &= abx+b^2x+bx^2+a^2x+abx+ax^2+a^2b+ab^2+abx\\
(a+b)x^2+(a^2+2ab+b^2)x &= -a^2b-ab^2\\
x^2+(a+b)x &= -ab\\
x^2+(a+b)x+\left(\frac{a+b}{2}\right)^2 &= \frac{a^2-2ab+b^2}{4}\\
x+\frac{a+b}{2} &= \pm\frac{a-b}{2}\\
x &= -b,\text{ or }-a
\end{aligned}$$

28.
$$\frac{(3x-a)(2x+a)}{2}+\frac{a^2-4x^2}{3}=2a^2$$
$$\begin{aligned}
3(3x-a)(2x+a)+2(a^2-4x^2) &= 12a^2\\
18x^2+3ax-3a^2+2a^2-8x^2 &= 12a^2\\
10x^2+3ax &= 13a^2\\
100x^2+30ax+\tfrac{9}{4}a^2 &= \tfrac{529}{4}a^2\\
10x+\tfrac{3}{2}a &= \pm\tfrac{23}{2}a\\
10x &= 10a,\text{ or }-13a\\
\therefore\ x &= a,\text{ or }-\tfrac{13}{10}a
\end{aligned}$$

29.
$$\begin{aligned}
9x^2-3(a+2b)x+2ab &= 0\\
9x^2-3(a+2b)x+\tfrac{1}{4}(a+2b)^2 &= \frac{a^2-4ab+4b^2}{4}\\
3x-\tfrac{1}{2}(a+2b) &= \pm\tfrac{1}{2}(a-2b)\\
3x &= a,\text{ or }2b\\
\therefore\ x &= \frac{a}{3},\text{ or }\frac{2b}{3}
\end{aligned}$$

30
$$\begin{aligned}
(2a+1)x^2+3a^2x+a^3-a^2 &= 0\\
4(2a+1)^2x^2+12a^2(2a+1)x &= 4(2a+1)(a^2-a^3)\\
4(2a+1)^2x^2+12a^2(2a+1)x+9a^4 &= 9a^4-8a^4+4a^3+4a^2\\
&= a^4+4a^3+4a^2\\
2(2a+1)x+3a^2 &= \pm(a^2+2a)\\
2(2a+1)x &= 2a-2a^2,\text{ or }-4a^2-2a\\
\therefore\ x &= \frac{a-a^2}{2a+1},\text{ or }-a.
\end{aligned}$$

31.

$$
\begin{aligned}
(1-a^2)x^2-2(1+a^2)x+1-a^2&=0\\
(1-a^2)^2x^2-2(1-a^2)(1+a^2)x&=-(1-a^2)^2\\
(1-a^2)^2x^2-2(1-a^2)(1+a^2)x+(1+a^2)^2&=4a^2\\
(1-a^2)x-(1+a^2)&=\pm 2a\\
(1-a^2)x&=1\pm 2a+a^2\\
\therefore\ x&=\frac{1\pm 2a+a^2}{1-a^2}\\
&=\frac{1+a}{1-a},\ \text{or}\ \frac{1-a}{1+a}
\end{aligned}
$$

32.

$$
\begin{aligned}
(a+b)^2x^2-(a^2-b^2)x&=ab\\
4(a+b)^2x^2-4(a^2-b^2)x&=4ab\\
4(a+b)^2x^2-4(a^2-b^2)x+(a-b)^2&=a^2+2ab+b^2\\
2(a+b)x-(a-b)&=\pm(a+b)\\
2(a+b)x&=2a,\ \text{or}\ -2b\\
\therefore\ x&=\frac{a}{a+b},\text{or}-\frac{b}{a+b}
\end{aligned}
$$

33.

$$
\begin{aligned}
a^2x^2+2bcx+a^2&=c^2x^2+2a^2x+b^2\\
(a^2-c^2)x^2+2(bc-a^2)x&=b^2-a^2\\
(a^2-c^2)^2x^2+2(a^2-c^2)(bc-a^2)x&=(a^2-c^2)(b^2-a^2)\\
(a^2-c^2)^2x^2+2(a^2-c^2)(bc-a^2)x+(bc-a^2)^2&=a^2b^2-2a^2bc+a^2c^2\\
(a^2-c^2)x+bc-a^2&=\pm(ab-ac)\\
(a^2-c^2)x&=a^2+ab-ac-bc\ \text{or}\ a^2-ab+ac-bc\\
(a^2-c^2)x&=(a-c)(a+b)\ \text{or}\ (a+c)(a-b)\\
\therefore\ x&=\frac{a+b}{a+c},\ \text{or}\ \frac{a-b}{a-c}
\end{aligned}
$$

34.

$$
\begin{aligned}
(a+b)x^2-(2a+b)x+a&=0\\
4(a+b)^2x^2-4(a+b)(2a+b)x&=-4(a+b)a\\
4(a+b)^2x^2-4(a+b)(2a+b)x+(2a+b)^2&=b^2\\
2(a+b)x-(2a+b)&=\pm b\\
2(a+b)x&=2a+2b,\ \text{or}\ 2a\\
\therefore\ x&=1,\ \text{or}\ \frac{a}{a+b}
\end{aligned}
$$

35.

$$
\frac{2x-3b}{x-2b}-\frac{3x}{x+2a}=\frac{a}{2(a-b)}
$$

$$
\begin{aligned}
&2(a-b)(2x-3b)(x+2a)-6x(a-b)(x-2b)\\
&\qquad=a(x-2b)(x+2a)\\
&4(a-b)x^2+(8a-6b)(a-b)x-12ab(a-b)-6(a-b)x^2+12b(a-b)x\\
&\qquad=ax^2+a(2a-2b)x-4a^2b
\end{aligned}
$$

$$(2b-3a)x^2+(6a+6b)(a-b)x=8a^2b-12ab^2$$

$$(2b-3a)^2x^2+6(2b-3a)(a^2-b^2)x=(2b-3a)(8a^2b-12ab^2)$$

$$(2b-3a)^2x^2+6(2b-3a)(a^2-b^2)x+9(a^2-b^2)^2$$
$$=9a^4-24a^3b+34a^2b^2-24ab^3+9b^4$$

$$(2b-3a)x+3(a^2-b^2)=\pm(3a^2-4ab+3b^2)$$

$$(2b-3a)x=6b^2-4ab,\text{ or }-6a^2+4ab$$

$$\therefore\ x=\frac{6b^2-4ab}{2b-3a},\text{ or }2a$$

36.
$$\frac{3x-2b}{x-b}-\frac{4x+2b}{x+a}=\frac{a-b}{a+b}$$

$$(a+b)(3x-2b)(x+a)-(a+b)(4x+2b)(x-b)=(a-b)(x-b)(x+a)$$

$$3(a+b)x^2+(a+b)(3a-2b)x-2ab(a+b)-4(a+b)x^2+2b(a+b)x+2b^2(a+b)$$
$$=(a-b)x^2+(a-b)^2x-ab(a-b)$$

$$-2ax^2+(2a^2+5ab-b^2)x=a^2b+ab^2-2b^3$$

$$4a^2x^2-2a(2a^2+5ab-b^2)x=-2a^3b-2a^2b^2+4ab^3$$

$$4a^2x^2-2a(2a^2+5ab-b^2)x+\frac{(2a^2+5ab-b^2)^2}{4}=\frac{4a^4+12a^3b+13a^2b^2+6ab^3+b^4}{4}$$

$$2ax-\frac{2a^2+5ab-b^2}{2}=\pm\frac{2a^2+3ab+b^2}{2}$$

$$2ax=2a^2+4ab,\text{ or }ab-b^2$$

$$\therefore\ x=a+2b,\text{ or }\frac{ab-b^2}{2a}$$

37.
$$(3a^2+b^2)(x^2-x+1)=(a^2+3b^2)(x^2+x+1)$$

$$2(a^2-b^2)x^2-4(a^2+b^2)x+2(a^2-b^2)=0$$

$$x^2-2\frac{a^2+b^2}{a^2-b^2}x=-1$$

$$x^2-2\frac{a^2+b^2}{a^2-b^2}x+\left(\frac{a^2+b^2}{a^2-b^2}\right)^2=\frac{4a^2b^2}{(a^2-b^2)^2}$$

$$x-\frac{a^2+b^2}{a^2-b^2}=\pm\frac{2ab}{a^2-b^2}$$

$$\therefore\ x=\frac{a^2\pm 2ab+b^2}{a^2-b^2}=\frac{a+b}{a-b},\text{ or }\frac{a-b}{a+b}$$

38.
$$x^2-(a+b)x+ac+bc-c^2=0$$

$$x^2-(a+b)x+\left(\frac{a+b}{2}\right)^2=\frac{a^2+2ab+b^2-4ac-4bc+4c^2}{4}$$

$$x-\frac{a+b}{2}=\pm\frac{a+b-2c}{2}$$

$$\therefore\ x=a+b-c,\text{ or }c$$

39.

$$x^2 - px = (p + q + r)(q + r)$$

$$x^2 - px + \frac{p^2}{4} = pq + pr + q^2 + 2qr + r^2 + \frac{p^2}{4}$$

$$x - \frac{p}{2} = \pm\left(\frac{p}{2} + q + r\right)$$

$$\therefore\ x = p + q + r, \text{ or } -(q + r)$$

40.

$$(a^2 + ab)(x^2 - 1) - (a^2 + b^2)x + (a + b)(a - 2b) = 0$$

$$a(a + b)x^2 - (a^2 + b^2)x = 2ab + 2b^2$$

$$ax^2 - \frac{a^2 + b^2}{a + b}x = 2b$$

$$a^2x^2 - \frac{a(a^2 + b^2)}{a + b}x + \frac{(a^2 + b^2)^2}{4(a + b)^2} = \frac{a^4 + 8a^3b + 18a^2b^2 + 8ab^3 + b^4}{4(a + b)^2}$$

$$ax - \frac{a^2 + b^2}{2(a + b)} = \pm\frac{a^2 + 4ab + b^2}{2(a + b)}$$

$$ax = -\frac{2ab}{a + b}, \text{ or } (a + b)$$

$$\therefore\ x = -\frac{2b}{a + b}, \text{ or } \frac{(a + b)}{a}$$

Exercise 96 Page 265.

Find all the roots of

1.

$$x^2 - 5x + 4 = 0$$

$$(x - 1)(x - 4) = 0$$

$$\therefore\ x = 1, \text{ or } 4$$

2.

$$6x^2 - 5x - 6 = 0$$

$$(2x - 3)(3x + 2) = 0$$

$$\therefore\ x = \tfrac{3}{2}, \text{ or } -\tfrac{2}{3}$$

3.

$$2x^2 - x - 3 = 0$$

$$(2x - 3)(x + 1) = 0$$

$$\therefore\ x = \tfrac{3}{2}, \text{ or } -1$$

4.

$$10x^2 + x - 3 = 0$$

$$(5x + 3)(2x - 1) = 0$$

$$\therefore\ x = -\tfrac{3}{5}, \text{ or } \tfrac{1}{2}$$

5.

$$x^3 + x^2 - 6x = 0$$

$$x(x^2 + x - 6) = 0$$

$$x(x + 3)(x - 2) = 0$$

$$\therefore\ x = 0, -3, \text{ or } 2$$

6.

$$x^3 - 8 = 0$$

$$(x - 2)(x^2 + 2x + 4) = 0$$

$$\therefore\ x = 2$$

or

$$x^2 + 2x + 4 = 0$$

$$x^2 + 2x + 1 = -3$$

$$x + 1 = \pm\sqrt{-3}$$

$$\therefore\ x = -1 \pm \sqrt{-3}$$

7.

$$x^3 + 8 = 0$$

$$(x + 2)(x^2 - 2x + 4) = 0$$

$$\therefore\ x = -2$$

or

$$x^2 - 2x + 4 = 0$$

$$x^2 - 2x + 1 = -3$$

$$x - 1 = \pm\sqrt{-3}$$

$$\therefore\ x = 1 \pm \sqrt{-3}$$

8.
$$x^4 - 16 = 0$$
$$(x^2 + 4)(x^2 - 4) = 0$$
$$x^2 + 4 = 0$$
$$x^2 = -4$$
$$\therefore\ x = \pm 2\sqrt{-1}$$

or

$$x^2 - 4 = 0$$
$$x^2 = 4$$
$$x = \pm 2$$
$$\therefore\ x = \pm 2, \text{ or } \pm 2\sqrt{-1}$$

9.
$$(x-1)(x-3)(x^2+5x+6) = 0$$
$$(x-1)(x-3)(x+2)(x+3) = 0$$
$$\therefore\ x = 1, 3, -2, \text{ or } -3$$

10.
$$(2x-1)(x-2)(3x^2-5x-2) = 0$$
$$(2x-1)(x-2)(3x+1)(x-2) = 0$$
$$\therefore\ x = \tfrac{1}{2}, 2, -\tfrac{1}{3}, \text{ or } 2$$

11.
$$(x^2+x-2)(2x^2+3x-5) = 0$$
$$(x-1)(x+2)(2x+5)(x-1) = 0$$
$$\therefore\ x = 1, -2, -\tfrac{5}{2}, \text{ or } 1$$

12.
$$x^3 + x^2 - 4(x+1) = 0$$
$$x^2(x+1) - 4(x+1) = 0$$
$$(x^2-4)(x+1) = 0$$
$$(x+2)(x-2)(x+1) = 0$$
$$\therefore\ x = -2, 2, \text{ or } -1$$

13.
$$3x^3 + 2x^2 - (3x+2) = 0$$
$$x^2(3x+2) - (3x+2) = 0$$
$$(x^2-1)(3x+2) = 0$$
$$(x-1)(x+1)(3x+2) = 0$$
$$\therefore\ x = 1, -1, \text{ or } -\tfrac{2}{3}$$

14.
$$x^3 - 27 - 13x + 39 = 0$$
$$x^3 - 27 - 13(x-3) = 0$$
$$(x^2+3x+9)(x-3) - 13(x-3) = 0$$
$$(x^2+3x-4)(x-3) = 0$$
$$(x-1)(x+4)(x-3) = 0$$
$$\therefore\ x = 1, -4, \text{ or } 3$$

15.
$$x^3 + 8 + 3(x^2-4) = 0$$
$$(x^2-2x+4)(x+2) + 3(x-2)(x+2) = 0$$
$$(x^2-2x+4+3x-6)(x+2) = 0$$
$$(x^2+x-2)(x+2) = 0$$
$$(x-1)(x+2)(x+2) = 0$$
$$\therefore\ x = 1, -2, \text{ or } -2$$

16.
$$\begin{aligned} x(x^2-1)-6(x-1)&=0\\ x(x+1)(x-1)-6(x-1)&=0\\ (x^2+x-6)(x-1)&=0\\ (x-2)(x+3)(x-1)&=0\\ \therefore\ x&=2,\ -3,\text{ or }1 \end{aligned}$$

17.
$$\begin{aligned} 2x^3-2x^2-(x^2-1)&=0\\ 2x^2(x-1)-(x+1)(x-1)&=0\\ (2x^2-x-1)(x-1)&=0\\ (2x+1)(x-1)(x-1)&=0\\ \therefore\ x&=-\tfrac{1}{2},\ 1,\text{ or }1 \end{aligned}$$

18.
$$\begin{aligned} x^3-3x-2&=0\\ (x+1)(x^2-x-2)&=0\\ (x+1)(x+1)(x-2)&=0\\ \therefore\ x&=-1,\ -1,\text{ or }2 \end{aligned}$$

19.
$$\begin{aligned} 2x^3+2x^2+(x^2-5x-6)&=0\\ 2x^2(x+1)+(x+1)(x-6)&=0\\ (2x^2+x-6)(x+1)&=0\\ (2x-3)(x+2)(x+1)&=0\\ \therefore\ x&=\tfrac{3}{2},\ -2,\text{ or }-1 \end{aligned}$$

20.
$$\begin{aligned} x^4-4x^3-x^2+16x-12&=0\\ (x-1)(x^3-3x^2-4x+12)&=0\\ (x-1)[x^2(x-3)-4(x-3)]&=0\\ (x-1)(x^2-4)(x-3)&=0\\ (x-1)(x+2)(x-2)(x-3)&=0\\ \therefore\ x&=1,\ -2,\ 2,\text{ or }3 \end{aligned}$$

Exercise 97. Page 266.

Solve:

1.
$$\begin{aligned} 2x^2+3x&=14\\ 2x^2+3x-14&=0\\ a=2,\ b=3,\ c&=-14\\ \therefore\ x&=\frac{-3\pm\sqrt{9+112}}{4}\\ &=\frac{-3\pm11}{4}\\ &=2,\text{ or }-\tfrac{7}{2} \end{aligned}$$

2.
$$\begin{aligned} 3x^2-5x&=12\\ 3x^2-5x-12&=0\\ a=3,\ b=-5,\ c&=-12\\ \therefore\ x&=\frac{5\pm\sqrt{25+144}}{6}\\ &=\frac{5\pm13}{6}\\ &=3,\text{ or }-\tfrac{4}{3} \end{aligned}$$

3.
$$\begin{aligned} x^2-7x&=18\\ x^2-7x-18&=0\\ a=1,\ b=-7,\ c&=-18\\ \therefore\ x&=\frac{7\pm\sqrt{49+72}}{2}\\ &=\frac{7\pm11}{2}\\ &=9,\text{ or }-2 \end{aligned}$$

4.
$$\begin{aligned} 5x^2-x&=42\\ 5x^2-x-42&=0\\ a=5,\ b=-1,\ c&=-42\\ \therefore\ x&=\frac{1\pm\sqrt{1+840}}{10}\\ &=\frac{1\pm29}{10}\\ &=3,\text{ or }-\tfrac{14}{5} \end{aligned}$$

5.
$$\begin{gathered}6x^2 - 7x = 10\\ 6x^2 - 7x - 10 = 0\\ a = 6,\ b = -7,\ c = -10\\ \therefore\ x = \frac{7 \pm \sqrt{49 + 240}}{12}\\ = \frac{7 \pm 17}{12}\\ = 2, \text{ or } -\tfrac{5}{6}\end{gathered}$$

6.
$$\begin{gathered}3x^2 - 11x = -6\\ 3x^2 - 11x + 6 = 0\\ a = 3,\ b = -11,\ c = 6\\ \therefore\ x = \frac{11 \pm \sqrt{121 - 72}}{6}\\ = \frac{11 \pm 7}{6}\\ = 3, \text{ or } \tfrac{2}{3}\end{gathered}$$

7.
$$\begin{gathered}5x^2 - 7x = -2\\ 5x^2 - 7x + 2 = 0\\ a = 5,\ b = -7,\ c = 2\\ \therefore\ x = \frac{7 \pm \sqrt{49 - 40}}{10}\\ = \frac{7 \pm 3}{10}\\ = 1, \text{ or } \tfrac{2}{5}\end{gathered}$$

8.
$$\begin{gathered}4x^2 - 9x = 28\\ 4x^2 - 9x - 28 = 0\\ a = 4,\ b = -9,\ c = -28\\ \therefore\ x = \frac{9 \pm \sqrt{81 + 448}}{8}\\ = \frac{9 \pm 23}{8}\\ = 4, \text{ or } -\tfrac{7}{4}\end{gathered}$$

9.
$$\begin{gathered}5x^2 + 7x = 12\\ 5x^2 + 7x - 12 = 0\\ a = 5,\ b = 7,\ c = -12\\ \therefore\ x = \frac{-7 \pm \sqrt{49 + 240}}{10}\\ = \frac{-7 \pm 17}{10}\\ = 1, \text{ or } -\tfrac{12}{5}\end{gathered}$$

10.
$$\begin{gathered}11x^2 - 9x = -\tfrac{10}{9}\\ 11x^2 - 9x + \tfrac{10}{9} = 0\\ a = 11,\ b = -9,\ c = \tfrac{10}{9}\\ \therefore\ x = \frac{9 \pm \sqrt{81 - \frac{440}{9}}}{22}\\ = \frac{9 \pm \frac{17}{3}}{22}\\ = \tfrac{2}{3}, \text{ or } \tfrac{5}{33}\end{gathered}$$

11.
$$\begin{gathered}7x^2 + 5x = 38\\ 7x^2 + 5x - 38 = 0\\ a = 7,\ b = 5,\ c = -38\\ \therefore\ x = \frac{-5 \pm \sqrt{25 + 1064}}{14}\\ = \frac{-5 \pm 33}{14}\\ = 2, \text{ or } -\tfrac{19}{7}\end{gathered}$$

12.
$$\begin{gathered}5x^2 - 7x = 6\\ 5x^2 - 7x - 6 = 0\\ a = 5,\ b = -7,\ c = -6\\ \therefore\ x = \frac{7 \pm \sqrt{49 + 120}}{10}\\ = \frac{7 \pm 13}{10}\\ = 2, \text{ or } -\tfrac{3}{5}\end{gathered}$$

Exercise 98. Page 268.

1. $x^4 - 5x^2 + 4 = 0$

$x^4 - 5x^2 + \frac{25}{4} = \frac{9}{4}$

$x^2 - \frac{5}{2} = \pm \frac{3}{2}$

$x^2 = 4$, or 1

$\therefore\ x = \pm 2$, or ± 1

2. $x^4 - 13x^2 + 36 = 0$

$x^4 - 13x^2 + \frac{169}{4} = \frac{25}{4}$

$x^2 - \frac{13}{2} = \pm \frac{5}{2}$

$x^2 = 9$, or 4

$\therefore\ x = \pm 3$, or ± 2

3. $x^4 - 21x^2 = 100$

$x^4 - 21x^2 + \frac{441}{4} = \frac{841}{4}$

$x^2 - \frac{21}{2} = \pm \frac{29}{2}$

$x^2 = 25$, or -4

$\therefore\ x = \pm 5$, or $\pm 2\sqrt{-1}$

4. $4x^6 - 3x^3 = 27$

$4x^6 - 3x^3 + \frac{9}{16} = \frac{441}{16}$

$2x^3 - \frac{3}{4} = \pm \frac{21}{4}$

$2x^3 = 6$, or $-\frac{9}{2}$

$x^3 = 3$, or $-\frac{9}{4}$

$\therefore\ x = \sqrt[3]{3}$, or $-\sqrt[3]{\frac{9}{4}}$

5. $2x^4 + 5x^2 = 21\frac{3}{8}$

$4x^4 + 10x^2 + \frac{25}{4} = \frac{196}{4}$

$2x^2 + \frac{5}{2} = \pm \frac{14}{2}$

$2x^2 = \frac{9}{2}$, or $-\frac{19}{2}$

$x^2 = \frac{9}{4}$, or $-\frac{19}{4}$

$\therefore\ x = \pm \frac{3}{2}$, or $\pm \frac{1}{2}\sqrt{-19}$

6. $10x^4 - 21 = x^2$

$10x^4 - x^2 = 21$

$100x^4 - 10x^2 + \frac{1}{4} = \frac{841}{4}$

$10x^2 - \frac{1}{2} = \pm \frac{29}{2}$

$10x^2 = 15$, or -14

$x^2 = \frac{3}{2}$, or $-\frac{7}{5}$

$\therefore\ x = \pm \sqrt{\frac{3}{2}}$, or $\pm \sqrt{-\frac{7}{5}}$

7. $\sqrt[3]{x^2} + 3\sqrt[3]{x} = 1\frac{3}{4}$

$x^{\frac{2}{3}} + 3x^{\frac{1}{3}} = \frac{7}{4}$

$x^{\frac{2}{3}} + 3x^{\frac{1}{3}} + \frac{9}{4} = 4$

$x^{\frac{1}{3}} + \frac{3}{2} = \pm 2$

$x^{\frac{1}{3}} = \frac{1}{2}$, or $-\frac{7}{2}$

$\therefore\ x = \frac{1}{8}$, or $-\frac{343}{8}$

8. $3\sqrt[4]{x} - 2\sqrt{x} = -20$

$2x^{\frac{1}{2}} - 3x^{\frac{1}{4}} = 20$

$4x^{\frac{1}{2}} - 6x^{\frac{1}{4}} + \frac{9}{4} = \frac{169}{4}$

$2x^{\frac{1}{4}} - \frac{3}{2} = \pm \frac{13}{2}$

$2x^{\frac{1}{4}} = 8$, or -5

$x^{\frac{1}{4}} = 4$, or $-\frac{5}{2}$

$\therefore\ x = 256$, or $\frac{625}{16}$

9. $5x^{2n} + 3x^n = 6\frac{3}{4}$

$100x^{2n} + 60x^n = 135$

$100x^{2n} + 60x^n + 9 = 144$

$10x^n + 3 = \pm 12$

$10x^n = 9$, or -15

$x^n = \frac{9}{10}$, or $-\frac{3}{2}$

$\therefore\ x = \sqrt[n]{\frac{9}{10}}$, or $\sqrt[n]{-\frac{3}{2}}$

10. $(8x + 3)^2 + (8x + 3) = 30$

$(8x + 3)^2 + (8x + 3) + \frac{1}{4} = \frac{121}{4}$

$(8x + 3) + \frac{1}{2} = \pm \frac{11}{2}$

$8x + 3 = 5$, or -6

$8x = 2$, or -9

$\therefore\ x = \frac{1}{4}$, or $-\frac{9}{8}$

11.

$$2(x^2-x+1)-\sqrt{x^2-x+1}=1$$
$$4(x^2-x+1)-2\sqrt{x^2-x+1}+\tfrac{1}{4}=\tfrac{9}{4}$$
$$2\sqrt{x^2-x+1}-\tfrac{1}{2}=\pm\tfrac{3}{2}$$
$$2\sqrt{x^2-x+1}=2, \text{ or } -1$$
$$\sqrt{x^2-x+1}=1, \text{ or } -\tfrac{1}{2}$$
$$x^2-x+1=1, \text{ or } \tfrac{1}{4}$$

if

$$x^2-x+1=1$$
$$x^2-x=0$$
$$x(x-1)=0$$
$$\therefore\ x=0, \text{ or } 1$$

if

$$x^2-x+1=\tfrac{1}{4}$$
$$x^2-x+\tfrac{1}{4}=-\tfrac{1}{2}$$
$$x-\tfrac{1}{2}=\pm\sqrt{-\tfrac{1}{2}}$$
$$\therefore\ x=\tfrac{1}{2}\pm\sqrt{-\tfrac{1}{2}}$$

12.

$$x^6-9x^3+8=0$$
$$x^6-9x^3+\tfrac{81}{4}=\tfrac{49}{4}$$
$$x^3-\tfrac{9}{2}=\pm\tfrac{7}{2}$$
$$x^3=8, \text{ or } 1$$
$$\therefore\ x=2, \text{ or } 1$$

13.

$$x^{\frac{1}{2}}+x^{-\frac{1}{2}}=\tfrac{13}{6}$$

Multiply through by $x^{\frac{1}{2}}$.

$$x+1=\tfrac{13}{6}x^{\frac{1}{2}}$$
$$x-\tfrac{13}{6}x^{\frac{1}{2}}=-1$$
$$x-\tfrac{13}{6}x^{\frac{1}{2}}+\tfrac{169}{144}=\tfrac{25}{144}$$
$$x^{\frac{1}{2}}-\tfrac{13}{12}=\pm\tfrac{5}{12}$$
$$x^{\frac{1}{2}}=\tfrac{3}{2}, \text{ or } \tfrac{2}{3}$$
$$\therefore\ x=\tfrac{9}{4}, \text{ or } \tfrac{4}{9}$$

14.

$$x+1+\sqrt{x+1}=6$$
$$x+1+\sqrt{x+1}+\tfrac{1}{4}=\tfrac{25}{4}$$
$$\sqrt{x+1}+\tfrac{1}{2}=\pm\tfrac{5}{2}$$
$$\sqrt{x+1}=2, \text{ or } -3$$
$$x+1=4, \text{ or } 9$$
$$\therefore\ x=3, \text{ or } 8$$

15.

$$x^4-13x^2=-36$$
$$x^4-13x^2+\tfrac{169}{4}=\tfrac{25}{4}$$
$$x^2-\tfrac{13}{2}=\pm\tfrac{5}{2}$$
$$x^2=9, \text{ or } 4$$
$$\therefore\ x=\pm 3, \text{ or } \pm 2$$

16.

$$2x^2+4x+9+3\sqrt{2x^2+4x+9}=40$$
$$(2x^2+4x+9)+3\sqrt{2x^2+4x+9}+\tfrac{9}{4}=\tfrac{169}{4}$$
$$\sqrt{2x^2+4x+9}+\tfrac{3}{2}=\pm\tfrac{13}{2}$$
$$\sqrt{2x^2+4x+9}=5, \text{ or } -8$$
$$2x^2+4x+9=25, \text{ or } 64$$

if

$$2x^2+4x+9=25$$
$$2x^2+4x=16$$
$$x^2+2x=8$$

$$x^2 + 2x + 1 = 9$$
$$x + 1 = \pm 3$$
$$\therefore\ x = 2, \text{ or } -4$$

if
$$2x^2 + 4x + 9 = 64$$
$$2x^2 + 4x = 55$$
$$4x^2 + 8x + 4 = 114$$
$$2x + 2 = \pm\sqrt{114}$$
$$\therefore\ x = -1 \pm \tfrac{1}{2}\sqrt{114}$$

17.
$$\frac{12x^3 - 11x^2 + 10x - 78}{8x^2 - 7x + 6} = 1\tfrac{1}{2}x - \tfrac{1}{2}$$
$$12x^3 - 11x^2 + 10x - 78 = (\tfrac{3}{2}x - \tfrac{1}{2})(8x^2 - 7x + 6)$$
$$24x^3 - 22x^2 + 20x - 156 = (3x - 1)(8x^2 - 7x + 6)$$
$$24x^3 - 22x^2 + 20x - 156 = 24x^3 - 29x^2 + 25x - 6$$
$$7x^2 - 5x - 150 = 0$$
$$49x^2 - 35x + \tfrac{25}{4} = \tfrac{4225}{4}$$
$$7x - \tfrac{5}{2} = \pm\tfrac{65}{2}$$
$$7x = 35, \text{ or } -30$$
$$\therefore\ x = 5, \text{ or } -\tfrac{30}{7}$$

Exercise 99. Page 270.

Solve:

1.
$$\sqrt{9x + 40} - 2\sqrt{x + 7} = \sqrt{x}$$
$$9x + 40 - 4\sqrt{9x + 40}\,\sqrt{x + 7} + 4x + 28 = x$$
$$4\sqrt{9x + 40}\,\sqrt{x + 7} = 12x + 68$$
$$\sqrt{9x + 40}\,\sqrt{x + 7} = 3x + 17$$
$$(9x + 40)(x + 7) = 9x^2 + 102x + 289$$
$$9x^2 + 103x + 280 = 9x^2 + 102x + 289$$
$$\therefore\ x = 9$$

2.
$$\sqrt{a + x} + \sqrt{a - x} = \sqrt{b}$$
$$a + x + 2\sqrt{a + x}\,\sqrt{a - x} + a - x = b$$
$$2\sqrt{a + x}\,\sqrt{a - x} = b - 2a$$
$$4(a^2 - x^2) = b^2 - 4ab + 4a^2$$
$$4x^2 = 4ab - b^2$$
$$x^2 = \frac{4ab - b^2}{4}$$
$$\therefore\ x = \pm\tfrac{1}{2}\sqrt{4ab - b^2}$$

3.

$$\frac{3x+\sqrt{4x-x^2}}{3x-\sqrt{4x-x^2}}=2$$

$$3x+\sqrt{4x-x^2}=6x-2\sqrt{4x-x^2}$$

$$3\sqrt{4x-x^2}=3x$$

$$\sqrt{4x-x^2}=x$$

$$4x-x^2=x^2$$

$$2x^2-4x=0$$

$$x^2-2x=0$$

$$x(x-2)=0$$

$$\therefore\ x=0,\text{ or }2$$

4.

$$\sqrt{x-3}-\sqrt{x-14}=\sqrt{4x-155}$$

$$x-3-2\sqrt{x-3}\sqrt{x-14}+x-14=4x-155$$

$$-2\sqrt{x-3}\sqrt{x-14}=2x-138$$

$$-\sqrt{x-3}\sqrt{x-14}=x-69$$

$$(x-3)(x-14)=x^2-138x+4761$$

$$x^2-17x+42=x^2-138x+4761$$

$$121x=4719$$

$$\therefore\ x=39$$

5.

$$\sqrt{x+4}-\sqrt{x}=\sqrt{x+\tfrac{3}{2}}$$

$$x+4-2\sqrt{x}\sqrt{x+4}+x=x+\tfrac{3}{2}$$

$$-2\sqrt{x}\sqrt{x+4}=-x-\tfrac{5}{2}$$

$$4x(x+4)=x^2+5x+\tfrac{25}{4}$$

$$4x^2+16x=x^2+5x+\tfrac{25}{4}$$

$$3x^2+11x=\tfrac{25}{4}$$

$$9x^2+33x+\tfrac{121}{4}=\tfrac{196}{4}$$

$$3x+\tfrac{11}{2}=\pm\tfrac{14}{2}$$

$$3x=\tfrac{3}{2},\text{ or }-\tfrac{25}{2}$$

$$\therefore\ x=\tfrac{1}{2},\text{ or }-\tfrac{25}{6}$$

6.

$$\frac{3\sqrt{x}-4}{2+\sqrt{x}}-\frac{15+3\sqrt{x}}{40+\sqrt{x}}=0$$

$$(3\sqrt{x}-4)(40+\sqrt{x})-(2+\sqrt{x})(15+3\sqrt{x})=0$$

$$116\sqrt{x}-160+3x-30-21\sqrt{x}-3x=0$$

$$95\sqrt{x}=190$$

$$\sqrt{x}=2$$

$$\therefore\ x=4$$

7.
$$\begin{aligned}
\sqrt{14x+9}+2\sqrt{x+1}+\sqrt{3x+1}&=0\\
\sqrt{14x+9}&=-2\sqrt{x+1}-\sqrt{3x+1}\\
14x+9&=4x+4+4\sqrt{x+1}\,\sqrt{3x+1}+3x+1\\
7x+4&=4\sqrt{x+1}\,\sqrt{3x+1}\\
49x^2+56x+16&=48x^2+64x+16\\
x^2-8x&=0\\
x(x-8)&=0\\
\therefore\ x&=0,\text{ or }8
\end{aligned}$$

8.
$$\begin{aligned}
\sqrt{5x+1}-2-\sqrt{x+1}&=0\\
\sqrt{5x+1}&=2+\sqrt{x+1}\\
5x+1&=4+4\sqrt{x+1}+x+1\\
4x-4&=4\sqrt{x+1}\\
x-1&=\sqrt{x+1}\\
x^2-2x+1&=x+1\\
x^2-3x&=0\\
x(x-3)&=0\\
\therefore\ x&=0,\text{ or }3
\end{aligned}$$

9.
$$\begin{aligned}
\sqrt{x-2}+\sqrt{x+3}-\sqrt{4x+1}&=0\\
\sqrt{x-2}+\sqrt{x+3}&=\sqrt{4x+1}\\
x-2+2\sqrt{x-2}\,\sqrt{x+3}+x+3&=4x+1\\
2\sqrt{x-2}\,\sqrt{x+3}&=2x\\
\sqrt{x-2}\,\sqrt{x+3}&=x\\
x^2+x-6&=x^2\\
\therefore\ x&=6
\end{aligned}$$

10.
$$\begin{aligned}
\sqrt{7-x}+\sqrt{3x+10}+\sqrt{x+3}&=0\\
\sqrt{7-x}+\sqrt{3x+10}&=-\sqrt{x+3}\\
7-x+2\sqrt{7-x}\,\sqrt{3x+10}+3x+10&=x+3\\
2\sqrt{7-x}\,\sqrt{3x+10}&=-x-14\\
4(7-x)(3x+10)&=(x+14)^2\\
-12x^2+44x+280&=x^2+28x+196\\
13x^2-16x&=84\\
169x^2-208x&=1092\\
169x^2-208x+64&=1156\\
13x-8&=\pm 34\\
13x&=42,\text{ or }-26\\
\therefore\ x&=\tfrac{42}{13},\text{ or }-2
\end{aligned}$$

11. $$3\sqrt{x^3+17}+\sqrt{x^3+1}+2\sqrt{5x^3+41}=0$$
$$3\sqrt{x^3+17}+\sqrt{x^3+1}=-2\sqrt{5x^3+41}$$
$$9(x^3+17)+6\sqrt{x^3+17}\sqrt{x^3+1}+x^3+1=4(5x^3+41)$$
$$6\sqrt{x^3+17}\sqrt{x^3+1}=10x^3+10$$
$$3\sqrt{x^3+17}\sqrt{x^3+1}=5x^3+5$$
$$9x^6+162x^3+153=25x^6+50x^3+25$$
$$16x^6-112x^3=128$$
$$x^6-7x^3=8$$
$$x^6-7x^3+\tfrac{49}{4}=\tfrac{81}{4}$$
$$x^3-\tfrac{7}{2}=\pm\tfrac{9}{2}$$
$$x^3=8, \text{ or } -1$$
$$\therefore\ x=2, \text{ or } -1$$

12. $$2x-\sqrt{2x-1}=x+2$$
$$x-2=\sqrt{2x-1}$$
$$x^2-4x+4=2x-1$$
$$x^2-6x=-5$$
$$x^2-6x+9=4$$
$$x-3=\pm 2$$
$$\therefore\ x=5, \text{ or } 1$$

13. $$\sqrt{x+2}-\sqrt{x-2}-\sqrt{2x}=0$$
$$\sqrt{x+2}-\sqrt{x-2}=\sqrt{2x}$$
$$x+2-2\sqrt{x+2}\sqrt{x-2}+x-2=2x$$
$$-2\sqrt{x+2}\sqrt{x-2}=0$$
$$(x+2)(x-2)=0$$
$$\therefore\ x=-2, \text{ or } 2$$

14. $$\frac{1}{x+\sqrt{x^2-1}}+\frac{1}{x-\sqrt{x^2-1}}=12$$
$$x-\sqrt{x^2-1}+x+\sqrt{x^2-1}=12(x+\sqrt{x^2-1})(x-\sqrt{x^2-1})$$
$$2x=12[x^2-(x^2-1)]$$
$$2x=12$$
$$\therefore\ x=6$$

15. $$\sqrt{3x}+\sqrt{3x+13}=\frac{91}{\sqrt{3x+13}}$$
$$\sqrt{3x}\sqrt{3x+13}+3x+13=91$$
$$\sqrt{3x}\sqrt{3x+13}=78-3x$$
$$3x(3x+13)=(78-3x)^2$$
$$9x^2+39x=6084-468x+9x^2$$
$$507x=6084$$
$$\therefore\ x=12$$

16.

$$
\begin{aligned}
x - \sqrt[3]{x^3 - 2x^2} - 2 &= 0\\
\sqrt[3]{x^3 - 2x^2} &= x - 2\\
x^3 - 2x^2 &= (x-2)^3\\
x^3 - 2x^2 &= x^3 - 6x^2 + 12x - 8\\
4x^2 - 12x &= -8\\
4x^2 - 12x + 9 &= 1\\
2x - 3 &= \pm 1\\
2x &= 4, \text{ or } 2\\
\therefore\ x &= 2, \text{ or } 1
\end{aligned}
$$

17.

$$
\begin{aligned}
\frac{3x - \sqrt{x^2-8}}{x - \sqrt{x^2-8}} &= x + \sqrt{x^2-8}\\
3x - \sqrt{x^2-8} &= (x - \sqrt{x^2-8})(x + \sqrt{x^2-8})\\
3x - \sqrt{x^2-8} &= 8\\
\sqrt{x^2-8} &= 3x - 8\\
x^2 - 8 &= 9x^2 - 48x + 64\\
8x^2 - 48x &= -72\\
x^2 - 6x &= -9\\
x^2 - 6x + 9 &= 0\\
x - 3 &= 0\\
\therefore\ x &= 3
\end{aligned}
$$

18.

$$
\begin{aligned}
2x^2 + 3x - 5\sqrt{2x^2+3x+9} &= -3\\
(2x^2+3x+9) - 5\sqrt{2x^2+3x+9} + \tfrac{25}{4} &= \tfrac{49}{4}\\
\sqrt{2x^2+3x+9} - \tfrac{5}{2} &= \pm\tfrac{7}{2}\\
\sqrt{2x^2+3x+9} &= 6, \text{ or } -1\\
2x^2+3x+9 &= 36, \text{ or } 1
\end{aligned}
$$

if

$$
\begin{aligned}
2x^2+3x+9 &= 36\\
2x^2+3x &= 27\\
4x^2+6x+\tfrac{9}{4} &= \tfrac{225}{4}\\
2x+\tfrac{3}{2} &= \pm\tfrac{15}{2}\\
2x &= 6, \text{ or } -9\\
\therefore\ x &= 3, \text{ or } -\tfrac{9}{2}
\end{aligned}
$$

if

$$
\begin{aligned}
2x^2+3x+9 &= 1\\
2x^2+3x &= -8\\
4x^2+6x+\tfrac{9}{4} &= -\tfrac{55}{4}\\
2x+\tfrac{3}{2} &= \pm\tfrac{1}{2}\sqrt{-55}\\
2x &= \frac{-3\pm\sqrt{-55}}{2}\\
\therefore\ x &= \frac{-3\pm\sqrt{-55}}{4}
\end{aligned}
$$

19.

$$\begin{aligned}
3x^2+15x-2\sqrt{x^2+5x+1}&=2\\
3(x^2+5x+1)-2\sqrt{x^2+5x+1}&=5\\
9(x^2+5x+1)-6\sqrt{x^2+5x+1}&=15\\
9(x^2+5x+1)-6\sqrt{x^2+5x+1}+1&=16\\
3\sqrt{x^2+5x+1}-1&=\pm 4\\
3\sqrt{x^2+5x+1}&=5,\text{ or }-3\\
\sqrt{x^2+5x+1}&=\tfrac{5}{3},\text{ or }-1
\end{aligned}$$

if

$$\begin{aligned}
\sqrt{x^2+5x+1}&=\tfrac{5}{3}\\
x^2+5x+1&=\tfrac{25}{9}\\
x^2+5x&=\tfrac{16}{9}\\
x^2+5x+\tfrac{25}{4}&=\tfrac{289}{36}\\
x+\tfrac{5}{2}&=\pm\tfrac{17}{6}\\
\therefore\ x&=\tfrac{1}{3},\text{ or }-\tfrac{16}{3}
\end{aligned}$$

if

$$\begin{aligned}
\sqrt{x^2+5x+1}&=-1\\
x^2+5x+1&=1\\
x^2+5x&=0\\
x(x+5)&=0\\
\therefore\ x&=0,\text{ or }-5
\end{aligned}$$

20.

$$\begin{aligned}
x^2-\tfrac{3}{2}x+3\sqrt{2x^2-3x+2}&=7\\
2x^2-3x+6\sqrt{2x^2-3x+2}&=14\\
2x^2-3x+2+6\sqrt{2x^2-3x+2}+9&=25\\
\sqrt{2x^2-3x+2}+3&=\pm 5\\
\sqrt{2x^2-3x+2}&=2,\text{ or }-8\\
2x^2-3x+2&=4,\text{ or }64
\end{aligned}$$

if

$$\begin{aligned}
2x^2-3x+2&=4\\
2x^2-3x&=2\\
4x^2-6x+\tfrac{9}{4}&=\tfrac{25}{4}\\
2x-\tfrac{3}{2}&=\pm\tfrac{5}{2}\\
2x&=4,\text{ or }-1\\
\therefore\ x&=2,\text{ or }-\tfrac{1}{2}
\end{aligned}$$

if

$$\begin{aligned}
2x^2-3x+2&=64\\
2x^2-3x&=62\\
4x^2-6x+\tfrac{9}{4}&=\tfrac{505}{4}\\
2x-\tfrac{3}{2}&=\pm\tfrac{1}{2}\sqrt{505}\\
2x&=\frac{3\pm\sqrt{505}}{2}\\
\therefore\ x&=\frac{3\pm\sqrt{505}}{4}
\end{aligned}$$

21.

$$\begin{aligned}
2x^2-\sqrt{x^2-2x-3}&=4x+9\\
2x^2-4x-\sqrt{x^2-2x-3}&=9\\
2(x^2-2x-3)-\sqrt{x^2-2x-3}&=3\\
4(x^2-2x-3)-2\sqrt{x^2-2x-3}+\tfrac{1}{4}&=\tfrac{25}{4}\\
2\sqrt{x^2-2x-3}-\tfrac{1}{2}&=\pm\tfrac{5}{2}\\
2\sqrt{x^2-2x-3}&=3,\text{ or }-2\\
\sqrt{x^2-2x-3}&=\tfrac{3}{2},\text{ or }-1\\
x^2-2x-3&=\tfrac{9}{4},\text{ or }1
\end{aligned}$$

if
$$\begin{aligned} x^2-2x-3&=\tfrac{9}{4}\\ x^2-2x&=\tfrac{21}{4}\\ x^2-2x+1&=\tfrac{25}{4}\\ x-1&=\pm\tfrac{5}{2}\\ \therefore\ x&=\tfrac{7}{2},\text{ or }-\tfrac{3}{2}\end{aligned}$$

if
$$\begin{aligned} x^2-2x-3&=1\\ x^2-2x&=4\\ x^2-2x+1&=5\\ x-1&=\pm\sqrt{5}\\ \therefore\ x&=1\pm\sqrt{5}\end{aligned}$$

22.
$$\begin{aligned} 3x^2-4x+\sqrt{3x^2-4x-6}&=18\\ 3x^2-4x-6+\sqrt{3x^2-4x-6}+\tfrac{1}{4}&=\tfrac{49}{4}\\ \sqrt{3x^2-4x-6}+\tfrac{1}{2}&=\pm\tfrac{7}{2}\\ \sqrt{3x^2-4x-6}&=3,\text{ or }-4\\ 3x^2-4x-6&=9,\text{ or }16\end{aligned}$$

if
$$\begin{aligned} 3x^2-4x-6&=9\\ 3x^2-4x&=15\\ 9x^2-12x+4&=49\\ 3x-2&=\pm 7\\ 3x&=9,\text{ or }-5\\ \therefore\ x&=3,\text{ or }-\tfrac{5}{3}\end{aligned}$$

if
$$\begin{aligned} 3x^2-4x-6&=16\\ 3x^2-4x&=22\\ 9x^2-12x+4&=70\\ 3x-2&=\pm\sqrt{70}\\ 3x&=2\pm\sqrt{70}\\ \therefore\ x&=\frac{2\pm\sqrt{70}}{3}\end{aligned}$$

23.
$$\begin{aligned} 3x^2-7+3\sqrt{3x^2-16x+21}&=16x\\ 3x^2-16x-7+3\sqrt{3x^2-16x+21}&=0\\ 3x^2-16x+21+3\sqrt{3x^2-16x+21}+\tfrac{9}{4}&=\tfrac{121}{4}\\ \sqrt{3x^2-16x+21}+\tfrac{3}{2}&=\pm\tfrac{11}{2}\\ \sqrt{3x^2-16x+21}&=4,\text{ or }-7\\ 3x^2-16x+21&=16,\text{ or }49\end{aligned}$$

if
$$\begin{aligned} 3x^2-16x+21&=16\\ 3x^2-16x&=-5\\ 9x^2-48x+64&=49\\ 3x-8&=\pm 7\\ 3x&=15,\text{ or }1\\ \therefore\ x&=5,\text{ or }\tfrac{1}{3}\end{aligned}$$

if
$$\begin{aligned} 3x^2-16x+21&=49\\ 3x^2-16x&=28\\ 9x^2-48x+64&=148\\ 3x-8&=\pm\sqrt{148}\\ 3x&=8\pm\sqrt{148}\\ \therefore\ x&=\frac{8\pm\sqrt{148}}{3}\\ &=\frac{8\pm 2\sqrt{37}}{3}\end{aligned}$$

Exercise 100. Page 274.

1. The sum of the squares of two consecutive integers is 761. Find the numbers.

Let $x =$ the smaller integer;
then $x + 1 =$ the greater integer.

$$(x+1)^2 + x^2 = 761$$
$$x^2 + 2x + 1 + x^2 = 761$$
$$2x^2 + 2x = 760$$
$$x^2 + x = 380$$
$$x^2 + x + \tfrac{1}{4} = \tfrac{1521}{4}$$
$$x + \tfrac{1}{2} = \pm \tfrac{39}{2}$$
$$\therefore\ x = 19, \text{ or } -20$$
$$x + 1 = 20, \text{ or } -19$$

Therefore the integers are 19 and 20.

2. The sum of the squares of two consecutive numbers exceeds the product of the numbers by 13. Find the numbers.

Let $x =$ the smaller number;
then $x + 1 =$ the greater number.

$$x^2 + (x+1)^2 = x(x+1) + 13$$
$$x^2 + x^2 + 2x + 1 = x^2 + x + 13$$
$$x^2 + x = 12$$
$$x^2 + x + \tfrac{1}{4} = \tfrac{49}{4}$$
$$x + \tfrac{1}{2} = \pm \tfrac{7}{2}$$
$$\therefore\ x = 3, \text{ or } -4$$
$$x + 1 = 4, \text{ or } -3$$

Therefore the numbers are 3 and 4.

3. The square of the sum of two consecutive even numbers exceeds the sum of their squares by 336. Find the numbers.

Let $x =$ the smaller number;
then $x + 2 =$ the greater number.

$$(2x+2)^2 = x^2 + (x+2)^2 + 336$$
$$4x^2 + 8x + 4 = x^2 + x^2 + 4x + 4 + 336$$
$$2x^2 + 4x = 336$$
$$x^2 + 2x = 168$$
$$x^2 + 2x + 1 = 169$$
$$x + 1 = \pm 13$$
$$\therefore\ x = 12, \text{ or } -14$$
$$x + 2 = 14, \text{ or } -12$$

Therefore the numbers are 12 and 14.

4. Twice the product of two consecutive numbers exceeds the sum of the numbers by 49. Find the numbers.

Let $x =$ the smaller number;
then $x + 1 =$ the greater number.

$$2(x+1)x = 2x + 1 + 49$$
$$2x^2 + 2x = 2x + 50$$
$$2x^2 = 50$$
$$x^2 = 25$$
$$\therefore\ x = \pm 5$$
$$x + 1 = 6, \text{ or } -4$$

Therefore the numbers are 5 and 6.

5. The sum of the squares of three consecutive numbers is 110. Find the numbers.

Let $x =$ the smallest number;
then $x + 1 =$ the middle number,
and $x + 2 =$ the greatest number.

$$x^2 + (x+1)^2 + (x+2)^2 = 110$$
$$x^2 + x^2 + 2x + 1 + x^2 + 4x + 4 = 110$$
$$3x^2 + 6x = 105$$
$$x^2 + 2x = 35$$
$$x^2 + 2x + 1 = 36$$
$$x + 1 = \pm 6$$
$$\therefore\ x = 5, \text{ or } -7$$
$$x + 1 = 6, \text{ or } -6$$
$$x + 2 = 7, \text{ or } -5$$

Therefore the numbers are 5, 6, and 7.

6. The difference of the cubes of two successive odd numbers is 602. Find the numbers.

Let $x =$ the smaller number;
then $x + 2 =$ the greater number.

$$(x+2)^3 - x^3 = 602$$
$$x^3 + 6x^2 + 12x + 8 - x^3 = 602$$
$$6x^2 + 12x = 594$$
$$x^2 + 2x = 99$$
$$x^2 + 2x + 1 = 100$$
$$x + 1 = \pm 10$$
$$\therefore\ x = 9, \text{ or } -11$$
$$x + 2 = 11, \text{ or } -9$$

Therefore the numbers are 9 and 11.

7. The length of a rectangular field exceeds its breadth by 2 rods. If the length and breadth of the field were each increased by 4 rods, the area would be 80 square rods. Find the dimensions of the field.

Let $x =$ the breadth of the field in rods;
then $x + 2 =$ the length of the field in rods,
$x + 4 =$ the increased breadth of the field in rods,
$x + 6 =$ the increased length of the field in rods.

$$(x+4)(x+6) = 80$$
$$x^2 + 10x + 24 = 80$$
$$x^2 + 10x + 25 = 81$$
$$x + 5 = \pm 9$$
$$\therefore\ x = 4, \text{ or } -14$$
$$x + 2 = 6, \text{ or } -12$$

Therefore the field is 4 rods wide and 6 rods long.

8. The area of a square may be doubled by increasing its length by 10 feet and its breadth by 3 feet. Determine its side.

Let $x =$ the length of a side of the square in feet;
then $x^2 =$ the area of the square in square feet,
$x + 10 =$ the increased length of the square in feet,
$x + 3 =$ the increased breadth of the square in feet,
$(x + 10)(x + 3) =$ the area of the increased square in square feet.

$$(x+10)(x+3) = 2x^2$$
$$x^2 + 13x + 30 = 2x^2$$
$$x^2 - 13x = 30$$
$$x^2 - 13x + \tfrac{169}{4} = \tfrac{289}{4}$$
$$x - \tfrac{13}{2} = \pm \tfrac{17}{2}$$
$$\therefore\ x = 15, \text{ or } -2$$

Therefore the side of the square is 15 feet long.

9. A grass plot 12 yards long and 9 yards wide has a path around it. The area of the path is $\frac{2}{3}$ of the area of the plot. Find the width of the path.

Let $x =$ the width of the path in yards;
then $2x + 12 =$ the length in yards of the plot including the path.
$2x + 9 =$ the breadth in yards of the plot including the path.
$(2x + 9)(2x + 12) =$ the area in square yards of the plot and path.

$(2x + 9)(2x + 12) - 108 =$ the area in square yards of the path.

$$\begin{aligned}
(2x + 9)(2x + 12) - 108 &= \tfrac{2}{3} \times 108 \\
4x^2 + 42x + 108 - 108 &= 72 \\
4x^2 + 42x &= 72 \\
4x^2 + 42x + \tfrac{441}{4} &= \tfrac{729}{4} \\
2x + \tfrac{21}{2} &= \pm \tfrac{27}{2} \\
2x &= 3, \text{ or } -24 \\
\therefore\ x &= \tfrac{3}{2}, \text{ or } -12
\end{aligned}$$

Therefore the path is $1\frac{1}{2}$ yards wide.

10. The perimeter of a rectangular field is 60 rods. Its area is 200 square rods. Find its dimensions.

Let $x =$ the length of the field in rods;

then $30 - x =$ the breadth of the field in rods.

$$\begin{aligned}
x(30 - x) &= 200 \\
30x - x^2 &= 200 \\
x^2 - 30x + 225 &= 25 \\
x - 15 &= \pm 5 \\
\therefore\ x &= 20, \text{ or } 10 \\
30 - x &= 10, \text{ or } 20
\end{aligned}$$

Therefore the field is 20 rods long and 10 rods wide.

11. The length of a rectangular plot is 10 rods more than twice its width, and the length of a diagonal of the plot is 25 rods. What are the dimensions of the field?

Let $x =$ the width of the plot in rods;

then $2x + 10 =$ the length of the plot in rods,

$\sqrt{(2x + 10)^2 + x^2} =$ the length of the diagonal of the plot in rods.

$$\begin{aligned}
\sqrt{(2x + 10)^2 + x^2} &= 25 \\
(2x + 10)^2 + x^2 &= 625 \\
4x^2 + 40x + 100 + x^2 &= 625 \\
5x^2 + 40x &= 525 \\
x^2 + 8x &= 105 \\
x^2 + 8x + 16 &= 121 \\
x + 4 &= \pm 11 \\
\therefore\ x &= 7, \text{ or } -15 \\
2x + 10 &= 24, \text{ or } -20
\end{aligned}$$

Therefore the plot is 24 rods long and 7 rods wide.

12. The denominator of a certain fraction exceeds the numerator by 3. If both numerator and denominator be increased by 4, the fraction will be increased by $\frac{1}{8}$. Determine the fraction.

Let $x =$ the numerator of the fraction;

then $x + 3 =$ the denominator of the fraction.

$$\frac{x}{x+3} = \text{the fraction.}$$

$$\frac{x+4}{x+7} = \text{the second fraction.}$$

$$\frac{x+4}{x+7} = \frac{x}{x+3} + \frac{1}{8}$$

$$8(x+4)(x+3) = 8x(x+7) + (x+3)(x+7)$$

$$8x^2 + 56x + 96 = 8x^2 + 56x + x^2 + 10x + 21$$

$$x^2 + 10x = 75$$

$$x^2 + 10x + 25 = 100$$

$$x + 5 = \pm 10$$

$$\therefore x = 5, \text{ or } -15$$

$$x + 3 = 8, \text{ or } -12$$

Therefore the fraction is $\frac{5}{8}$.

13. The numerator of a fraction exceeds twice the denominator by 1. If the numerator be decreased by 3, and the denominator increased by 3, the resulting fraction will be the reciprocal of the given fraction. Find the fraction.

Let $x =$ the denominator of the fraction;

then $2x + 1 =$ the numerator of the fraction.

$$\frac{2x+1}{x} = \text{the fraction.}$$

$$\frac{2x-2}{x+3} = \text{the second fraction.}$$

$$\frac{2x-2}{x+3} = \frac{x}{2x+1}$$

$$(2x-2)(2x+1) = x(x+3)$$

$$4x^2 - 2x - 2 = x^2 + 3x$$

$$3x^2 - 5x = 2$$

$$9x^2 - 15x + \frac{25}{4} = \frac{49}{4}$$

$$3x - \frac{5}{2} = \pm \frac{7}{2}$$

$$3x = 6, \text{ or } -1$$

$$\therefore x = 2, \text{ or } -\frac{1}{3}$$

$$2x + 1 = 5, \text{ or } \frac{1}{3}$$

Therefore the fraction is $\frac{5}{2}$.

14. A farmer sold a number of sheep for $120. If he had sold 5 less for the same money, he would have received $2 more per sheep. How much did he receive per sheep?

State the problem to which the negative solution applies.

Let $x =$ the number of dollars received for each sheep;

then $\frac{120}{x} =$ the number of sheep.

$\frac{120}{x} - 5 =$ the number of sheep less 5.

$\frac{120}{\frac{120}{x} - 5} =$ the number of dollars he would have received for each sheep had he sold 5 less for the same money.

$$\frac{120}{\frac{120}{x} - 5} = x + 2$$

$$\frac{120x}{120 - 5x} = x + 2$$

$$120x = (x + 2)(120 - 5x)$$

$$120x = -5x^2 + 110x + 240$$

$$5x^2 + 10x = 240$$

$$x^2 + 2x = 48$$

$$x^2 + 2x + 1 = 49$$

$$x + 1 = \pm 7$$

$$\therefore x = 6, \text{ or } -8$$

Therefore he received $6 for each sheep.

Had the problem read: "If he had sold 5 *more* for the same money he would have received $2 *less* per sheep," the solutions would have been -6 and 8, the negatives of the present solutions.

15. A merchant sold a certain number of yards of silk for $40.50. If he had sold 9 yards more for the same money, he would have received 75 cents less per yard. How many yards did he sell?

Let $x =$ the number of yards he sold;

then $\frac{4050}{x} =$ the number of cents received per yard.

$\frac{4050}{x + 9} =$ the number of cents he would have received per yard if he had sold 9 yards more at the same price.

$$\frac{4050}{x}=\frac{4050}{x+9}+75$$
$$4050(x+9)=4050x+75x(x+9)$$
$$4050\times 9=75(x^2+9x)$$
$$x^2+9x=54\times 9$$
$$x^2+9x+\tfrac{81}{4}=\tfrac{2025}{4}$$
$$x+\tfrac{9}{2}=\pm\tfrac{45}{2}$$
$$\therefore\ x=18,\text{ or }-27$$

Therefore he sold 18 yards.

16. A man bought a number of geese for $27. He sold all but 2 for $25, thus gaining 25 cents on each goose sold. How many geese did he buy?

Let $x =$ the number of geese bought;

then $\frac{2700}{x} =$ the number of cents paid for each goose.

$\frac{2500}{x-2} =$ the number of cents received for each goose sold.

$$\frac{2500}{x-2}-\frac{2700}{x}=25$$
$$2500x-2700(x-2)=25x(x-2)$$
$$2500x-2700x+5400=25x^2-50x$$
$$25x^2+150x=5400$$
$$x^2+6x=216$$
$$x^2+6x+9=225$$
$$x+3=\pm 15$$
$$\therefore\ x=12,\text{ or }-18$$

Therefore he bought 12 geese.

17. A man agrees to do a piece of work for $48. It takes him four days longer than he expected, and he finds that he has earned $1 less per day than he expected. In how many days did he expect to do the work?

Let $x =$ the number of days in which he expected to do the work;

then $\frac{48}{x} =$ the number of dollars he expected to receive per day.

$\frac{48}{x+4} =$ the number of dollars he actually receives per day.

$$\frac{48}{x}-\frac{48}{x+4}=1$$
$$48(x+4)-48x=x(x+4)$$
$$48x+192-48x=x^2+4x$$
$$x^2+4x=192$$
$$x^2+4x+4=196$$
$$x+2=\pm 14$$
$$\therefore\ x=12, \text{ or } -16$$

Therefore he expected to do the work in 12 days.

18. Find the price of eggs per dozen when 10 more in one dollar's worth lowers the price 4 cents a dozen.

Let $x=$ the price per dozen in cents;

then $12\left(\frac{100}{x}\right)=$ the number of eggs in one dollar's worth.

$\frac{100}{\frac{1200}{x}+10}=$ the cost of one egg if there were 10 more in a dollar's worth.

$\frac{1200}{\frac{1200}{x}+10}=$ the cost per dozen if there were 10 more in a dollar's worth.

$$x-\frac{1200}{\frac{1200}{x}+10}=4$$
$$x-\frac{120x}{120+x}=4$$
$$x(120+x)-120x=4(120+x)$$
$$120x+x^2-120x=480+4x$$
$$x^2-4x=480$$
$$x^2-4x+4=484$$
$$x-2=\pm 22$$
$$\therefore\ x=24, \text{ or } -20$$

Therefore the price is 24 cents per dozen.

19. A man sold a horse for $171, and gained as many per cent on the sale as the horse cost dollars. How much did the horse cost?

Let $x=$ the cost of the horse in dollars;

then $x\left(1+\frac{x}{100}\right)=$ the price in dollars for which the horse was sold.

$$x\left(1+\frac{x}{100}\right)=171$$

$$x+\frac{x^2}{100}=171$$

$$x^2+100x=17100$$

$$x^2+100x+2500=19600$$

$$x+50=\pm 140$$

$$\therefore\ x=90,\text{ or }-210$$

Therefore the horse cost $90.

20. A drover bought a certain number of sheep for $160. He kept 4, and sold the remainder for $10.60 per head, and made on his investment $\frac{3}{4}$ as many per cent as he paid dollars for each sheep bought. How many sheep did he buy?

Let $x =$ the number of sheep he bought;

then $\frac{160}{x} =$ the number of dollars paid for each sheep.

$10\frac{3}{5}(x-4) =$ the number of dollars received for the sheep sold.

$\dfrac{\frac{3}{4}\times\frac{160}{x}}{100}\times 160 =$ the number of dollars profit.

$$\therefore\ 10\tfrac{3}{5}(x-4)=160+\frac{\frac{3}{4}\times\frac{160}{x}}{100}\times 160$$

$$\tfrac{53}{5}(x-4)=160+\frac{192}{x}$$

$$53x(x-4)=800x+960$$

$$53x^2-212x=800x+960$$

$$53x^2-1012x=960$$

$$53^2x^2-53\times 1012x+506^2=306916$$

$$53x-506=\pm 554$$

$$53x=1060,\text{ or }-48$$

$$\therefore\ x=20,\text{ or }-\tfrac{48}{53}$$

Therefore he bought 20 sheep.

21. Two pipes running together can fill a cistern in $5\frac{5}{6}$ hours. The larger pipe will fill the cistern in 4 hours less time than the smaller. How long will it take each pipe running alone to fill the cistern?

Let $x =$ the number of hours in which the smaller pipe can fill the cistern;

then $x - 4 =$ the number of hours in which the larger pipe can fill the cistern.

$\frac{1}{x} + \frac{1}{x-4} =$ the part that both pipes together can fill in one hour.

But $\frac{1}{5\frac{5}{6}} = \frac{6}{35} =$ this part.

$$\therefore \frac{1}{x} + \frac{1}{x-4} = \frac{6}{35}$$

$$35(x-4) + 35x = 6x(x-4)$$

$$35x - 140 + 35x = 6x^2 - 24x$$

$$6x^2 - 94x = -140$$

$$9x^2 - 141x = -210$$

$$9x^2 - 141x + \frac{47^2}{4} = \frac{1369}{4}$$

$$3x - \tfrac{47}{2} = \pm \tfrac{37}{2}$$

$$3x = 42, \text{ or } 5$$

$$\therefore x = 14, \text{ or } \tfrac{5}{3}$$

$$x - 4 = 10, \text{ or } -\tfrac{7}{3}$$

Therefore the smaller pipe can fill the cistern in 14 hours, and the larger in 10 hours.

22. A and B can do a piece of work together in 18 days, and it takes B 15 days longer to do it alone than it does A. In how many days can each do it alone?

Let $x =$ the number of days in which A can do the work alone;

then $x + 15 =$ the number of days in which B can do the work alone.

$\frac{1}{x} + \frac{1}{x+15} =$ the part of the work A and B can do together in one day.

But $\frac{1}{18} =$ this part.

$$\therefore \frac{1}{x} + \frac{1}{x+15} = \frac{1}{18}$$

$$18(x+15) + 18x = x(x+15)$$

$$18x + 270 + 18x = x^2 + 15x$$

$$x^2 - 21x = 270$$

$$x^2 - 21x + \tfrac{441}{4} = \tfrac{1521}{4}$$

$$x - \tfrac{21}{2} = \pm \tfrac{39}{2}$$

$$\therefore x = 30, \text{ or } -9$$

$$x + 15 = 45, \text{ or } 6$$

Therefore A can do the work in 30 days, and B in 45 days.

23. A boat's crew row 4 miles down a river and back again in 1 hour and 30 minutes. Their rate in still water is 2 miles an hour faster than twice the rate of the current. Find the rate of the crew and the rate of the current.

Let $x =$ the rate of the current in miles per hour;

then $2x + 2 =$ the rate of the crew in still water.

$3x + 2 =$ their rate down stream.

$x + 2 =$ their rate up stream.

$\frac{4}{3x+2} =$ the number of hours it takes them to row 4 miles down stream.

$\frac{4}{x+2} =$ the number of hours it takes them to row back.

$$\frac{4}{3x+2} + \frac{4}{x+2} = \frac{3}{2}$$

$$8(x+2) + 8(3x+2) = 3(x+2)(3x+2)$$

$$8x + 16 + 24x + 16 = 9x^2 + 24x + 12$$

$$9x^2 - 8x = 20$$

$$9x^2 - 8x + \tfrac{16}{9} = \tfrac{196}{9}$$

$$3x - \tfrac{4}{3} = \pm \tfrac{14}{3}$$

$$3x = 6, \text{ or } -\tfrac{10}{3}$$

$$\therefore\ x = 2, \text{ or } -\tfrac{10}{9}$$

$$2x + 2 = 6$$

Therefore the rate of the current is 2 miles per hour, and the rate of the crew in still water is 6 miles per hour.

24. A number is formed by two digits. The units' digit is 2 more than the square of half the tens' digit, and if 18 be added to the number, the order of the digits will be reversed. Find the number.

Let $x =$ the tens' digit;

then $\frac{x^2}{4} + 2 =$ the units' digit.

$10x + \frac{x^2}{4} + 2 =$ the number.

$10\left(\frac{x^2}{4} + 2\right) + x =$ the number reversed.

$$10x + \frac{x^2}{4} + 2 + 18 = 10\left(\frac{x^2}{4} + 2\right) + x$$

$$9\left(\frac{x^2}{4}\right) - 9x = 0$$

$$x^2 - 4x = 0$$
$$x(x-4) = 0$$
$$\therefore\ x = 0, \text{ or } 4$$
$$\frac{x^2}{4} + 2 = 2, \text{ or } 6$$

Therefore the number is 46.

25. A circular grass plot is surrounded by a path of a uniform width of 3 feet. The area of the path is $\frac{7}{9}$ the area of the plot. Find the radius of the plot.

Let $x =$ the radius of the plot in feet;

then $x + 3 =$ the radius of the plot and path together.

$\pi x^2 =$ the area of the plot in square feet.

$\pi(x+3)^2 =$ the area of the plot and path in square feet.

$\pi(x+3)^2 - \pi x^2 =$ the area of the path in square feet.

$$\pi(x+3)^2 - \pi x^2 = \tfrac{7}{9}\pi x^2$$
$$(x+3)^2 - x^2 = \tfrac{7}{9}x^2$$
$$x^2 + 6x + 9 - x^2 = \tfrac{7}{9}x^2$$
$$\tfrac{7}{9}x^2 - 6x = 9$$
$$7x^2 - 54x = 81$$
$$49x^2 - 378x + 729 = 1296$$
$$7x - 27 = \pm 36$$
$$7x = 63, \text{ or } -9$$
$$\therefore\ x = 9, \text{ or } -\tfrac{9}{7}$$

Therefore the radius of the plot is 9 feet.

26. If a carriage wheel 11 feet round took $\frac{1}{4}$ of a second less to revolve, the rate of the carriage would be five miles more per hour. At what rate is the carriage travelling?

Let $x =$ the number of feet the carriage travels per second;

then $\frac{11}{x} =$ the number of seconds it takes the wheel to revolve once.

$\frac{11}{x} - \frac{1}{4} =$ the number of seconds it would take the wheel to revolve in the supposed case.

$\dfrac{11}{\frac{11}{x} - \frac{1}{4}} =$ the number of feet the carriage would move in one second.

$$\frac{11}{\frac{11}{x}-\frac{1}{4}}-x = \text{the number of feet gained per second.}$$

Also, 5 miles per hour $= 5 \times 5280$ feet per 3600 seconds

$= \frac{22}{3}$ feet per second.

$$\therefore \frac{11}{\frac{11}{x}-\frac{1}{4}}-x = \frac{22}{3}$$

$$\frac{44x}{44-x}-x = \frac{22}{3}$$

$$132x - 132x + 3x^2 = 22(44-x)$$

$$3x^2 = 968 - 22x$$

$$3x^2 + 22x = 968$$

$$9x^2 + 66x + 121 = 3025$$

$$3x + 11 = \pm 55$$

$$3x = 44, \text{ or } -66$$

$$\therefore x = \frac{44}{3}, \text{ or } -22$$

Therefore the carriage is travelling $\frac{44}{3}$ feet per second, or 10 miles per hour.

Exercise 101. Page 282.

1. $\begin{cases} x + y = 7 & (1) \\ xy = 10 & (2) \end{cases}$

Square (1), $x^2 + 2xy + y^2 = 49$

$4 \times (2)$ is $4xy = 40$

Subtract, $x^2 - 2xy + y^2 = 9$

Extract root, $x - y = \pm 3$

(1) is $x + y = 7$

Add, $2x = 10, \text{ or } 4$

$\therefore x = 5, \text{ or } 2$

$y = 2, \text{ or } 5.$

2. $\begin{cases} x + y = 12 & (1) \\ xy = 27 & (2) \end{cases}$

Square (1), $x^2 + 2xy + y^2 = 144$

$4 \times (2)$ is $4xy = 108$

Subtract, $x^2 - 2xy + y^2 = 36$

$$\begin{array}{ll}
\text{Extract root,} & x - y = \pm 6 \\
\text{(1) is} & x + y = 12 \\
\hline
\text{Add,} & 2x = 18, \text{ or } 6 \\
 & \therefore x = 9, \text{ or } 3 \\
 & y = 3, \text{ or } 9
\end{array}$$

3.

$$\begin{cases} x - y = 6 & (1) \\ xy = -8 & (2) \end{cases}$$

$$\begin{array}{ll}
\text{Square (1),} & x^2 - 2xy + y^2 = 36 \\
4 \times \text{(2) is} & 4xy = -32 \\
\hline
\text{Add,} & x^2 + 2xy + y^2 = 4 \\
\text{Extract root,} & x + y = \pm 2 \\
\text{(1) is} & x - y = 6 \\
\hline
\text{Add,} & 2x = 8, \text{ or } 4 \\
 & \therefore x = 4, \text{ or } 2 \\
 & y = -2, \text{ or } -4
\end{array}$$

4.

$$\begin{cases} x - y = 10 & (1) \\ xy = 11 & (2) \end{cases}$$

$$\begin{array}{ll}
\text{Square (1),} & x^2 - 2xy + y^2 = 100 \\
4 \times \text{(2) is} & 4xy = 44 \\
\hline
\text{Add,} & x^2 + 2xy + y^2 = 144 \\
\text{Extract root,} & x + y = \pm 12 \\
\text{(1) is} & x - y = 10 \\
\hline
\text{Add,} & 2x = 22, \text{ or } -2 \\
 & \therefore x = 11, \text{ or } -1 \\
 & y = 1, \text{ or } -11
\end{array}$$

5.

$$\begin{cases} x + y = 12 & (1) \\ x^2 + y^2 = 80 & (2) \end{cases}$$

$$\begin{array}{lll}
\text{Square (1),} & x^2 + 2xy + y^2 = 144 & \\
\text{(2) is} & x^2 + y^2 = 80 & \\
\hline
\text{Subtract,} & 2xy = 64 & (3) \\
\text{Subtract (3) from (2),} & x^2 - 2xy + y^2 = 16 & \\
\text{Extract root,} & x - y = \pm 4 & \\
\text{(1) is} & x + y = 12 & \\
\hline
\text{Add,} & 2x = 16, \text{ or } 8 & \\
 & \therefore x = 8, \text{ or } 4 & \\
 & y = 4, \text{ or } 8 &
\end{array}$$

6.
$$\begin{cases} x + y = 3 & (1) \\ x^2 + y^2 = 29 & (2) \end{cases}$$

Square (1), $x^2 + 2xy + y^2 = 9$

(2) is $x^2 + y^2 = 29$

Subtract, $2xy = -20$ (3)

Subtract (3) from (2), $x^2 - 2xy + y^2 = 49$

Extract root, $x - y = \pm 7$

(1) is $x + y = 3$

Add, $2x = 10, \text{ or } -4$

$\therefore x = 5, \text{ or } -2$

$y = -2, \text{ or } 5$

7.
$$\begin{cases} x - y = 9 & (1) \\ x^2 + y^2 = 45 & (2) \end{cases}$$

Square (1), $x^2 - 2xy + y^2 = 81$

(2) is $x^2 + y^2 = 45$

Subtract, $-2xy = 36$ (3)

Subtract (3) from (2), $x^2 + 2xy + y^2 = 9$

Extract root, $x + y = \pm 3$

(1) is $x - y = 9$

$2x = 12, \text{ or } 6$

$\therefore x = 6, \text{ or } 3$

$y = -3, \text{ or } -6$

8.
$$\begin{cases} x + 2y = 7 & (1) \\ x^2 + y^2 = 10 & (2) \end{cases}$$

From (1), $x = 7 - 2y$

Substitute in (2), $(7 - 2y)^2 + y^2 = 10$

$49 - 28y + 5y^2 = 10$

$5y^2 - 28y + 39 = 0$

$(5y - 13)(y - 3) = 0$

$\therefore y = 3, \text{ or } \frac{13}{5}$

$x = 1, \text{ or } \frac{9}{5}$

9.
$$\begin{cases} 3x - y = 12 & (1) \\ x^2 - y^2 = 16 & (2) \end{cases}$$

From (1), $y = 3x - 12$

Substitute in (2), $x^2 - (3x - 12)^2 = 16$

$$\begin{aligned}
x^2-9x^2+72x-144&=16\\
8x^2-72x+160&=0\\
x^2-9x+20&=0\\
(x-4)(x-5)&=0\\
\therefore\ x&=4,\text{ or }5\\
y&=0,\text{ or }3
\end{aligned}$$

10.

$$\begin{cases} y=3x+1 & (1)\\ x^2+xy=33 & (2)\end{cases}$$

Substitute the value of y from (1) in (2),

$$\begin{aligned}
x^2+x(3x+1)&=33\\
4x^2+x-33&=0\\
(x+3)(4x-11)&=0\\
\therefore\ x&=-3,\text{ or }\tfrac{11}{4}\\
y&=-8,\text{ or }\tfrac{37}{4}
\end{aligned}$$

11.

$$\begin{cases} 5x-4y=10 & (1)\\ 3x^2-4y^2=\ \ 8 & (2)\end{cases}$$

From (1), $y=\frac{5}{4}(x-2)$

Substitute in (2), $3x^2-4\left[\frac{25}{16}(x-2)^2\right]=8$

$$\begin{aligned}
3x^2-\tfrac{25}{4}(x^2-4x+4)&=8\\
-13x^2+100x-132&=0\\
13x^2-100x+132&=0\\
(13x-22)(x-6)&=0\\
\therefore\ x&=6,\text{ or }\tfrac{22}{13}\\
y&=\tfrac{5}{4}(x-2)=5,\text{ or }-\tfrac{5}{13}
\end{aligned}$$

12.

$$\begin{cases} x+7y=23 & (1)\\ xy=\ \ 6 & (2)\end{cases}$$

From (1), $x=23-7y$

Substitute in (2), $(23-7y)y=6$

$$\begin{aligned}
23y-7y^2&=6\\
7y^2-23y+6&=0\\
(7y-2)(y-3)&=0\\
\therefore\ y&=3,\text{ or }\tfrac{2}{7}\\
x&=\frac{6}{y}=2,\text{ or }21
\end{aligned}$$

13.

$$\begin{cases} 2x-3y=\ \ \ 2 & (1)\\ x^2-2xy=-7 & (2)\end{cases}$$

From (1), $y=\frac{2}{3}(x-1)$

Substitute in (2), $x^2 - \frac{4x}{3}(x-1) = -7$

$$3x^2 - 4x^2 + 4x = -21$$
$$x^2 - 4x - 21 = 0$$
$$(x+3)(x-7) = 0$$
$$\therefore\ x = -3, \text{ or } 7$$
$$y = \tfrac{2}{3}(x-1) = -\tfrac{8}{3}, \text{ or } 4$$

14. $\begin{cases} 2x - 3y = 1 & (1) \\ 3x^2 - 4xy = 32 & (2) \end{cases}$

From (1), $y = \frac{2x-1}{3}$

Substitute in (2), $3x^2 - \frac{4x(2x-1)}{3} = 32$

$$9x^2 - 8x^2 + 4x = 96$$
$$x^2 + 4x - 96 = 0$$
$$(x-8)(x+12) = 0$$
$$\therefore\ x = 8, \text{ or } -12$$
$$y = \frac{2x-1}{3} = 5, \text{ or } -\tfrac{25}{3}$$

15. $\begin{cases} x^2 - xy + y^2 = 21 & (1) \\ x + y = 9 & (2) \end{cases}$

Square (2), $x^2 + 2xy + y^2 = 81$

(1) is $x^2 - xy + y^2 = 21$

Subtract, $3xy = 60$

$$xy = 20 \qquad (3)$$

Subtract (3) from (1), $x^2 - 2xy + y^2 = 1$

Extract root, $x - y = \pm 1$

(2) is $x + y = 9$

Add, $2x = 10, \text{ or } 8$

$$\therefore\ x = 5, \text{ or } 4$$
$$y = 4, \text{ or } 5$$

16. $\begin{cases} x^2 - 3xy + 2y^2 = 0 & (1) \\ 2x + 3y = 7 & (2) \end{cases}$

Factor (1), $(x-y)(x-2y) = 0$

$$\therefore\ x = y, \text{ or } 2y$$

Substitute in (2), $2y + 3y = 7,$

or $4y + 3y = 7$

$$\therefore\ y = \tfrac{7}{5}, \text{ or } 1$$
$$x = \tfrac{7}{5}, \text{ or } 2$$

17.
$$\begin{cases} x^2 - y^2 = 9 & (1) \\ x - y = 1 & (2) \end{cases}$$

Divide (1) by (2),

$$x + y = 9$$

(2) is $x - y = 1$

Add, $2x = 10$

$$\therefore x = 5$$
$$y = 4$$

18.
$$\begin{cases} x^2+3y+17 = 0 & (1) \\ 3x - y = 3 & (2) \end{cases}$$

From (2), $y = 3x - 3$

Substitute in (1),

$$x^2 + 9x - 9 + 17 = 0$$
$$x^2 + 9x + 8 = 0$$
$$(x+1)(x+8) = 0$$
$$\therefore x = -1, \text{ or } -8$$
$$y = 3x - 3$$
$$= -6, \text{or } -27$$

19.
$$\begin{cases} \dfrac{1}{x} + \dfrac{1}{y} = 5 & (1) \\ \dfrac{1}{x^2} - \dfrac{1}{y^2} = 5 & (2) \end{cases}$$

Divide (2) by (1),

$$\frac{1}{x} - \frac{1}{y} = 1$$

(1) is $\dfrac{1}{x} + \dfrac{1}{y} = 5$

Add, $\dfrac{2}{x} = 6$

$$\therefore x = \tfrac{1}{3}$$
$$y = \tfrac{1}{2}$$

20.
$$\begin{cases} \dfrac{1}{x} + \dfrac{1}{y} = \dfrac{5}{6} & (1) \\ \dfrac{1}{x^2} + \dfrac{1}{y^2} = \dfrac{13}{36} & (2) \end{cases}$$

Square (1), $\dfrac{1}{x^2} + \dfrac{2}{xy} + \dfrac{1}{y^2} = \dfrac{25}{36}$

(2) is $\dfrac{1}{x^2} + \dfrac{1}{y^2} = \dfrac{13}{36}$

Subtract, $\dfrac{2}{xy} = \dfrac{12}{36}$ (3)

Subtract (3) from (2), $\dfrac{1}{x^2} - \dfrac{2}{xy} + \dfrac{1}{y^2} = \dfrac{1}{36}$

Extract root, $\dfrac{1}{x} - \dfrac{1}{y} = \pm\dfrac{1}{6}$

(1) is $\dfrac{1}{x} + \dfrac{1}{y} = \dfrac{5}{6}$

Add, $\dfrac{2}{x} = 1, \text{ or } \tfrac{2}{3}$

$$\therefore x = 2, \text{ or } 3$$
$$y = 3, \text{ or } 2$$

21. $$\left\{\begin{aligned} \frac{2}{x}+\frac{3}{y}&=2 \qquad (1)\\ xy&=6 \qquad (2)\end{aligned}\right.$$

From (2), $$y=\frac{6}{x}$$

Substitute in (1), $$\frac{2}{x}+\frac{x}{2}=2$$

$$4+x^2=4x$$

$$x^2-4x+4=0$$

Extract root, $$x-2=0$$

$$\therefore x=2$$

$$y=\frac{6}{x}=3$$

22. $$\left\{\begin{aligned} \frac{1}{x}+\frac{1}{y}&=11 \qquad (1)\\ \frac{1}{x^2}+\frac{1}{y^2}&=61 \qquad (2)\end{aligned}\right.$$

Square (1), $$\frac{1}{x^2}+\frac{2}{xy}+\frac{1}{y^2}=121$$

(2) is $$\frac{1}{x^2}\qquad+\frac{1}{y^2}=\ 61$$

Subtract, $$\frac{2}{xy}\qquad=\ 60 \qquad (3)$$

Subtract (3) from (2), $$\frac{1}{x^2}-\frac{2}{xy}+\frac{1}{y^2}=1$$

Extract root, $$\frac{1}{x}-\frac{1}{y}=\pm 1$$

(1) is $$\frac{1}{x}+\frac{1}{y}=\ 11$$

Add, $$\frac{2}{x}\qquad=\ 12, \text{ or } 10$$

$$\therefore x=\tfrac{1}{6}, \text{ or } \tfrac{1}{5}$$
$$y=\tfrac{1}{5}, \text{ or } \tfrac{1}{6}$$

23. $$\left\{\begin{aligned} \frac{6}{x}+\frac{8}{y}&=4 \qquad (1)\\ xy&=16 \qquad (2)\end{aligned}\right.$$

From (2), $$y=\frac{16}{x}$$

Substitute in (1), $\frac{6}{x}+\frac{8x}{16}=4$

$12+x^2=8x$

$x^2-8x=-12$

$x^2-8x+16=4$

Extract root, $x-4=\pm 2$

$\therefore x=6, \text{ or } 2$

$y=\frac{8}{3}, \text{ or } 8$

24. $\begin{cases} x^3+y^3=35 & (1) \\ x+y=5 & (2) \end{cases}$

Divide (1) by (2), $x^2-xy+y^2=7$ (3)

Square (2), $x^2+2xy+y^2=25$

Subtract, $3xy=18$

$xy=6$ (4)

Subtract (4) from (3), $x^2-2xy+y^2=1$

Extract root, $x-y=\pm 1$

(2) is $x+y=5$

Add, $2x=6, \text{ or } 4$

$\therefore x=3, \text{ or } 2$

$y=2, \text{ or } 3$

25. $\begin{cases} x^3-y^3=61 & (1) \\ x-y=1 & (2) \end{cases}$

Divide (1) by (2), $x^2+xy+y^2=61$ (3)

Square (2), $x^2-2xy+y^2=1$

Subtract, $3xy=60$

$xy=20$ (4)

Add (4) to (3), $x^2+2xy+y^2=81$

Extract root, $x+y=\pm 9$

(2) is $x-y=1$

Add, $2x=10, \text{ or } -8$

$\therefore x=5, \text{ or } -4$

$y=4, \text{ or } -5$

26. $\begin{cases} x^3 + y^3 = 65 & (1) \\ x + y = 5 & (2) \end{cases}$

Divide (1) by (2), $x^2 - xy + y^2 = 13$ (3)

Square (2), $x^2 + 2xy + y^2 = 25$

Subtract, $3xy = 12$

$xy = 4$ (4)

Subtract (4) from (3), $x^2 - 2xy + y^2 = 9$

Extract root, $x - y = \pm 3$

(2) is $x + y = 5$

Add, $2x = 8, \text{ or } 2$

$\therefore\ x = 4, \text{ or } 1$

$y = 1, \text{ or } 4$

27. $\begin{cases} x^2y + xy^2 = 120 & (1) \\ x + y = 8 & (2) \end{cases}$

Divide (1) by (2), $xy = 15$ (3)

Square (2), $x^2 + 2xy + y^2 = 64$

4 × (3) is $4xy = 60$

Subtract, $x^2 - 2xy + y^2 = 4$

Extract root, $x - y = \pm 2$

(2) is $x + y = 8$

Add, $2x = 10, \text{ or } 6$

$\therefore\ x = 5, \text{ or } 3$

$y = 3, \text{ or } 5$

28. $\begin{cases} x^3 - y^3 = \frac{7}{64} & (1) \\ x - y = \frac{1}{4} & (2) \end{cases}$

Divide (1) by (2), $x^2 + xy + y^2 = \frac{7}{16}$ (3)

Square (2), $x^2 - 2xy + y^2 = \frac{1}{16}$

Subtract, $3xy = \frac{3}{8}$

$xy = \frac{1}{8}$ (4)

Add (4) to (3), $x^2 + 2xy + y^2 = \frac{9}{16}$

Extract root, $x + y = \pm \frac{3}{4}$

(2) is $x - y = \frac{1}{4}$

Add, $2x = 1, \text{ or } -\frac{1}{2}$

$\therefore\ x = \frac{1}{2}, \text{ or } -\frac{1}{4}$

$y = \frac{1}{4}, \text{ or } -\frac{1}{2}$

29. $$\left\{\begin{array}{r} x^3 + y^3 = 126 \qquad (1) \\ x^2 - xy + y^2 = 21 \qquad (2) \end{array}\right.$$

Divide (1) by (2), $x + y = 6$ (3)

Square (3), $x^2 + 2xy + y^2 = 36$

(2) is $x^2 - xy + y^2 = 21$

Subtract, $3xy = 15$

$xy = 5$ (4)

Subtract (4) from (2), $x^2 - 2xy + y^2 = 16$

Extract root, $x - y = \pm 4$

(3) is $x + y = 6$

Add, $2x = 10$, or 2

$\therefore x = 5$, or 1

$y = 1$, or 5

30. $$\left\{\begin{array}{r} x^3 - y^3 = 56 \qquad (1) \\ x^2 + xy + y^2 = 28 \qquad (2) \end{array}\right.$$

Divide (1) by (2), $x - y = 2$ (3)

Square (3), $x^2 - 2xy + y^2 = 4$

(2) is $x^2 + xy + y^2 = 28$

$3xy = 24$

$xy = 8$ (4)

Add (4) to (2), $x^2 + 2xy + y^2 = 36$

Extract root, $x + y = \pm 6$

(3) is $x - y = 2$

Add, $2x = 8$, or -4

$\therefore x = 4$, or -2

$y = 2$, or -4

31. $$\left\{\begin{array}{r} \dfrac{x^2}{y} + \dfrac{y^2}{x} = \dfrac{35}{3} \qquad (1) \\[2ex] \dfrac{1}{x} + \dfrac{1}{y} = \dfrac{5}{12} \qquad (2) \end{array}\right.$$

From (1), $\dfrac{x^3 + y^3}{xy} = \dfrac{35}{3}$ (3)

From (2), $\dfrac{x + y}{xy} = \dfrac{5}{12}$ (4)

Divide (3) by (4), $x^2 - xy + y^2 = 28$ (5)

From (4), $x + y = \tfrac{5}{12}xy$

Square, $x^2 + 2xy + y^2 = \frac{25}{144}x^2y^2$

(5) is $x^2 - xy + y^2 = 28$

Subtract, $3xy = \frac{25}{144}x^2y^2 - 28$

$$\frac{25x^2y^2}{144} - 3xy = 28$$

$$\frac{25x^2y^2}{144} - 3xy + \left(\frac{18}{5}\right)^2 = \frac{1024}{25}$$

Extract root, $\frac{5xy}{12} - \frac{18}{5} = \pm\frac{32}{5}$

$$\frac{5xy}{12} = 10, \text{ or } -\frac{14}{5}$$

$$\therefore\ xy = 24, \text{ or } -\tfrac{168}{25}$$

Substitute the value of xy in (4), $x + y = 10, \text{ or } -\frac{14}{5}$ (6)

Subtract the value of xy from (5),

$$x^2 - 2xy + y^2 = 4, \text{ or } \tfrac{868}{25}$$

Extract root, $x - y = \pm 2, \text{ or } \pm\frac{\sqrt{868}}{5}$

$$x - y = \pm 2, \text{ or } \pm\tfrac{2}{5}\sqrt{217}$$

(6) is $x + y = 10, \text{ or } -\frac{14}{5}$

Add, $2x = 12, 8, \text{ or } \frac{-14 \pm 2\sqrt{217}}{5}$

$$\therefore\ x = 6, 4, \text{ or } \frac{-7 \pm \sqrt{217}}{5}$$

$$y = 4, 6, \text{ or } \frac{-7 \mp \sqrt{217}}{5}$$

32. $$\begin{cases} \dfrac{x^2}{y} - \dfrac{y^2}{x} = \dfrac{19}{2} & (1) \\[2ex] \dfrac{1}{y} - \dfrac{1}{x} = \dfrac{1}{18} & (2) \end{cases}$$

From (1), $x^3 - y^3 = \frac{19}{2}xy$ (3)

From (2), $x - y = \frac{1}{18}xy$ (4)

Divide (3) by (4), $x^2 + xy + y^2 = 171$ (5)

Square (4), $x^2 - 2xy + y^2 = \frac{1}{324}x^2y^2$

Subtract, $3xy = 171 - \frac{1}{324}x^2y^2$

$$\frac{x^2y^2}{324} + 3xy + 729 = 900$$

Extract root, $\frac{xy}{18} + 27 = \pm 30$

$\frac{xy}{18} = 3, \text{ or } -57$

$xy = 54, \text{ or } -1026$

Substitute in (4), $x - y = 3, \text{ or } -57$ (6)

Add the value of xy to (5),

Extract root, $x^2 + 2xy + y^2 = 225, \text{ or } -855$

$x + y = \pm 15, \text{ or } \pm \sqrt{-855}$

(6) is $x - y = 3, \text{ or } -57$

Add, $2x = 18, -12, \text{ or } -57 \pm \sqrt{-855}$

$\therefore\ x = 9, -6, \text{ or } \frac{-57 \pm \sqrt{-855}}{2}$

$y = 6, -9, \text{ or } \frac{-57 \mp \sqrt{-855}}{2}$

33. $\begin{cases} x^2 + xy = 24 & (1) \\ xy + y^2 = 40 & (2) \end{cases}$

Add, $x^2 + 2xy + y^2 = 64$

Extract root, $x + y = \pm 8$ (3)

Divide (1) by (3), $\therefore\ x = \pm 3$

Divide (2) by (3), $y = \pm 5$

34. $\begin{cases} x^2 - xy = 8 & (1) \\ xy - y^2 = 7 & (2) \end{cases}$

Subtract, $x^2 - 2xy + y^2 = 1$

Extract root, $x - y = \pm 1$ (3)

Divide (1) by (3), $\therefore\ x = \pm 8$

Divide (2) by (3), $y = \pm 7$

35. $\begin{cases} x^2 + 2xy = 24 & (1) \\ 2xy + 4y^2 = 120 & (2) \end{cases}$

Add, $x^2 + 4xy + 4y^2 = 144$

Extract root, $x + 2y = \pm 12$ (3)

Divide (1) by (3), $\therefore\ x = \pm 2$

Divide (2) by (3), $2y = \pm 10$

$y = \pm 5$

36. $$\begin{cases} 4x^2 + 5xy = 14 & (1) \\ 7xy + 9y^2 = 50 & (2) \end{cases}$$

Add, $$4x^2 + 12xy + 9y^2 = 64$$

Extract root, $$2x + 3y = \pm 8$$

$$\therefore x = \frac{\pm 8 - 3y}{2}$$

Substitute in (2), $$\frac{7(\pm 8 - 3y)y}{2} + 9y^2 = 50$$

$$\frac{\pm 56y - 21y^2}{2} + 9y^2 = 50$$

$$-\tfrac{3}{2}y^2 \pm 28y = 50$$

$$9y^2 \mp 168y + 784 = 484$$

Extract root, $$3y \mp 28 = \pm 22$$

$$\therefore y = \pm 2, \pm \tfrac{50}{3}$$

$$x = \pm 1, \mp 21$$

37. $$\begin{cases} x^2 + xy + y^2 = 39 & (1) \\ 2x^2 + 3xy + y^2 = 63 & (2) \end{cases}$$

Let $$y = vx$$

From (1), $$x^2 + vx^2 + v^2x^2 = 39 \qquad (3)$$

From (2), $$2x^2 + 3vx^2 + v^2x^2 = 63 \qquad (4)$$

Divide (3) by (4), $$\frac{1 + v + v^2}{2 + 3v + v^2} = \frac{13}{21}$$

$$21(1 + v + v^2) = 13(2 + 3v + v^2)$$

Reduce, $$8v^2 - 18v = 5$$

$$16v^2 - 36v + \tfrac{81}{4} = \tfrac{121}{4}$$

$$4v - \tfrac{9}{2} = \pm \tfrac{11}{2}$$

$$\therefore v = \tfrac{5}{2}, \text{ or } -\tfrac{1}{4}$$

Substitute these values in (3)

$$x^2(1 + \tfrac{5}{2} + \tfrac{25}{4}) = 39$$

$$x^2 = 4$$

$$\therefore x = \pm 2$$

$$y = \pm 5$$

or $$x^2(1 - \tfrac{1}{4} + \tfrac{1}{16}) = 39$$

$$x^2 = 48$$

$$\therefore x = \pm 4\sqrt{3}$$

$$y = \mp \sqrt{3}$$

38. $\begin{cases} x^2 + 3y^2 = 52 & (1) \\ xy + 2y^2 = 40 & (2) \end{cases}$

Let $y = vx$

From (1),

$x^2(1 + 3v^2) = 52 \quad (3)$

From (2),

$x^2(v + 2v^2) = 40 \quad (4)$

Divide (3) by (4),

$$\frac{1 + 3v^2}{v + 2v^2} = \frac{13}{10}$$

$$10 + 30v^2 = 13v + 26v^2$$

$$4v^2 - 13v = -10$$

$$4v^2 - 13v + \tfrac{169}{16} = \tfrac{9}{16}$$

$$2v - \tfrac{13}{4} = \pm\tfrac{3}{4}$$

$$2v = 4, \text{ or } \tfrac{5}{2}$$

$$v = 2, \text{ or } \tfrac{5}{4}$$

Substitute these values in (3),

$$x^2(1 + 12) = 52$$

$$\therefore\ x = \pm 2$$

$$y = \pm 4$$

or $x^2(1 + \tfrac{75}{16}) = 52$

$$\therefore\ x = \pm\frac{8}{\sqrt{7}}$$

$$y = \pm\frac{10}{\sqrt{7}}$$

39. $\begin{cases} 2x^2 - y^2 = 46 & (1) \\ xy + y^2 = 14 & (2) \end{cases}$

Let $y = vx$

From (1),

$x^2(2 - v^2) = 46 \quad (3)$

From (2),

$x^2(v + v^2) = 14 \quad (4)$

Divide (3) by (4),

$$\frac{2 - v^2}{v + v^2} = \frac{23}{7}$$

$$14 - 7v^2 = 23v + 23v^2$$

$$30v^2 + 23v = 14$$

$$900v^2 + 690v + \tfrac{529}{4} = \tfrac{2209}{4}$$

$$30v + \tfrac{23}{2} = \pm\tfrac{47}{2}$$

$$\therefore\ v = -\tfrac{7}{6}, \text{ or } \tfrac{2}{5}$$

Substitute these values in (3),

$$x^2(2 - \tfrac{49}{36}) = 46$$

$$\therefore\ x = \pm 6\sqrt{2}$$

$$y = \mp 7\sqrt{2}$$

or $x^2(2 - \tfrac{4}{25}) = 46$

$$\therefore\ x = \pm 5$$

$$y = \pm 2$$

40. $\begin{cases} x^2 + xy + 2y^2 = 44 & (1) \\ 2x^2 - 3xy + 2y^2 = 16 & (2) \end{cases}$

Let $y = vx$

From (1), $x^2(1 + v + 2v^2) = 44 \quad (3)$

From (2), $x^2(2 - 3v + 2v^2) = 16 \quad (4)$

Divide (3) by (4), $\dfrac{1 + v + 2v^2}{2 - 3v + 2v^2} = \dfrac{11}{4}$

$$4 + 4v + 8v^2 = 22 - 33v + 22v^2$$

$$14v^2 - 37v + 18 = 0$$

$$(v - 2)(14v - 9) = 0$$

$$\therefore\ v = 2, \text{ or } \tfrac{9}{14}$$

Substitute these values in (3), $x^2(1 + 2 + 8) = 44$

$$\therefore\ x = \pm 2$$

$$y = \pm 4$$

or $$x^2(1 + \tfrac{9}{14} + \tfrac{81}{98}) = 44$$

$$\therefore\ x = \pm \frac{14}{\sqrt{11}}$$

$$y = \pm \frac{9}{\sqrt{11}}$$

41. $$\begin{cases} x^2 + 3y^2 = 31 & (1) \\ 4xy + y^2 = 33 & (2) \end{cases}$$

Let $$y = vx$$

From (1), $$x^2(1 + 3v^2) = 31 \qquad (3)$$

From (2), $$x^2(4v + v^2) = 33 \qquad (4)$$

Divide (3) by (4), $$\frac{1 + 3v^2}{4v + v^2} = \frac{31}{33}$$

$$33 + 99v^2 = 124v + 31v^2$$

$$68v^2 - 124v + 33 = 0$$

$$(2v - 3)(34v - 11) = 0$$

$$\therefore\ v = \tfrac{3}{2}, \text{ or } \tfrac{11}{34}$$

Substitute these values in (3),

$$x^2(1 + \tfrac{27}{4}) = 31$$

$$\therefore\ x = \pm 2$$

$$y = \pm 3$$

or $$x^2(1 + \tfrac{363}{1156}) = 31$$

$$\therefore\ x = \pm \tfrac{34}{7}$$

$$y = \pm \tfrac{11}{7}$$

42. $$\begin{cases} 3x^2 + 7xy = 82 & (1) \\ x^2 + 5xy + 9y^2 = 279 & (2) \end{cases}$$

Add, $$4x^2 + 12xy + 9y^2 = 361$$

Extract root, $$2x + 3y = \pm 19$$

$$y = \frac{\pm 19 - 2x}{3}$$

Substitute in (1), $$3x^2 + \frac{7x(\pm 19 - 2x)}{3} = 82$$

$$9x^2 \pm 133x - 14x^2 = 246$$

$$5x^2 \mp 133x + 246 = 0$$

$$(5x \mp 123)(x \mp 2) = 0$$

$$\therefore\ x = \pm 2, \text{ or } \pm \tfrac{123}{5}$$

$$y = \pm 5, \text{ or } \mp \tfrac{151}{15}$$

43.
$$\begin{cases} x^4 + y^4 = 97 & (1) \\ x + y = 5 & (2) \end{cases}$$

Raise (2) to the fourth power,

$$x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 = 625$$

(1) is
$$x^4 \qquad\qquad\qquad\qquad\qquad + y^4 = 97$$

Subtract,
$$4x^3y + 6x^2y^2 + 4xy^3 \qquad = 528$$

$$2x^3y + 3x^2y^2 + 2xy^3 = 264$$

$$xy(2x^2 + 3xy + 2y^2) = 264 \tag{3}$$

Square (2),
$$x^2 + 2xy + y^2 = 25$$

Hence,
$$x^2 + y^2 = 25 - 2xy$$

$$2(x^2 + y^2) = 50 - 4xy$$

$$2x^2 + 3xy + 2y^2 = 50 - xy$$

Substitute in (3),
$$xy(50 - xy) = 264$$

$$x^2y^2 - 50xy + 625 = 361$$

Extract root,
$$xy - 25 = \pm 19$$

$$xy = 44, \text{ or } 6 \tag{4}$$

From (2),
$$y = 5 - x$$

Substitute in (4),
$$5x - x^2 = 44, \text{ or } 6$$

$$x^2 - 5x + \tfrac{25}{4} = -\tfrac{151}{4}, \text{ or } \tfrac{1}{4}$$

Extract root,
$$x - \tfrac{5}{2} = \pm \tfrac{1}{2}\sqrt{-151}, \text{ or } \pm \tfrac{1}{2}$$

$$\therefore\ x = \frac{5 \pm \sqrt{-151}}{2}, \ 3, \text{ or } 2$$

$$y = \frac{5 \mp \sqrt{-151}}{2}, \ 2, \text{ or } 3$$

44.
$$\begin{cases} x^4 + y^4 = 17 & (1) \\ x + y = 3 & (2) \end{cases}$$

Raise (2) to the fourth power,

$$x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 = 81$$

(1) is
$$x^4 \qquad\qquad\qquad\qquad\qquad + y^4 = 17$$

Subtract,
$$4x^3y + 6x^2y^2 + 4xy^3 \qquad = 64$$

$$2x^3y + 3x^2y^2 + 2xy^3 = 32$$

$$xy(2x^2 + 3xy + 2y^2) = 32 \tag{3}$$

Square (2),
$$x^2 + 2xy + y^2 = 9$$

$$x^2 + y^2 = 9 - 2xy$$

$$2x^2 + 2y^2 = 18 - 4xy$$

$$2x^2 + 3xy + 2y^2 = 18 - xy$$

Substitute in (3),
$$xy(18 - xy) = 32$$

$$18xy - x^2y^2 = 32$$

$$x^2y^2 - 18xy + 32 = 0$$

$$(xy - 16)(xy - 2) = 0$$

$$\therefore\ xy = 16, \text{ or } 2 \qquad (4)$$

From (2), $y = 3 - x$

Substitute in (4), $x(3 - x) = 16, \text{ or } 2$

$$x^2 - 3x = -16, \text{ or } -2$$

$$x^2 - 3x + \tfrac{9}{4} = -\tfrac{55}{4}, \text{ or } \tfrac{1}{4}$$

Extract root, $x - \frac{3}{2} = \pm \frac{1}{2}\sqrt{-55}, \text{ or } \pm \frac{1}{2}$

$$\therefore\ x = \frac{3 \pm \sqrt{-55}}{2}, \ 2, \text{ or } 1$$

$$y = 3 - x = \frac{3 \mp \sqrt{-55}}{2}, \ 1, \text{ or } 2$$

45.

$$\begin{cases} x^4 + y^4 = 881 & (1) \\ x - y = \ \ 1 & (2) \end{cases}$$

The fourth power of (2) is,

$$x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + \ \ y^4 = \ \ 1$$

Add (1) $x^4 \qquad\qquad + \ \ y^4 = 881$

Divide by 2, $2x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + 2y^4 = 882$

Extract root, $x^4 - 2x^3y + 3x^2y^2 - 2xy^3 + \ \ y^4 = 441$

Subtract $(2)^2$ from (3), $x^2 - xy + y^2 = \pm 21 \qquad (3)$

Add (4) to $(2)^2$, $xy = 20, \text{ or } -22 \qquad (4)$

$$x^2 + 2xy + y^2 = 81, \text{ or } -89$$

$$\therefore\ x + y = \pm 9, \text{ or } \pm \sqrt{-89} \qquad (5)$$

From (5) and (2), $x = 5, \text{ or } -4, \text{ or } \frac{1}{2}(1 \pm \sqrt{-89})$

$$y = 4, \text{ or } -5, \text{ or } \tfrac{1}{2}(-1 \pm \sqrt{-89})$$

46.

$$\begin{cases} x^5 + y^5 = 211 & (1) \\ x + y = \ \ 1 & (2) \end{cases}$$

Divide (1) by (2),

$$x^4 - x^3y + x^2y^2 - xy^3 + y^4 = 211$$

$$x^4 + y^4 - xy(x^2 + y^2) + x^2y^2 = 211 \qquad (3)$$

Square (2), $x^2 + 2xy + y^2 = 1$

$$x^2 + y^2 = 1 - 2xy \qquad (4)$$

Square (4), $x^4 + 2x^2y^2 + y^4 = 1 - 4xy + 4x^2y^2$

$$x^4 + y^4 = 1 - 4xy + 2x^2y^2 \qquad (5)$$

Substitute from (4) and (5) in (3),

$$(1 - 4xy + 2x^2y^2) - xy(1 - 2xy) + x^2y^2 = 211$$

$$5x^2y^2 - 5xy = 210$$

$$x^2y^2 - xy + \tfrac{1}{4} = \tfrac{169}{4}$$

Extract root, $$xy - \tfrac{1}{2} = \pm\tfrac{13}{2}$$

$$xy = 7, \text{ or } -6 \qquad (6)$$

From (2), $$y = 1 - x$$

Substitute in (6), $$x(1 - x) = 7, \text{ or } -6$$

$$x^2 - x = -7, \text{ or } 6$$

$$x^2 - x + \tfrac{1}{4} = -\tfrac{27}{4}, \text{ or } \tfrac{25}{4}$$

Extract root, $$x - \tfrac{1}{2} = \pm\tfrac{3}{2}\sqrt{-3}, \text{ or } \pm\tfrac{5}{2}$$

$$\therefore\ x = \frac{1 \pm 3\sqrt{-3}}{2}, 3, \text{ or } -2$$

$$y = \frac{1 \mp 3\sqrt{-3}}{2}, -2, \text{ or } 3$$

47. $$\begin{cases} x^5 - y^5 = 242 & (1) \\ x - y = 2 & (2) \end{cases}$$

Divide (1) by (2), $$x^4 + x^3y + x^2y^2 + xy^3 + y^4 = 121$$

Raise (2) to the fourth power, $$x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4 = 16$$

Subtract, $$5x^3y - 5x^2y^2 + 5xy^3 = 105$$

$$x^3y - x^2y^2 + xy^3 = 21$$

$$xy(x^2 - xy + y^2) = 21 \qquad (3)$$

Square (2), $$x^2 - 2xy + y^2 = 4$$

$$x^2 - xy + y^2 = 4 + xy$$

Substitute in (3), $$xy(4 + xy) = 21$$

$$x^2y^2 + 4xy = 21$$

$$\therefore\ xy = 3, \text{ or } -7 \qquad (4)$$

From (2), $$y = x - 2$$

Substitute in (4), $$x(x - 2) = 3, \text{ or } -7$$

$$x^2 - 2x = 3, \text{ or } -7$$

$$\therefore\ x = -1, 3, \text{ or } 1 \pm \sqrt{-6}$$

$$y = -3, 1, \text{ or } -1 \pm \sqrt{-6}$$

48. $$\begin{cases} x^2 + y^2 = xy + 7 & (1) \\ x + y = xy - 1 & (2) \end{cases}$$

Square (2), $$x^2 + 2xy + y^2 = x^2y^2 - 2xy + 1$$

(1) is $$x^2 + y^2 = xy + 7$$

Subtract, $$2xy = x^2y^2 - 3xy - 6$$

$$x^2y^2 - 5xy - 6 = 0$$

$$\therefore\ xy = 6, \text{ or } -1 \qquad (3)$$

Substitute in (2), $x + y = 5, \text{ or } -2$ (4)

From (4), $y = 5 - x, \text{ or } -2 - x$

Substitute in (3), $x(5 - x) = 6,$

or $x(-2 - x) = -1$

$x^2 - 5x + 6 = 0$

$\therefore x = 2, \text{ or } 3$

$y = 3, \text{ or } 2$

or $x^2 + 2x - 1 = 0$

$\therefore x = -1 \pm \sqrt{2}$

$y = -1 \mp \sqrt{2}$

49. $\begin{cases} x^3 - y^3 = 7xy & (1) \\ x - y = 2 & (2) \end{cases}$

Divide (1) by (2), $x^2 + xy + y^2 = \frac{7}{2}xy$

Square (2), $x^2 - 2xy + y^2 = 4$

Subtract, $3xy = \frac{7}{2}xy - 4$

$xy = 8$ (3)

From (2), $y = x - 2$

Substitute in (3), $x(x - 2) = 8$

$x^2 - 2x = 8$

$\therefore x = 4, \text{ or } -2$

$y = 2, \text{ or } -4$

50. $\begin{cases} x^3 + y^3 = 36xy & (1) \\ x + y = 24 & (2) \end{cases}$

Divide (1) by (2), $x^2 - xy + y^2 = \frac{3}{2}xy$

Square (2), $x^2 + 2xy + y^2 = 576$

Subtract, $3xy = 576 - \frac{3}{2}xy$

$xy = 128$ (3)

From (2), $y = 24 - x$

Substitute in (3), $x(24 - x) = 128$

$x^2 - 24x + 128 = 0$

$\therefore x = 8, \text{ or } 16$

$y = 16, \text{ or } 8$

51. $\begin{cases} x^3 + 3xy^2 = 62 & (1) \\ 3x^2y + y^3 = 63 & (2) \end{cases}$

Add, $x^3 + 3x^2y + 3xy^2 + y^3 = 125$

Extract cube root, $x + y = 5$ (3)

Subtract (2) from (1),

$$x^3 - 3x^2y + 3xy^2 - y^3 = -1$$

Extract cube root, $x - y = -1$ (4)

Add (3) and (4), $2x = 4$

$\therefore x = 2$

$y = 3$

52. $\begin{cases} x^2 + xy + y^2 = 61 & (1) \\ x^4 + x^2y^2 + y^4 = 1281 & (2) \end{cases}$

Divide (2) by (1), $x^2 - xy + y^2 = 21$ (3)

Subtract (3) from (1), $2xy = 40$

$xy = 20$ (4)

Add (4) to (1), $x^2 + 2xy + y^2 = 81$

Extract root, $x + y = \pm 9$ (5)

Subtract (4) from (3), $x^2 - 2xy + y^2 = 1$

Extract root, $x - y = \pm 1$ (6)

From (5) and (6), $\therefore x = \pm 5$, or ± 4

$y = \pm 4$, or ± 5

53. $\begin{cases} x^2 - xy + y^2 = 3 & (1) \\ x^4 + x^2y^2 + y^4 = 21 & (2) \end{cases}$

Divide (2) by (1), $x^2 + xy + y^2 = 7$ (3)

Subtract (1) from (3), $2xy = 4$

$xy = 2$ (4)

Subtract (4) from (1), $x^2 - 2xy + y^2 = 1$

Extract root, $x - y = \pm 1$ (5)

Add (4) to (3), $x^2 + 2xy + y^2 = 9$

Extract root, $x + y = \pm 3$ (6)

From (5) and (6), $\therefore x = \pm 2$, or ± 1

$y = \pm 1$, or ± 2

54. $\begin{cases} \dfrac{x+y}{x-y} + \dfrac{x-y}{x+y} = \dfrac{10}{3} & (1) \\ x^2 + y^2 = 20 & (2) \end{cases}$

From (1), $(x + y)^2 + (x - y)^2 = \frac{10}{3}(x^2 - y^2)$

$$2(x^2 + y^2) = \tfrac{10}{3}(x^2 - y^2)$$

$$2x^2 - 8y^2 = 0$$

$$x^2 = 4y^2$$

$$\therefore x = \pm 2y$$

Substitute in (2), $4y^2 + y^2 = 20$

$y = \pm 2$

$x = \pm 4$

55.
$$\begin{cases} \dfrac{x-y}{x+y} - \dfrac{x+y}{x-y} = \dfrac{24}{5}. & (1) \\ 3x + 4y = 36 & (2) \end{cases}$$

From (1),
$$\begin{aligned} (x-y)^2 - (x+y)^2 &= \tfrac{24}{5}(x^2-y^2) \\ -4xy &= \tfrac{24}{5}(x^2-y^2) \\ -5xy &= 6(x^2-y^2) \\ 6x^2 + 5xy - 6y^2 &= 0 \\ (3x-2y)(2x+3y) &= 0 \\ \therefore\ 3x - 2y &= 0 \\ x &= \frac{2y}{3} \end{aligned}$$

or
$$\begin{aligned} 2x + 3y &= 0 \\ x &= -\frac{3y}{2} \end{aligned}$$

Substitute in (2),
$$\begin{aligned} 2y + 4y &= 36 \\ \therefore\ y &= 6 \\ x &= \frac{2y}{3} = 4 \end{aligned}$$

or
$$\begin{aligned} -\frac{9y}{2} + 4y &= 36 \\ \therefore\ y &= -72 \\ x &= -\frac{3y}{2} = 108 \end{aligned}$$

56.
$$\begin{cases} x^2 + y^2 + x + y = 32 & (1) \\ xy + 16 = 0 & (2) \end{cases}$$

From (2), $$2xy = -32 \qquad (3)$$

Add (3) to (1), $$x^2 + 2xy + y^2 + x + y = 0 \qquad (4)$$

$$\begin{aligned} (x+y)^2 + (x+y) &= 0 \\ (x+y)(x+y+1) &= 0 \\ \therefore\ x + y &= 0 && (5) \end{aligned}$$

or
$$x + y + 1 = 0 \qquad (6)$$

From (5), $$x = -y$$

Substitute in (2),
$$\begin{aligned} y^2 &= 16 \\ y &= \pm 4 \\ \therefore\ x &= \mp 4 \end{aligned}$$

From (6), $$y = -x - 1$$

Substitute in (2),
$$\begin{aligned} x(-x-1) &= -16 \\ x^2 + x - 16 &= 0 \\ \therefore\ x &= -\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{65} \\ y &= -x - 1 = -\tfrac{1}{2} \mp \tfrac{1}{2}\sqrt{65} \end{aligned}$$

57.
$$\begin{cases} x - y - 3 = 0 & (1) \\ 2(x^2 - y^2) = 3xy & (2) \end{cases}$$

From (1), $x - y = 3$ (3)

Divide (2) by (3), $2(x + y) = xy$ (4)

From (3), $y = x - 3$

Substitute in (4), $2(2x - 3) = x(x - 3)$

$$4x - 6 = x^2 - 3x$$
$$x^2 - 7x + 6 = 0$$
$$\therefore x = 1, \text{ or } 6$$
$$y = x - 3 = -2, \text{ or } -3$$

58.
$$\begin{cases} \dfrac{1}{x} + \dfrac{1}{y} = 7 & (1) \\[2ex] \dfrac{1}{x + 1} + \dfrac{1}{y + 1} = \dfrac{31}{20} & (2) \end{cases}$$

From (1), $x + y = 7xy$ (3)

From (2), $y + 1 + x + 1 = \frac{31}{20}(x + 1)(y + 1)$

$$x + y + 2 = \tfrac{31}{20}(xy + x + y + 1)$$
$$\tfrac{31}{20}xy + \tfrac{11}{20}(x + y) - \tfrac{9}{20} = 0$$
$$31xy + 11(x + y) - 9 = 0 \qquad (4)$$

Substitute the value of $x + y$ from (3) in (4),

$$31xy + 77xy - 9 = 0$$
$$108xy = 9$$
$$xy = \tfrac{1}{12} \qquad (5)$$

Substitute in (3), $x + y = \frac{7}{12}$ (6)

From (6), $y = \frac{7}{12} - x$

Substitute in (5), $x(\frac{7}{12} - x) = \frac{1}{12}$

$$x^2 - \tfrac{7}{12}x + \tfrac{1}{12} = 0$$
$$\therefore x = \tfrac{1}{3}, \text{ or } \tfrac{1}{4}$$
$$y = \tfrac{1}{4}, \text{ or } \tfrac{1}{3}$$

59.
$$\begin{cases} x^4 + y^4 = 272 & (1) \\ x^2 + y^2 = 3xy - 4 & (2) \end{cases}$$

Square (2), $x^4 + 2x^2y^2 + y^4 = 9x^2y^2 - 24xy + 16$

(1) is, $x^4 \qquad + y^4 = \qquad 272$

Subtract, $2x^2y^2 = 9x^2y^2 - 24xy - 256$

$$7x^2y^2 - 24xy - 256 = 0$$
$$(7xy + 32)(xy - 8) = 0$$

$\therefore\ xy = 8, \text{ or } -\frac{32}{7}$ (3)

Substitute in (2), $x^2 + y^2 = 20, \text{ or } -\frac{124}{7}$ (4)

Add 2 × (3) to (4), $x^2 + 2xy + y^2 = 36, \text{ or } -\frac{188}{7}$

Extract root, $x + y = \pm 6, \text{ or } \pm\sqrt{-\frac{188}{7}}$ (5)

Subtract 2 × (3) from (4), $x^2 - 2xy + y^2 = 4, \text{ or } -\frac{60}{7}$

Extract root, $x - y = \pm 2, \text{ or } \pm\sqrt{-\frac{60}{7}}$ (6)

From (5) and (6), $x = \pm 4, \pm 2, \frac{1}{2}\left(\pm\sqrt{-\frac{188}{7}} \pm \sqrt{-\frac{60}{7}}\right)$

$y = \pm 2, \pm 4, \frac{1}{2}\left(\pm\sqrt{-\frac{188}{7}} \mp \sqrt{-\frac{60}{7}}\right)$

60. $$\begin{cases} x^2 + y^2 = x^2y^2 + 1 & (1) \\ x + y = 2xy - 1 & (2) \end{cases}$$

Square (2), $x^2 + 2xy + y^2 = 4x^2y^2 - 4xy + 1$

(1) is $x^2 \qquad + y^2 = x^2y^2 \qquad + 1$

Subtract, $2xy \qquad = 3x^2y^2 - 4xy$

$$3x^2y^2 - 6xy = 0$$

$\therefore\ xy = 0, \text{ or } 2$ (3)

Substitute in (2), $x + y = -1, \text{ or } 3$ (4)

From (3) and (4), $x = 0,\ y = 1$

$y = 0,\ x = 1$

or $xy = 2$

$$x + y = 3$$

$$y = -x + 3$$

$$x(-x + 3) = 2$$

$$x^2 - 3x + 2 = 0$$

$$(x - 1)(x - 2) = 0$$

$$\therefore\ x = 1, \text{ or } 2$$

$$y = 2, \text{ or } 1$$

61. $$\begin{cases} x^3 - y^3 = a^3 & (1) \\ x - y = a & (2) \end{cases}$$

From (2), $y = x - a$

Substitute in (1), $x^3 - (x - a)^3 = a^3$

$$x^3 - (x^3 - 3ax^2 + 3a^2x - a^3) = a^3$$

$$3ax^2 - 3a^2x = 0$$

$$x^2 - ax = 0$$

$$\therefore\ x = 0, \text{ or } a$$

$$y = -a, \text{ or } 0$$

62. $$\begin{cases} \dfrac{x}{a}+\dfrac{y}{b}=1 & (1)\\ \dfrac{a}{x}+\dfrac{b}{y}=4 & (2)\end{cases}$$

From (1), $$bx+ay=ab \qquad (3)$$

From (2), $$bx+ay=4xy \qquad (4)$$

$$\therefore\ 4xy=ab \qquad (5)$$

From (3), $$ay=ab-bx$$

$$y=\frac{ab-bx}{a}$$

Substitute in (5), $$4x\left(\frac{ab-bx}{a}\right)=ab$$

$$4ax-4x^2=a^2$$

$$4x^2-4ax+a^2=0$$

$$2x-a=0$$

$$\therefore\ x=\frac{a}{2}$$

$$y=\frac{b}{2}$$

63. $$\begin{cases} x^2=ax+by & (1)\\ y^2=bx+ay & (2)\end{cases}$$

Subtract, $$x^2-y^2=a(x-y)+b(y-x)$$

$$(x-y)(x+y)=a(x-y)-b(x-y)$$

$$(x-y)(x+y-a+b)=0$$

$$\therefore\ x-y=0, \text{ or } x+y-a+b=0$$

$$\therefore\ x=y, \text{ or } x=a-b-y$$

Substitute in (2),

$$y^2=by+ay, \qquad \text{or } y^2=b(a-b-y)+ay$$

$$y^2-(a+b)y=0, \quad \text{or } y^2+(b-a)y=b(a-b)$$

$$\therefore\ y=0, \text{ or } a+b, \text{ or } y=\frac{a-b}{2}\pm\tfrac{1}{2}\sqrt{a^2+2ab-3b^2}$$

$$x=0, \text{ or } a+b \qquad x=\frac{a-b}{2}\mp\tfrac{1}{2}\sqrt{a^2+2ab-3b^2}$$

64. $$\begin{cases} x^2+y^2=2(a^2+b^2) & (1)\\ xy=a^2-b^2 & (2)\end{cases}$$

Add 2 × (2) to (1), $$x^2+2xy+y^2=4a^2$$

Extract root, $$x+y=\pm 2a \qquad (3)$$

Subtract 2 × (2) from (1), $$x^2-2xy+y^2=4b^2$$

Extract root, $$x-y=\pm 2b \qquad (4)$$

From (3) and (4), $$x=\pm a\pm b$$

$$y=\pm a\mp b$$

65.

$$\begin{cases} x^2 + y^2 = \dfrac{a^4 + b^4}{a^2b^2} & (1) \\ xy = 1 & (2) \end{cases}$$

Add 2 × (2) to (1),

$$x^2 + 2xy + y^2 = \frac{a^4 + 2a^2b^2 + b^4}{a^2b^2}$$

$$x + y = \pm\frac{a^2 + b^2}{ab} \qquad (3)$$

Subtract 2 × (2) from (1),

$$x^2 - 2xy + y^2 = \frac{a^4 - 2a^2b^2 + b^4}{a^2b^2}$$

$$x - y = \pm\frac{a^2 - b^2}{ab} \qquad (4)$$

From (3) and (4),

$$x = \pm\frac{a}{b}, \text{ or } \pm\frac{b}{a}$$

$$y = \pm\frac{b}{a}, \text{ or } \pm\frac{a}{b}$$

66.

$$\begin{cases} x^2 - y^2 = \dfrac{a - b}{a + b} & (1) \\ xy = \dfrac{ab}{(a + b)^2} & (2) \end{cases}$$

From (2),

$$y = \frac{ab}{x(a + b)^2}$$

Substitute in (1),

$$x^2 - \frac{a^2b^2}{x^2(a + b)^4} = \frac{a - b}{a + b}$$

$$x^4 - \left(\frac{a - b}{a + b}\right)x^2 = \frac{a^2b^2}{(a + b)^4}$$

$$x^4 - \left(\frac{a - b}{a + b}\right)x^2 + \frac{(a - b)^2}{4(a + b)^2} = \frac{a^4 + 2a^2b^2 + b^4}{4(a + b)^4}$$

$$x^2 - \frac{a - b}{2(a + b)} = \pm\frac{a^2 + b^2}{2(a + b)^2}$$

$$\therefore\ x^2 = \frac{a^2}{(a + b)^2}, \text{ or } \frac{-b^2}{(a + b)^2}$$

$$\therefore\ x = \pm\frac{a}{a + b}, \text{ or } \pm\frac{b\sqrt{-1}}{a + b}$$

$$y = \pm\frac{b}{a + b}, \text{ or } \mp\frac{a\sqrt{-1}}{a + b}$$

67.

$$\begin{cases} x^2 - xy = 2\,ab + 2\,b^2 & (1) \\ xy - y^2 = 2\,ab - 2\,b^2 & (2) \end{cases}$$

Subtract,
$$x^2 - 2\,xy + y^2 = 4\,b^2$$

Extract root,
$$x - y = \pm 2\,b \qquad (3)$$

Divide (1) by (3),
$$x = \pm (a + b)$$

Divide (2) by (3),
$$y = \pm (a - b)$$

68.

$$\begin{cases} x^2 - y^2 = a^2 & (1) \\ xy = b^2 & (2) \end{cases}$$

From (2),
$$y = \frac{b^2}{x}$$

Substitute in (1),
$$x^2 - \frac{b^4}{x^2} = a^2$$

$$x^4 - a^2x^2 = b^4$$

$$x^4 - a^2x^2 + \frac{a^4}{4} = \frac{4\,b^4 + a^4}{4}$$

Extract root,
$$x^2 - \frac{a^2}{2} = \pm \tfrac{1}{2}\sqrt{4\,b^4 + a^4}$$

$$\therefore\ x^2 = \frac{a^2 \pm \sqrt{4\,b^4 + a^4}}{2}$$

$$\therefore\ x = \pm\sqrt{\frac{a^2 \pm \sqrt{4\,b^4 + a^4}}{2}}$$

$$y = \pm\sqrt{\frac{-\,a^2 \pm \sqrt{4\,b^4 + a^4}}{2}}$$

69.

$$\begin{cases} x^2 - y^2 = 8\,ab & (1) \\ xy = a^2 - 4\,b^2 & (2) \end{cases}$$

From (2),
$$y = \frac{a^2 - 4\,b^2}{x}$$

Substitute in (1),
$$x^2 - \frac{(a^2 - 4\,b^2)^2}{x^2} = 8\,ab$$

$$x^4 - 8\,abx^2 = (a^2 - 4\,b^2)^2$$

$$x^4 - 8\,abx^2 + 16\,a^2b^2 = a^4 + 8\,a^2b^2 + 16\,b^4$$

Extract root,
$$x^2 - 4\,ab = \pm (a^2 + 4\,b^2)$$

$$x^2 = \pm (a^2 + 4\,ab + 4\,b^2)$$

$$x = \pm (a + 2\,b),\ \text{or}\ \pm (a - 2\,b)\sqrt{-1}$$

$$y = \pm (a - 2\,b),\ \text{or}\ \mp (a + 2\,b)\sqrt{-1}$$

70. $$\begin{cases} x^3 + y^3 = a^3 + b^3 & (1) \\ x + y = a + b & (2) \end{cases}$$

Divide (1) by (2), $x^2 - xy + y^2 = a^2 - ab + b^2$ (3)

Square (2), $x^2 + 2xy + y^2 = a^2 + 2ab + b^2$

Subtract, $3xy = 3ab$

$$xy = ab \qquad (4)$$

Subtract (4) from (3), $x^2 - 2xy + y^2 = a^2 - 2ab + b^2$

$$x - y = \pm(a - b)$$

(2) is $x + y = a + b$

$$2x = 2a, \text{ or } 2b$$

$$x = a, \text{ or } b$$

$$y = b, \text{ or } a$$

Exercise 102. Page 285.

1. The area of a rectangle is 60 square feet, and its perimeter is 34 feet. Find the length and breadth of the rectangle.

Let $x =$ the length of the rectangle in feet,

$y =$ the breadth of the rectangle in feet.

Then, $2(x + y) =$ the perimeter of the rectangle in feet,

$xy =$ the area of the rectangle in square feet.

$$xy = 60 \qquad (1)$$

$$2(x + y) = 34 \qquad (2)$$

From (2), $y = 17 - x$

Substitute in (1), $x(17 - x) = 60$

$$x^2 - 17x + 60 = 0$$

$$\therefore x = 5, \text{ or } 12$$

$$y = 12, \text{ or } 5$$

Therefore the rectangle is 12 feet long and 5 feet wide.

2. The area of a rectangle is 108 square feet. If the length and breadth of the rectangle are each increased by 3 feet, the area will be 180 square feet. Find the length and breadth of the rectangle.

Let $x =$ the length of the rectangle in feet,

$y =$ the breadth of the rectangle in feet.

Then, $xy =$ the area of the rectangle in square feet,

$(x + 3)(y + 3) =$ the area of the enlarged rectangle in square feet.

$$xy = 108 \qquad (1)$$

$$(x + 3)(y + 3) = 180 \qquad (2)$$

From (2), $xy + 3x + 3y = 171$

$$108 + 3x + 3y = 171$$

$$x + y = 21$$

$$y = 21 - x$$

Substitute in (1), $x(21 - x) = 108$

$$x^2 - 21x + 108 = 0$$

$$\therefore x = 9, \text{ or } 12$$

$$y = 12, \text{ or } 9$$

Therefore the rectangle is 12 feet long and 9 feet wide.

3. If the length and breadth of a rectangular plot are each increased by 10 feet, the area will be increased by 400 square feet. But if the length and breadth are each diminished by 5 feet, the area will be 75 square feet. Find the length and breadth of the plot.

Let $x =$ the length of the rectangle in feet,
$y =$ the breadth of the rectangle in feet.
Then, $xy =$ the area of the rectangle in square feet,
$(x + 10)(y + 10) =$ the area of the increased rectangle in square feet,
$(x - 5)(y - 5) =$ the area of the diminished rectangle.

$$(x + 10)(y + 10) = xy + 400 \qquad (1)$$

$$(x - 5)(y - 5) = 75 \qquad (2)$$

From (1), $10x + 10y = 300$

$$x + y = 30 \qquad (3)$$

From (2), $xy - 5x - 5y = 50 \qquad (4)$

Substitute from (3) in (4),

$$xy - 150 = 50$$

$$xy = 200 \qquad (5)$$

From (3), $y = 30 - x$

Substitute in (5), $x(30 - x) = 200$

$$x^2 - 30x + 200 = 0$$

$$\therefore x = 10, \text{ or } 20$$

$$y = 20, \text{ or } 10$$

Therefore the plot is 20 feet long and 10 feet wide.

4. The area of a rectangle is 168 square feet, and the length of its diagonal is 25 feet. Find the length and breadth of the rectangle.

Let $x =$ the length of the rectangle in feet,
$y =$ the breadth of the rectangle in feet.

Then, $\sqrt{x^2+y^2}=$ the length of the diagonal in feet,

$xy=$ the area of the rectangle in square feet.

$$xy=168 \qquad (1)$$

$$\sqrt{x^2+y^2}=25 \qquad (2)$$

From (2), $x^2+y^2=625$

Add $2\times(1)$, $2xy=336$

$$x^2+2xy+y^2=961$$

$$x+y=\pm 31$$

Similarly, $x^2-2xy+y^2=289$

$$x-y=\pm 17$$

$$\therefore\ x=\pm 24, \text{ or } \pm 7$$

$$y=\pm 7, \text{ or } \pm 24$$

Therefore the rectangle is 24 feet long and 7 feet wide.

5. The diagonal of a rectangle is 25 inches. If the rectangle were 4 inches shorter and 8 inches wider, the diagonal would still be 25 inches. Find the area of the rectangle.

Let $x=$ the length of the rectangle in inches,

$y=$ the breadth of the rectangle in inches.

Then, $\sqrt{x^2+y^2}=$ the length of the diagonal in inches,

$\sqrt{(x-4)^2+(y+8)^2}=$ the length of the diagonal of the new rectangle in inches.

$$\sqrt{x^2+y^2}=25 \qquad (1)$$

$$\sqrt{(x-4)^2+(y+8)^2}=25 \qquad (2)$$

From (1), $x^2+y^2=625$ (3)

From (2), $x^2-8x+16+y^2+16y+64=625$

$$x^2+y^2-8x+16y=545$$

$$625-8x+16y=545$$

$$8x-16y=80$$

$$x-2y=10 \qquad (4)$$

$$x=10+2y$$

Substitute in (3), $(10+2y)^2+y^2=625$

$$5y^2+40y=525$$

$$y^2+8y=105$$

$$\therefore\ y=7, \text{ or } -15$$

$$x=24, \text{ or } -20$$

Therefore the rectangle is 24 inches long and 7 inches wide.

6. **A rectangular field, containing 180 square rods, is surrounded by a road 1 rod wide. The area of the road is 58 square rods. Find the dimensions of the field.**

Let $x =$ the length of the field in rods,
$y =$ the width of the field in rods.
Then, $xy =$ the area of the field in square rods,
$(x+2)(y+2) =$ the area of the field and road in square rods.

$$xy = 180 \quad (1)$$
$$(x+2)(y+2) = 238 \quad (2)$$

From (2), $xy + 2x + 2y + 4 = 238$

$$180 + 2x + 2y + 4 = 238$$
$$x + y = 27$$
$$y = 27 - x$$

Substitute in (1), $x(27 - x) = 180$

$$x^2 - 27x + 180 = 0$$
$$\therefore x = 12, \text{ or } 15$$
$$y = 15, \text{ or } 12$$

Therefore the field is 12 rods wide and 15 rods long.

7. Two square gardens have a total surface of 2137 square yards. A rectangular piece of land whose dimensions are respectively equal to the sides of the two squares, will have 1093 square yards less than the two gardens united. What are the sides of the two squares?

Let $x =$ the length in yards of a side of the first garden,
$y =$ the length in yards of a side of the second garden.
Then $x^2 =$ the area in square yards of the first garden,
$y^2 =$ the area in square yards of the second garden,
$xy =$ the area in square yards of the rectangle.

$$x^2 + y^2 = 2137 \quad (1)$$
$$xy = 2137 - 1093 \quad (2)$$

From (2), $xy = 1044 \quad (3)$

Add $2 \times (3)$ to (1), $x^2 + 2xy + y^2 = 4225$

$$x + y = \pm 65 \quad (5)$$

Similarly, $x^2 - 2xy + y^2 = 49$

$$x - y = \pm 7 \quad (6)$$

From (5) and (6), $x = \pm 36, \text{ or } \pm 29$

$$y = \pm 29, \text{ or } \pm 36$$

Therefore the one garden is 36 feet square and the other 29 feet square.

8. The sum of two numbers is 22, and the difference of their squares is 44. Find the numbers.

Let $x =$ the greater number,
$y =$ the less number.

$$x + y = 22 \qquad (1)$$
$$x^2 - y^2 = 44 \qquad (2)$$

Divide (2) by (1), $x - y = 2$ (3)

From (2) and (3), $x = 12$

$$y = 10$$

Therefore the numbers are 10 and 12.

9. The difference of two numbers is 6, and their product exceeds their sum by 39. Find the numbers.

Let $x =$ the greater number,
$y =$ the less number.

$$x - y = 6 \qquad (1)$$
$$xy = x + y + 39 \qquad (2)$$

From (1), $y = x - 6$

Substitute in (2), $x(x - 6) = x + x - 6 + 39$

$$x^2 - 8x = 33$$
$$\therefore x = -3, \text{ or } 11$$
$$y = -9, \text{ or } 5$$

Therefore the numbers are 5 and 11.

10. The sum of two numbers is equal to the difference of their squares, and the product of the numbers exceeds twice their sum by 2. Find the numbers.

Let $x =$ the greater number,
$y =$ the less number.

$$x + y = x^2 - y^2 \qquad (1)$$
$$xy = 2(x + y) + 2 \qquad (2)$$

From (1), $x + y - (x^2 - y^2) = 0$

$$(x + y)[1 - (x - y)] = 0$$
$$\therefore x + y = 0$$

or $x - y = 1$

The former solution $x = -y$ is to be rejected, since x and y are to be both positive.

Hence we have $x - y = 1$ (3)

$$xy = 2x + 2y + 2 \qquad (2)$$

From (3), $y = x - 1$

Substitute in (2), $x(x-1)=2x+2x-2+2$

$$x^2-x=4x$$
$$x^2-5x=0$$
$$\therefore\ x=0,\text{ or }5$$
$$y=-1,\text{ or }4$$

Therefore the numbers are 4 and 5.

11. The sum of two numbers is 20, and the sum of their cubes is 2060. Find the numbers.

Let $x=$ the greater number,
$y=$ the less number.

$$x+y=20 \qquad (1)$$
$$x^3+y^3=2060 \qquad (2)$$

Divide (2) by (1), $x^2-xy+y^2=103$ (3)

Square (1), $x^2+2xy+y^2=400$

Subtract, $3xy=297$

$$xy=99 \qquad (4)$$

Subtract (4) from (3), $x^2-2xy+y^2=4$

$$x-y=\pm 2$$

(1) is $x+y=20$

Hence, $x=11,\text{ or }9$

$$y=9,\text{ or }11$$

Therefore the numbers are 9 and 11.

12. The difference of two numbers is 5, and the difference of their cubes exceeds the difference of their squares by 1290. Find the numbers.

Let $x=$ the greater number,
$y=$ the less number.

$$x-y=5 \qquad (1)$$
$$x^3-y^3=x^2-y^2+1290 \qquad (2)$$

From (2), $(x-y)(x^2+xy+y^2)=(x-y)(x+y)+1290$

$$5(x^2+xy+y^2)=5(x+y)+1290$$
$$x^2+xy+y^2=x+y+258 \qquad (3)$$

From (1), $y=x-5$

Substitute in (3),

$$x^2+x(x-5)+(x-5)^2=x+x-5+258$$
$$3x^2-15x+25=2x-5+258$$

$$3x^2 - 17x = 228$$

$$\therefore\ x = 12, \text{ or } -\tfrac{19}{3}$$

$$y = 7, \text{ or } -\tfrac{34}{3}$$

Therefore the numbers are 7 and 12.

13. A number is formed of two digits. The sum of the squares of the digits is 58. If twelve times the units' digit be subtracted from the number, the order of the digits will be reversed. Find the number.

Let $x =$ the tens' digit,

$y =$ the units' digit.

Then, $10x + y =$ the number,

$10y + x =$ the number reserved.

$$\therefore\ x^2 + y^2 = 58 \qquad (1)$$

$$10x + y - 12y = 10y + x \qquad (2)$$

From (2),

$$9x - 21y = 0$$

$$3x - 7y = 0$$

$$x = \frac{7y}{3}$$

Substitute in (1),

$$\tfrac{49}{9}y^2 + y^2 = 58$$

$$y^2 = 9$$

$$\therefore\ y = \pm 3$$

$$x = \pm 7$$

Therefore the number is 73.

14. A number is formed of three digits, the third digit being twice the sum of the other two. The first digit plus the product of the other two digits is 25. If 180 be added to the number, the order of the first and second digits will be reversed. Find the number.

Let $x =$ the hundreds' digit,

$y =$ the tens' digit.

Then, $2(x + y) =$ the units' digit.

$100x + 10y + 2(x + y) =$ the number.

$100y + 10x + 2(x + y) =$ the number after the hundreds' and tens' digits are interchanged.

$$x + 2y(x + y) = 25 \qquad (1)$$

$$100x + 10y + 2(x + y) + 180 = 100y + 10x + 2(x + y) \qquad (2)$$

From (1),

$$x + 2xy + 2y^2 = 25 \qquad (3)$$

From (2),

$$90x - 90y = -180$$

$$x - y = -2 \qquad (4)$$

$$x = y - 2$$

Substitute in (3),

$$\begin{aligned}(y-2)+2(y-2)y+2y^2&=25\\ 4y^2-3y&=27\\ \therefore\ y&=3,\text{ or }-\tfrac{9}{4}\\ x&=1,\text{ or }-\tfrac{17}{4}\end{aligned}$$

Therefore the number is 138.

15. There are two numbers formed of the same two digits in reverse orders. The sum of the numbers is 33 times the difference of the two digits, and the difference of the squares of the numbers is 4752. Find the numbers.

Let $10x+y=$ the greater number.

Then, $10y+x=$ the smaller number.

$$10x+y+10y+x=33(x-y) \tag{1}$$

$$(10x+y)^2-(10y+x^2)=4752 \tag{2}$$

From (1), $22x-44y=0$

$$x=2y \tag{3}$$

From (2), $99x^2-99y^2=4752$

$$x^2-y^2=48 \tag{4}$$

From (3) and (4), $4y^2-y^2=48$

$$\begin{aligned}\therefore\ y&=\pm 4\\ x&=\pm 8\end{aligned}$$

Therefore the numbers are 84 and 48.

16. The sum of the numerator and denominator of a certain fraction is 5; and if the numerator and denominator be each increased by 3, the value of the fraction will be increased by $\frac{1}{6}$. Find the fraction.

Let $x=$ the numerator,

$y=$ the denominator.

Then, $\frac{x}{y}=$ the fraction,

$\frac{x+3}{y+3}=$ the fraction increased by $\frac{1}{6}$.

$$x+y=5 \tag{1}$$

$$\frac{x+3}{y+3}=\frac{x}{y}+\frac{1}{6} \tag{2}$$

From (2), $6y(x+3)=6x(y+3)+y(y+3)$

$$6xy+18y=6xy+18x+y^2+3y$$

$$y^2-15y+18x=0 \tag{3}$$

From (1), $x=5-y$

Substitute in (3),

$$y^2 - 15y + 18(5 - y) = 0$$
$$y^2 - 33y + 90 = 0$$
$$\therefore\ y = 3, \text{ or } 30$$
$$x = 2, \text{ or } -25$$

Therefore the fraction is $\frac{2}{3}$.

17. The fore wheel of a carriage turns in a mile 132 times more than the hind wheel; but if the circumferences were each increased by 2 feet, it would turn only 88 times more. Find the circumference of each.

Let x = the circumference of the hind wheel in feet,
y = the circumference of the fore wheel in feet.

Then, $\frac{5280}{x}$ = the number of times the hind wheel turns in going 1 mile.
$\frac{5280}{y}$ = the number of feet the fore wheel turns in going 1 mile.
$\frac{5280}{x+2}$ = the number of times the hind wheel would turn in going 1 mile if its circumference were 2 feet greater.
$\frac{5280}{y+2}$ = the number of times the fore wheel would turn in going 1 mile if its circumference were 2 feet greater.

$$\frac{5280}{y} - \frac{5280}{x} = 132 \qquad (1)$$
$$\frac{5280}{y+2} - \frac{5280}{x+2} = 88 \qquad (2)$$

From (1),

$$5280(x - y) = 132xy$$
$$40(x - y) = xy \qquad (3)$$

From (2),

$$5280(x + 2) - 5280(y + 2) = 88(y + 2)(x + 2)$$
$$5280(x - y) = 88(xy + 2x + 2y + 4)$$
$$60(x - y) = xy + 2x + 2y + 4$$
$$58x - 62y - 4 = xy \qquad (4)$$

Subtract (3) from (4),

$$18x - 22y - 4 = 0$$
$$9x - 11y = 2$$
$$x = \frac{2 + 11y}{9}$$

Substitute in (3),

$$40\left(\frac{2+11y}{9}-y\right)=\left(\frac{2+11y}{9}\right)y$$

$$40\left(\frac{2+2y}{9}\right)=\left(\frac{2+11y}{9}\right)y$$

$$80+80y=2y+11y^2$$

$$11y^2-78y-80=0$$

$$\therefore\ y=8,\text{ or }-\tfrac{10}{11}$$

$$x=10,\text{ or }-\tfrac{8}{9}$$

Therefore the wheels are 8 and 10 feet in circumference respectively.

Exercise 103. Page 289.

Determine without solving the character of the roots of each of the following equations:

1. $x^2+5x+6=0$

$$a=1,\ b=5,\ c=6$$
$$b^2-4ac=25-24$$
$$=1$$

$\therefore$ the roots are real, unequal, and rational.

2. $x^2+2x-15=0$

$$a=1,\ b=2,\ c=-15$$
$$b^2-4ac=4+60$$
$$=64$$

$\therefore$ the roots are real, unequal, and rational.

3. $x^2+2x+3=0$

$$a=1,\ b=2,\ c=3$$
$$b^2-4ac=4-12$$
$$=-8$$

$\therefore$ the roots are imaginary.

4. $3x^2+7x+2=0$

$$a=3,\ b=7,\ c=2$$
$$b^2-4ac=49-24$$
$$=25$$

$\therefore$ the roots are real, unequal, and rational.

5. $9x^2+6x+1=0$

$$a=9,\ b=6,\ c=1$$
$$b^2-4ac=36-36$$
$$=0$$

$\therefore$ the roots are real and equal.

6. $6x^2-7x-3=0$

$$a=6,\ b=-7,\ c=-3$$
$$b^2-4ac=49+72$$
$$=121$$

$\therefore$ the roots are real, unequal, and rational.

7. $5x^2-5x-3=0$

$$a=5,\ b=-5,\ c=-3$$
$$b^2-4ac=25+60$$
$$=85$$

$\therefore$ the roots are real, unequal surds.

8. $2x^2-x+5=0$

$$a=2,\ b=-1,\ c=5$$
$$b^2-4ac=1-40$$
$$=-39$$

$\therefore$ the roots are imaginary.

9. $6x^2 + x - 77 = 0$

$$a = 6,\ b = 1,\ c = -77$$
$$b^2 - 4ac = 1 + 1848$$
$$= 1849 = 43^2$$

$\therefore$ the roots are real, unequal, and rational.

10. $5x^2 + 8x + \frac{16}{5} = 0$

$$a = 5,\ b = 8,\ c = \tfrac{16}{5}$$
$$b^2 - 4ac = 64 - 64$$
$$= 0$$

$\therefore$ the roots are real and equal.

Determine the values of m for which the two roots of each of the following equations are equal:

11. $(m + 1)x^2 + (m - 1)x + m + 1 = 0$

$$a = m + 1,\quad b = m - 1,\quad c = m + 1$$
$$b^2 - 4ac = (m - 1)^2 - 4(m + 1)^2 = 0$$
$$m^2 - 2m + 1 - 4m^2 - 8m - 4 = 0$$
$$-3m^2 - 10m - 3 = 0$$
$$\therefore\ m = -3, \text{ or } -\tfrac{1}{3}$$

12. $(2m - 3)x^2 + mx + m - 1 = 0$

$$b^2 - 4ac = m^2 - 4(2m - 3)(m - 1) = 0$$
$$m^2 - 8m^2 + 20m - 12 = 0$$
$$7m^2 - 20m + 12 = 0$$
$$\therefore\ m = 2, \text{ or } \tfrac{6}{7}$$

13. $2mx^2 + x^2 + 4x + 2mx + 2m - 4 = 0$

$$b^2 - 4ac = (4 + 2m)^2 - 4(2m + 1)(2m - 4) = 0$$
$$16 + 16m + 4m^2 - 16m^2 + 24m + 16 = 0$$
$$12m^2 - 40m - 32 = 0$$
$$3m^2 - 10m - 8 = 0$$
$$\therefore\ m = 4, \text{ or } -\tfrac{2}{3}$$

14. $2mx^2 + 3mx - 6 = 3x - 2m - x^2$

$$(2m + 1)x^2 + (3m - 3)x - 6 + 2m = 0$$
$$b^2 - 4ac = (3m - 3)^2 - 4(2m + 1)(2m - 6) = 0$$
$$9m^2 - 18m + 9 - 16m^2 + 40m + 24 = 0$$
$$7m^2 - 22m - 33 = 0$$
$$\therefore\ m = \frac{11 \pm 4\sqrt{22}}{7}$$

15. $mx^2 + 9x - 10 = 3mx - 2x^2 + 2m$

$$(m + 2)x^2 + (9 - 3m)x - 2m - 10 = 0$$
$$b^2 - 4ac = (9 - 3m)^2 - 4(m + 2)(-2m - 10) = 0$$
$$81 - 54m + 9m^2 + 8m^2 + 56m + 80 = 0$$
$$17m^2 + 2m + 161 = 0$$
$$\therefore\ m = \frac{-1 \pm 4\sqrt{-171}}{17}$$

Exercise 104. Page 292.

Form the equations of which the roots are

1. $7, 6.$

$$(x-7)(x-6)=0$$
$$x^2-13x+42=0$$

2. $5, -3.$

$$(x-5)(x+3)=0$$
$$x^2-2x-15=0$$

3. $1\frac{1}{2}, -2.$

$$(x-1\tfrac{1}{2})(x+2)=0$$
$$x^2+\tfrac{1}{2}x-3=0$$

4. $4, 2\frac{1}{3}.$

$$(x-4)(x-2\tfrac{1}{3})=0$$
$$x^2-6\tfrac{1}{3}x+9\tfrac{1}{3}=0$$
$$3x^2-19x+28=0$$

5. $1\frac{1}{2}, -1\frac{1}{3}.$

$$(x-1\tfrac{1}{2})(x+1\tfrac{1}{3})=0$$
$$x^2-\tfrac{1}{6}x-2=0$$
$$6x^2-x-12=0$$

6. $-1\frac{1}{6}, -1\frac{2}{3}.$

$$(x+1\tfrac{1}{6})(x+1\tfrac{2}{3})=0$$
$$x^2+2\tfrac{5}{6}x+\tfrac{35}{18}=0$$
$$18x^2+51x+35=0$$

7. $13, -4\frac{1}{3}.$

$$(x-13)(x+4\tfrac{1}{3})=0$$
$$(x-13)(3x+13)=0$$
$$3x^2-26x-169=0$$

8. $\frac{3}{11}, \frac{2}{11}.$

$$(x-\tfrac{3}{11})(x-\tfrac{2}{11})=0$$
$$(11x-3)(11x-2)=0$$
$$121x^2-55x+6=0$$

9. $3+\sqrt{2},\ 3-\sqrt{2}.$

$$(x-3-\sqrt{2})(x-3+\sqrt{2})=0$$
$$x^2-6x+7=0$$

10. $1+\sqrt{-1},\ 1-\sqrt{-1}.$

$$(x-1-\sqrt{-1})(x-1+\sqrt{-1})=0$$
$$x^2-2x+2=0$$

11. $a, a-b.$

$$(x-a)(x-a+b)=0$$
$$x^2-2ax+bx+a^2-ab=0$$

12. $a+b,\ a-b.$

$$(x-a-b)(x-a+b)=0$$
$$x^2-2ax+a^2-b^2=0$$

Resolve into factors, real or imaginary:

13.

$$12x^2+x-1=0$$
$$\therefore\ x=\tfrac{1}{4}, \text{ or } -\tfrac{1}{3}$$
$$\therefore\ 12x^2+x-1=(4x-1)(3x+1)$$

14.

$$3x^2-14x-24=0$$
$$\therefore\ x=6, \text{ or } -\tfrac{4}{3}$$
$$\therefore\ 3x^2-14x-24=(x-6)(3x+4)$$

15.

$$x^2-2x-2=0$$
$$\therefore\ x=1\pm\sqrt{3}$$
$$\therefore\ x^2-2x-2=(x-1-\sqrt{3})(x-1+\sqrt{3})$$

16.
$$x^2 - 2x + 3 = 0$$
$$\therefore x = 1 + \sqrt{-2}, \text{ or } 1 - \sqrt{-2}$$
$$\therefore x^2 - 2x + 3 = (x - 1 - \sqrt{-2})(x - 1 + \sqrt{-2})$$

17.
$$x^2 + x + 1 = 0$$
$$\therefore x = \frac{-1 \pm \sqrt{-3}}{2}$$
$$\therefore x^2 + x + 1 = \left(x + \frac{1 - \sqrt{-3}}{2}\right)\left(x + \frac{1 + \sqrt{-3}}{2}\right)$$

18.
$$x^2 - 2x + 9 = 0$$
$$\therefore x = 1 \pm \sqrt{-8}$$
$$\therefore x^2 - 2x + 9 = (x - 1 - \sqrt{-8})(x - 1 + \sqrt{-8})$$

Exercise 105. Page 303.

1. Find the duplicate of the ratio 3 : 4.

$$3^2 : 4^2 = 9 : 16$$

2. Find the ratio compounded of the ratios 2 : 3, 3 : 4, 6 : 7, 14 : 8.

$$2 \times 3 \times 6 \times 14 : 3 \times 4 \times 7 \times 8 = 3 : 4$$

3. Find a third proportional to 21 and 28.

$$21 : 28 = 28 : x$$
$$21x = 28^2$$
$$\therefore x = \frac{28^2}{21} = \frac{112}{3}$$

4. Find a mean proportional between 6 and 24.

$$6 : x = x : 24$$
$$x^2 = 144$$
$$\therefore x = 12$$

5. Find a fourth proportional to 3, 5, and 42.

$$3 : 5 = 42 : x$$
$$3x = 210$$
$$\therefore x = 70$$

6. Find x if $5 + x : 11 - x = 3 : 5$.

$$5 + x : 11 - x = 3 : 5$$
$$25 + 5x = 33 - 3x$$
$$8x = 8$$
$$\therefore x = 1$$

7. Find the number which must be added to both the terms of the ratio 3 : 5, in order that the resulting ratio may be equal to the ratio 15 : 16.

$$3 + x : 5 + x = 15 : 16$$
$$48 + 16x = 75 + 15x$$
$$\therefore x = 27$$

If $a : b = c : d$, show that

8. $ao : bd = c^2 : d^2$.

$$a : b = c : d$$
$$\therefore \frac{a}{b} = \frac{c}{d}$$
$$\frac{ac}{bd} = \frac{c^2}{d^2}$$
$$\therefore ac : bd = c^2 : d^2$$

If $a:b=c:d$, show that

9. $ab:cd=a^2:c^2$.

$$\frac{a}{b}=\frac{c}{d}$$

$$\therefore\ \frac{a}{c}=\frac{b}{d}$$

$$\frac{ab}{cd}=\frac{a^2}{c^2}$$

$$\therefore\ ab:cd=a^2:c^2$$

10. $a^2-b^2:c^2-d^2=a^2:c^2$

$$\frac{a}{b}=\frac{c}{d}$$

$$\therefore\ \frac{a^2}{b^2}=\frac{c^2}{d^2}$$

$$\frac{b^2}{a^2}=\frac{d^2}{c^2}$$

$$1-\frac{b^2}{a^2}=1-\frac{d^2}{c^2}$$

$$\frac{a^2-b^2}{a^2}=\frac{c^2-d^2}{c^2}$$

$$\frac{a^2-b^2}{c^2-d^2}=\frac{a^2}{c^2}$$

$$\therefore\ a^2-b^2:c^2-d^2=a^2:c^2$$

11. $2a+b:2c+d=b:d$.

$$\frac{a}{b}=\frac{c}{d}$$

$$\therefore\ \frac{2a}{b}=\frac{2c}{d}$$

$$\frac{2a}{b}+1=\frac{2c}{d}+1$$

$$\frac{2a+b}{b}=\frac{2c+d}{d}$$

$$\frac{2a+b}{2c+d}=\frac{b}{d}$$

$$\therefore\ 2a+b:2c+d=b:d$$

12. $5a-b:5c-d=a:c$.

$$\frac{a}{b}=\frac{c}{d}$$

$$\frac{5a}{b}=\frac{5c}{d}$$

$$\frac{5a}{b}-1=\frac{5c}{d}-1$$

$$\frac{5a-b}{b}=\frac{5c-d}{d}$$

$$\frac{5a-b}{5c-d}=\frac{b}{d}=\frac{a}{c}$$

$$\therefore\ 5a-b:5c-d=a:c$$

13. $a-3b:a+3b=c-3d:c+3d$.

$$\frac{a}{b}=\frac{c}{d}$$

$$\frac{a}{b}-3=\frac{c}{d}-3$$

$$\frac{a-3b}{b}=\frac{c-3d}{d} \qquad (1)$$

Also, $$\frac{a}{b}+3=\frac{c}{d}+3$$

$$\frac{a+3b}{b}=\frac{c+3d}{d} \qquad (2)$$

Divide (1) by (2), $$\frac{a-3b}{a+3b}=\frac{c-3d}{c+3d}$$

$$\therefore\ a-3b:a+3b=c-3d:c+3d$$

14. $a^2 + ab + b^2 : a^2 - ab + b^2 = c^2 + cd + d^2 : c^2 - cd + d^2.$

$$\frac{a}{b} = \frac{c}{d}$$

$$\frac{a^3}{b^3} = \frac{c^3}{d^3}$$

$$\frac{a^3}{b^3} - 1 = \frac{c^3}{d^3} - 1$$

$$\frac{a^3 - b^3}{b^3} = \frac{c^3 - d^3}{d^3} \qquad (1)$$

Similarly, $$\frac{a^3 + b^3}{b^3} = \frac{c^3 + d^3}{d^3} \qquad (2)$$

Divide (1) by (2), $$\frac{a^3 - b^3}{a^3 + b^3} = \frac{c^3 - d^3}{c^3 + d^3} \qquad (3)$$

Also, $$\frac{a - b}{a + b} = \frac{c - d}{c + d} \qquad (4)$$

Divide (3) by (4), $$\frac{a^2 + ab + b^2}{a^2 - ab + b^2} = \frac{c^2 + cd + d^2}{c^2 - cd + d^2}$$

$$\therefore a^2 + ab + b^2 : a^2 - ab + b^2 = c^2 + cd + d^2 : c^2 - cd + d^2$$

Find x in the proportion

15. $45 : 68 = 90 : x$

$45\,x = 90 \times 68$

$\therefore x = 136$

6. $6 : 3 = x : 7$

$3\,x = 42$

$\therefore x = 14$

17. $x : 1\tfrac{1}{2} = 1\tfrac{5}{7} : 1\tfrac{4}{5}$

$x : \tfrac{3}{2} = \tfrac{12}{7} : \tfrac{9}{5}$

$\frac{9\,x}{5} = \frac{18}{7}$

$\therefore x = \tfrac{10}{7}$

18. $3 : x = 7 : 42$

$7\,x = 126$

$\therefore x = 18$

19. Find two numbers in the ratio 2 : 3, the sum of whose squares is 325.

Let $x =$ the smaller number,

$y =$ the greater number.

Then, $$x : y = 2 : 3 \qquad (1)$$

$$x^2 + y^2 = 325 \qquad (2)$$

From (1), $$3\,x = 2\,y$$

$$x = \frac{2\,y}{3}$$

Substitute in (2), $\frac{4y^2}{9} + y^2 = 325$

$$y^2 = 225$$
$$\therefore\ y = \pm 15$$
$$x = \pm 10$$

Therefore the numbers are ± 10 and ± 15.

20. Find two numbers in the ratio $5:3$, the difference of whose squares is 400.

Let $x =$ the smaller number,

$y =$ the larger number.

Then, $x : y = 3 : 5$ (1)

$y^2 - x^2 = 400$ (2)

From (1), $5x = 3y$

$$x = \tfrac{3}{5}y$$

Substitute in (2), $y^2 - \frac{9}{25}y^2 = 400$

$$\tfrac{16}{25}y^2 = 400$$
$$y^2 = 625$$
$$\therefore\ y = \pm 25$$
$$x = \pm 15$$

Therefore the numbers are ± 15 and ± 25.

21. Find three numbers which are to each other as $2:3:5$, such that half the sum of the greatest and least exceeds the other by 25.

Let $2x =$ the smallest number,

$3x =$ the middle number,

$5x =$ the greatest number.

Then, $\frac{2x + 5x}{2} = 3x + 25$

$$\therefore\ x = 50$$

Therefore the numbers are 100, 150, and 250.

22. Find x if $6x - a : 4x - b = 3x + b : 2x + a$.

$$6x - a : 4x - b = 3x + b : 2x + a$$
$$(6x - a)(2x + a) = (4x - b)(3x + b)$$
$$12x^2 + 4ax - a^2 = 12x^2 + bx - b^2$$
$$(4a - b)x = a^2 - b^2$$
$$\therefore\ x = \frac{a^2 - b^2}{4a - b}$$

23. Find x and y from the proportionals

$$x : y = x + y : 42 ;\quad x : y = x - y : 6.$$

$$x : y = x + y : 42 \qquad (1)$$

$$x : y = x - y : 6 \qquad (2)$$

Add (1) and (2), $2x : 2y = 2x : 48$

$$x : y = x : 24$$

$$\therefore\ y = 24$$

From (2), $x : 24 :: x - 24 : 6$

or, $x : 4 :: x - 24 : 1$

$$\therefore\ x = 32$$

24. Find x and y from the proportionals

$$2x + y : y = 3y : 2y - x ;\quad 2x + 1 : 2x + 6 = y : y + 2.$$

$$2x + y : y = 3y : 2y - x \qquad (1)$$

$$2x + 1 : 2x + 6 = y : y + 2 \qquad (2)$$

From (1), $(2x + y)(2y - x) = 3y^2$

$$3xy - 2x^2 + 2y^2 = 3y^2$$

$$2x^2 - 3xy + y^2 = 0$$

$$(2x - y)(x - y) = 0$$

$$\therefore\ y = 2x, \text{ or } x$$

From (2), $(2x + 1)(y + 2) = (2x + 6)y$

$$2xy + 4x + y + 2 = 2xy + 6y$$

$$4x - 5y = -2$$

$$4x - 10x = -2, \text{ or } 4x - 5x = -2$$

$$\therefore\ x = \tfrac{1}{3}, \text{ or } 2$$

$$y = \tfrac{2}{3}, \text{ or } 2$$

25. If $\dfrac{a+b+c+d}{a+b-c-d} = \dfrac{a-b+c-d}{a-b-c+d}$ show that $\dfrac{a}{b} = \dfrac{c}{d}$.

$$\frac{a+b+c+d}{a+b-c-d} = \frac{a-b+c-d}{a-b-c+d}$$

Add the two terms of each fraction for a new numerator and subtract each denominator from its numerator for a new denominator.

$$\frac{2(a+b)}{2(c+d)} = \frac{2(a-b)}{2(c-d)}$$

$$\frac{a+b}{c+d} = \frac{a-b}{c-d}$$

$$\frac{a+b}{a-b} = \frac{c+d}{c-d}$$

Repeat the same process.

$$\frac{2a}{2b}=\frac{2c}{2d}$$

$$\frac{a}{b}=\frac{c}{d}$$

Exercise 106. Page 308.

1. If $x \propto y$, and if $y=3$ when $x=5$, find x when y is 5.

$$x=my$$

$$5=m\times 3$$

$$\therefore m=\tfrac{5}{3}$$

$$x=\frac{5y}{3}$$

$$x=\frac{5\times 5}{3}$$

$$\therefore x=8\tfrac{1}{3}$$

2. If W varies inversely as P, and W is 4 when P is 15, find W when P is 12.

$$W=\frac{m}{P}$$

$$4=\frac{m}{15}$$

$$\therefore m=60$$

$$W=\frac{60}{P}$$

$$W=\tfrac{60}{12}$$

$$\therefore W=5$$

3. If $x \propto y$ and $y \propto z$, show that $xz \propto y^2$.

$$x=my \qquad (1)$$

$$y=m'z$$

$$\therefore z=\frac{y}{m'} \qquad (2)$$

From (1) and (2),

$$xz=\frac{my^2}{m'}$$

$$\therefore xz \propto y^2$$

4. If $x \propto \frac{1}{y}$ and $y \propto \frac{1}{z}$, show that $x \propto z$.

$$x=\frac{m}{y} \qquad (1)$$

$$y=\frac{m'}{z}$$

Put this value of y in (1),

$$x=\frac{mz}{m'}$$

$$\therefore x \propto z$$

5. If x varies inversely as y^2-1, and is equal to 24 when $y=10$, find x when $y=5$.

$$x=\frac{m}{y^2-1}$$

$$24=\frac{m}{100-1}$$

$$\therefore m=99\times 24$$

$$=2376$$

$$\therefore x=\frac{2376}{y^2-1}$$

$$\therefore x=\tfrac{2376}{24}$$

$$=99$$

6. If x varies as $\frac{1}{y}+\frac{1}{z}$, and is equal to 3 when $y=1$ and $z=2$, show that $xyz=2(y+z)$.

$$x=m\left(\frac{1}{y}+\frac{1}{z}\right)$$

$$3=m(\tfrac{1}{1}+\tfrac{1}{2})$$

$$\therefore\ m = 2$$

$$x = 2\left(\frac{1}{y} + \frac{1}{z}\right)$$

$$= \frac{2(y+z)}{yz}$$

$$\therefore\ xyz = 2(y+z)$$

7. If $x - y$ varies inversely as $z + \frac{1}{z}$, and $x + y$ varies inversely as $z - \frac{1}{z}$, find the relation between x and z if $x = 1$, $y = 3$, when $z = \frac{1}{2}$.

$$x - y = \frac{m}{z + \frac{1}{z}}$$

$$x + y = \frac{m'}{z - \frac{1}{z}}$$

$$1 - 3 = \frac{m}{\frac{1}{2} + 2}$$

$$\therefore\ m = -5$$

$$1 + 3 = \frac{m'}{\frac{1}{2} - 2}$$

$$m' = -6$$

$$\therefore\ x - y = \frac{-5}{z + \frac{1}{z}}$$

$$x + y = \frac{-6}{z - \frac{1}{z}}$$

$$\text{Add, } 2x = \frac{-11z^3 - z}{z^4 - 1}$$

$$\therefore\ x = \frac{-11z^3 - z}{2z^4 - 2}$$

8. The area of a circle varies as the square of its radius, and the area of a circle whose radius is 1 foot is 3.1416 square feet. Find the area of a circle whose radius is 20 feet.

Let $x =$ the area of a circle of radius r.

Then, $x = mr^2$

$$3.1416 = m$$

$$\therefore\ x = 3.1416\,r^2$$

$$x = 3.1416 \times 400$$

$$= 1256.64$$

Therefore the area is 1256.64 square feet.

9. The volume of a sphere varies as the cube of its radius, and the volume of a sphere whose radius is 1 foot is 4.188 cubic feet. Find the volume of a sphere whose radius is 2 feet.

Let $x =$ the volume of a sphere of radius r.

Then, $x = mr^3$

$$4.188 = m$$

$$\therefore\ x = 4.188\,r^3$$

$$x = 4.188 \times 8$$

$$= 33.504$$

Therefore the volume is 33.504 cubic feet.

10. If a sphere of given material 3 inches in diameter weighs 24 lbs., how much will a sphere of the same material weigh if its diameter is 5 inches?

The weights of spheres are proportional to the cubes of their diameter.

$$W = mr^3$$

$$24 = m\,27$$

$$\therefore\ m = \tfrac{8}{9}$$
$$\therefore\ W = \tfrac{8}{9} r^3$$
$$W = \tfrac{8}{9}\,125$$
$$= 111\tfrac{1}{9}$$

Therefore the sphere will weigh $111\frac{1}{9}$ pounds.

11. The *velocity* of a falling body varies as the time during which it has fallen from rest. If the velocity of a falling body at the end of 2 seconds is 64 feet, what is its velocity at the end of 8 seconds?

$$V = mt$$
$$64 = m \times z$$
$$\therefore\ m = 32$$
$$V = 32t$$
$$V = 32 \times 8$$
$$= 256$$

Therefore the required velocity will be 256 feet per second.

12. The *distance* a body falls from rest varies as the square of the time it is falling. If a body falls through 144 feet in 3 seconds, how far will it fall in 5 seconds?

$$d = mt^2$$
$$144 = m \times 9$$
$$m = 16$$
$$d = 16t^2$$
$$d = 16 \times 25$$
$$= 400$$

Therefore the body will fall 400 feet in 5 seconds.

13. Compare the volume of two cones, one of which is twice as high as the other, but with one-half its diameter.

$$V = mhr^2$$
$$66 = m \times 7 \times 9$$
$$\therefore\ m = \tfrac{22}{21}$$
$$V_1 = \tfrac{22}{21} hr^2$$
$$V_2 = \tfrac{22}{21} \times 2h \times \left(\frac{r}{2}\right)^2$$
$$= \tfrac{11}{21} hr^2$$
$$\therefore\ \frac{V_1}{V_2} = \frac{1}{2}$$

14. Find the volume of a cone 9 feet high with a base whose radius is 3 feet.

$$V = \tfrac{22}{21} hr^2$$
$$= \tfrac{22}{21} \times 9 \times 9$$
$$= 84\tfrac{6}{7} \text{ cu. ft.}$$

15. Find the volume of a cone 7 feet high with a base whose radius is 4 feet.

$$V = \tfrac{22}{21} hr^2$$
$$= \tfrac{22}{21} \times 7 \times 16$$
$$= 117\tfrac{1}{3} \text{ cu. ft.}$$

16. Find the volume of a cone 9 feet high with a base whose radius is 4 feet.

$$V = \tfrac{22}{21} hr^2$$
$$= \tfrac{22}{21} \times 9 \times 16$$
$$= 150\tfrac{6}{7} \text{ cu. ft.}$$

17. The volume of a sphere varies as the cube of its radius. If the volume is $179\frac{2}{3}$ cubic feet when the radius is $3\frac{1}{2}$ feet, find the volume when the radius is 1 foot 6 inches.

$$V = mr^3$$
$$179\tfrac{2}{3} = m(\tfrac{7}{2})^3$$
$$\tfrac{539}{3} = m\tfrac{343}{8}$$
$$\therefore\ m = \tfrac{88}{21}$$
$$\therefore\ V = \tfrac{88}{21}r^3$$
$$V = \tfrac{88}{21}(\tfrac{3}{2})^3$$
$$= \frac{88 \times 27}{21 \times 8}$$
$$= \tfrac{99}{7}$$

Therefore the required volume is $14\tfrac{1}{7}$ cubic feet.

18. Find the radius of a sphere whose volume is the sum of the volumes of two spheres with radii $3\tfrac{1}{2}$ feet and 6 feet respectively.

$$V = \tfrac{88}{21}r^3$$
$$V_1 = \tfrac{88}{21}(\tfrac{7}{2})^3$$
$$V_2 = \tfrac{88}{21}(6)^3$$
$$V_3 = V_1 + V_2 = \tfrac{88}{21}[(\tfrac{7}{2})^3 + 6^3]$$
$$\tfrac{88}{21}r_3{}^3 = \tfrac{88}{21}[(\tfrac{7}{2})^3 + 6^3]$$
$$\therefore\ r_3{}^3 = (\tfrac{7}{2})^3 + 6^3$$
$$= \tfrac{2071}{8}$$
$$r_3 = \tfrac{1}{2}\sqrt[3]{2071}$$

Therefore the required radius is 6.37 feet.

19. The distance of the offing at sea varies as the square root of the height of the eye above the sea-level, and the distance is 3 miles when the height is 6 feet. Find the distance when the height is 24 feet.

$$D = m\sqrt{h}$$
$$3 = m\sqrt{6}$$
$$\therefore\ m = \frac{3}{\sqrt{6}} = \frac{\sqrt{6}}{2}$$
$$D = \frac{\sqrt{6}}{2}\sqrt{h}$$
$$D = \frac{\sqrt{6}}{2}\sqrt{24}$$
$$= 6$$

Therefore the required distance is 6 miles.

Exercise 107. Page 314.

Find l and s, if

1. $a = 7,\ d = 4,\ n = 13.$

$$l = 7 + 12 \times 4$$
$$= 55$$
$$s = \tfrac{13}{2}(7 + 55)$$
$$= 403$$

2. $a = 5,\ d = 3,\ n = 12.$

$$l = 5 + 11 \times 3$$
$$= 38$$
$$s = \tfrac{12}{2}(5 + 38)$$
$$= 258$$

3. $a = \tfrac{1}{2},\ d = 6,\ n = 30.$

$$l = \tfrac{1}{2} + 29 \times 6$$
$$= 174\tfrac{1}{2}$$
$$s = \tfrac{30}{2}(\tfrac{1}{2} + 174\tfrac{1}{2})$$
$$= 2625$$

4. $a = 63,\ d = -5,\ n = 8.$

$$l = 63 + 7 \times (-5)$$
$$= 28$$
$$s = \tfrac{8}{2}(63 + 28)$$
$$= 364$$

5. $a = \frac{4}{7},\ d = \frac{5}{7},\ n = 15.$

$$\begin{aligned} l &= \tfrac{4}{7} + 14 \times \tfrac{5}{7} \\ &= 10\tfrac{4}{7} \\ s &= \tfrac{15}{2}(\tfrac{4}{7} + 10\tfrac{4}{7}) \\ &= \tfrac{585}{7} \end{aligned}$$

6. $a = 3n,\ d = 2n,\ n = 36.$

$$\begin{aligned} l &= 3n + 35 \times (2n) \\ &= 73n \\ s &= \tfrac{36}{2}(3n + 73n) \\ &= 1368n \end{aligned}$$

Find d and s, if

7. $a = 2,\ l = 134,\ n = 13.$

$$\begin{aligned} 134 &= 2 + 12d \\ \therefore\ d &= 11 \\ s &= \tfrac{13}{2}(2 + 134) \\ &= 884 \end{aligned}$$

8. $a = 0,\ l = 200,\ n = 51.$

$$\begin{aligned} 200 &= 0 + 50d \\ \therefore\ d &= 4 \\ s &= \tfrac{51}{2}(0 + 200) \\ &= 5100 \end{aligned}$$

9. $a = 169,\ l = 8,\ n = 24.$

$$\begin{aligned} 8 &= 169 + 23d \\ \therefore\ d &= -7 \\ s &= \tfrac{24}{2}(169 + 8) \\ &= 2124 \end{aligned}$$

10. $a = 5a^3,\ l = 145a^3,\ n = 21$

$$\begin{aligned} 145a^3 &= 5a^3 + 20d \\ \therefore\ d &= 7a^3 \\ s &= \tfrac{21}{2}(5a^3 + 145a^3) \\ &= 1575a^3 \end{aligned}$$

Insert eight arithmetical means between

11. 13 and 76.

$a = 13,\ l = 76,\ n = 10$

$$\begin{aligned} \therefore\ 76 &= 13 + 9d \\ \therefore\ d &= 7 \end{aligned}$$

∴ the series is 13, 20, 27, 34, 41, ··· 76.

12. 1 and 2.

$a = 1,\ l = 2,\ n = 10$

$$\begin{aligned} 2 &= 1 + 9d \\ \therefore\ d &= \tfrac{1}{9} \end{aligned}$$

∴ the series is 1, $\frac{10}{9}$, $\frac{11}{9}$, ··· 2.

13. $\frac{1}{2}$ and $\frac{3}{4}$.

$a = \frac{1}{2},\ l, = \frac{3}{4},\ n = 10$

$$\begin{aligned} \tfrac{3}{4} &= \tfrac{1}{2} + 9d \\ \therefore\ d &= \tfrac{1}{36} \end{aligned}$$

∴ the series is $\frac{1}{2}$, $\frac{19}{36}$, $\frac{5}{9}$ ··· $\frac{3}{4}$.

14. 47 and 2.

$a = 47,\ l = 2,\ n = 10$

$$\begin{aligned} 2 &= 47 + 9d \\ \therefore\ d &= -5 \end{aligned}$$

∴ the series is 47, 42, 37, ··· 2.

Find a and s, if

15. $d = 7,\ l = 149,\ n = 22.$

$$\begin{aligned} 149 &= a + 21 \times 7 \\ \therefore\ a &= 2 \\ s &= \tfrac{22}{2}(2 + 149) \\ &= 1661 \end{aligned}$$

16. $d = 21,\ l = 242,\ n = 12.$

$$\begin{aligned} 242 &= a + 11 \times 21 \\ \therefore\ a &= 11 \\ s &= \tfrac{12}{2}(11 + 242) \\ &= 1518 \end{aligned}$$

Find n and s, if

17. $a = 17, l = 350, d = 9.$

$$\begin{aligned} 350 &= 17 + 9(n-1) \\ \therefore\ n &= 38 \\ s &= \tfrac{38}{2}(17 + 350) \\ &= 6973 \end{aligned}$$

18. $a = 34, l = 10, d = -2.$

$$\begin{aligned} 10 &= 34 + (n-1)(-2) \\ n &= 13 \\ s &= \tfrac{13}{2}(34 + 10) \\ &= 286 \end{aligned}$$

Find d and n, if

19. $a = 1\tfrac{1}{2}, l = 54, s = 999.$

$$\begin{aligned} 999 &= \frac{n}{2}(1\tfrac{1}{2} + 54) \\ \therefore\ n &= 36 \\ 54 &= 1\tfrac{1}{2} + 35d \\ \therefore\ d &= \tfrac{3}{2} \end{aligned}$$

20. $a = 2, l = 87, s = 801.$

$$\begin{aligned} 801 &= \frac{n}{2}(2 + 87) \\ \therefore\ n &= 18 \\ 87 &= 2 + 17d \\ \therefore\ d &= 5 \end{aligned}$$

Find d and l, if

21. $a = 10; n = 14, s = 1050.$

$$\begin{aligned} 1050 &= \tfrac{14}{2}(10 + l) \\ \therefore\ l &= 140 \\ 140 &= 10 + 13d \\ \therefore\ d &= 10 \end{aligned}$$

22. $a = 1, n = 20, s = 305.$

$$\begin{aligned} 305 &= \tfrac{20}{2}(1 + l) \\ \therefore\ l &= 29\tfrac{1}{2} \\ 29\tfrac{1}{2} &= 1 + 19d \\ \therefore\ d &= 1\tfrac{1}{2} \end{aligned}$$

Find a and d, if

23. $l = 21,\ n = 7,\ s = 105.$

$$\begin{aligned} 105 &= \tfrac{7}{2}(a + 21) \\ \therefore\ a &= 9 \\ 21 &= 9 + 6d \\ \therefore\ d &= 2 \end{aligned}$$

24. $l = 105,\ n = 16,\ s = 840.$

$$\begin{aligned} 840 &= \tfrac{16}{2}(a + 105) \\ \therefore\ a &= 0 \\ 105 &= a + 15d \\ \therefore\ d &= 7 \end{aligned}$$

Find a and l, if

25. $n = 21,\ d = 4,\ s = 1197.$

$$\begin{aligned} 1197 &= \tfrac{21}{2}(2a + 20 \times 4) \\ \therefore\ a &= 17 \\ l &= 17 + 20 \times 4 \\ &= 97 \end{aligned}$$

26. $n = 25,\ s = -75,\ d = \tfrac{1}{2}.$

$$\begin{aligned} -75 &= \tfrac{25}{2}(2a + 24 \times \tfrac{1}{2}) \\ \therefore\ a &= -9 \\ l &= -9 + 24 \times \tfrac{1}{2} \\ &= 3 \end{aligned}$$

Find n and l, if

27. $s = 636,\ a = 9,\ d = 8.$

$$\begin{aligned} 636 &= \frac{n}{2}[18 + (n-1)8] \\ 636 &= n(9 + 4n - 4) \\ 636 &= 4n^2 + 5n \\ \therefore\ n &= 12, \text{ or } -\tfrac{53}{4} \\ l &= 9 + 11 \times 8 \\ &= 97 \end{aligned}$$

28. $s = 798,\ a = 18,\ d = 6.$

$$\begin{aligned} 798 &= \frac{n}{2}[36 + (n-1)6] \\ 798 &= n(18 + 3n - 3) \\ 798 &= 3n^2 + 15n \\ \therefore\ n &= 14, \text{ or } -19 \\ l &= 18 + 13 \times 6 \\ &= 96 \end{aligned}$$

Find a and n, if

29. $s = 623,\ d = 5,\ l = 77.$

$$\begin{aligned} 77 &= a + (n-1)5 \\ a + 5n &= 82 \\ 623 &= \frac{n}{2}(a + 77) \\ &= \frac{n}{2}(82 - 5n + 77) \\ &= \frac{n}{2}(159 - 5n) \\ \therefore\ 1246 &= 159n - 5n^2 \\ \therefore\ n &= 14, \text{ or } \tfrac{89}{5} \\ a &= 82 - 5n \\ &= 12 \end{aligned}$$

30. $s = 1008,\ d = 4,\ l = 88.$

$$\begin{aligned} 88 &= a + (n-1)4 \\ a + 4n &= 92 \\ 1008 &= \frac{n}{2}(a + 88) \\ 1008 &= \frac{n}{2}(92 - 4n + 88) \\ &= \frac{n}{2}(180 - 4n) \\ &= 90n - 2n^2 \\ n^2 - 45n + 504 &= 0 \\ \therefore\ n &= 21, \text{ or } 24 \\ a &= 92 - 4n \\ &= 8, \text{ or } -4 \end{aligned}$$

31. Find the arithmetical series in which the 15th term is 25, and the 29th term 46.

$$\begin{aligned} 25 &= a + 14d \\ 46 &= a + 28d \\ \hline 21 &= \quad 14d \\ \therefore\ d &= \tfrac{3}{2} \\ a &= 4 \end{aligned}$$

Therefore the series is $4,\ 5\tfrac{1}{2},\ 7,\ \cdots$.

How many terms must be taken of

32. The series $-16,\ -15,\ -14, \cdots$ to make -100?

$$a = -16,\ d = 1,\ s = -100$$

$$-100 = \frac{n}{2}(-32 + n - 1)$$

$$-100 = \frac{n}{2}(n - 33)$$

$$-200 = n^2 - 33n$$

$$\therefore\ n = 8, \text{ or } 25$$

33. The series 20, $18\frac{3}{4}$, $17\frac{1}{2}$, ⋯ to make $162\frac{1}{2}$?

$$a = 20,\ d = -\tfrac{5}{4},\ s = 162\tfrac{1}{2}$$

$$162\tfrac{1}{2} = \frac{n}{2}[40 + (n-1)(-\tfrac{5}{4})]$$

$$325 = n\left(40 - \frac{5n}{4} + \frac{5}{4}\right)$$

$$325 = -\frac{5n^2}{4} + \frac{165n}{4}$$

$$5n^2 - 165n + 1300 = 0$$

$$n^2 - 33n + 260 = 0$$

$$\therefore\ n = 13, \text{ or } 20$$

34. The sum of three numbers in arithmetical progression is 9, and the sum of their squares is 29. Find the numbers.

Let $a - d$, a, $a + d$ be the three numbers.

Then, $$3a = 9 \tag{1}$$

$$\therefore\ a = 3$$

$$(a-d)^2 + a^2 + (a+d)^2 = 29 \tag{2}$$

From (2), $$3a^2 + 2d^2 = 29$$

$$27 + 2d^2 = 29$$

$$\therefore\ d = \pm 1$$

Therefore the numbers are 2, 3, and 4.

35. The sum of three numbers in arithmetical progression is 12, and their product is 60. Find the numbers.

Let $a - d$, a, $a + d$ be the three numbers.

Then, $$3a = 12 \tag{1}$$

$$(a-d)a(a+d) = 60 \tag{2}$$

From (1), $$a = 4$$

From (2), $$a(a^2 - d^2) = 60$$

$$4(16 - d^2) = 60$$

$$\therefore\ d = \pm 1$$

Therefore the numbers are 3, 4, and 5.

Exercise 108. Page 320.

Find l and s, if

1. $a=4, r=2, n=7.$

$$l=4\times2^6=256$$

$$s=\frac{4(2^7-1)}{2-1}$$

$$=508$$

2. $a=9, r=\frac{1}{3}, n=11.$

$$l=9(\tfrac{1}{3})^{10}=\frac{1}{3^8}$$

$$s=\frac{9[(\frac{1}{3})^{11}-1]}{\frac{1}{3}-1}$$

$$=\frac{27}{2}\left(1-\frac{1}{3^{11}}\right)$$

$$=\frac{1}{2}\left(\frac{3^{11}-1}{3^8}\right)$$

Find r and s, if

3. $a=1, n=4, l=64.$

$$64=r^3$$

$$\therefore\ r=4$$

$$s=\frac{4\times64-1}{3}$$

$$=85$$

4. $a=7, n=8, l=896.$

$$896=7r^7$$

$$r^7=128$$

$$\therefore\ r=2$$

$$s=\frac{2\times896-7}{2-1}$$

$$=1785$$

5. Insert 1 geometrical mean between 14 and 686.

$$a=14,\ n=3,\ l=686$$

$$686=14r^2$$

$$r^2=49$$

$$\therefore\ r=7$$

$\therefore$ the series is 14, 98, 686.

6. Insert 3 geometrical means between 31 and 496.

$$a=31,\ n=5,\ l=496$$

$$496=31r^4$$

$$r^4=16$$

$$\therefore\ r=2$$

$\therefore$ the series is 31, 62, 124, 248, 496.

7. Find a and s, if

$l=128,\ r=2,\ n=7.$

$$128=a\times2^6$$

$$\therefore\ a=2$$

$$s=\frac{2\times128-2}{2-1}$$

$$=254$$

8. Find s and n, if

$a=9,\ l=2304,\ r=2.$

$$s=\frac{2\times2304-9}{2-1}$$

$$=4599$$

$$2304=9\times2^{n-1}$$

$$2^{n-1}=256$$

$$\therefore\ n=9$$

9. Find r and n, if $a=2, l=1458, s=2186.$

$$2186=\frac{1458r-2}{r-1}$$

$$2186r-2186=1458r-2$$

$$728r=2184$$

$$\therefore\ r=3$$

$$1458=2\times3^{n-1}$$

$$729=3^{n-1}$$

$$\therefore\ n=7$$

10. If the fifth term is $\frac{8}{9}$ and the 7th term $\frac{32}{81}$, find the series.

$$a = \tfrac{8}{9},\ n = 3,\ l = \tfrac{32}{81}.$$
$$\tfrac{32}{81} = \tfrac{8}{9} \times r^2$$
$$r^2 = \tfrac{4}{9}$$
$$r = \tfrac{2}{3}$$

$\therefore$ the series is $\frac{9}{2}, 3, 2, \frac{4}{3}, \frac{8}{9} \cdots$.

11. Find three numbers in geometrical progression whose sum is 14, and the sum of whose squares 84.

Let a, ar, ar^2 be the number.

Then, $$a + ar + ar^2 = 14 \tag{1}$$
$$a^2 + a^2r^2 + a^2r^4 = 84 \tag{2}$$

From (1), $$a(1 + r + r^2) = 14 \tag{3}$$

From (2), $$a^2(1 + r^2 + r^4) = 84 \tag{4}$$

Divide (4) by (3), $$a(1 - r + r^2) = 6$$

(3) is $$a(1 + r + r^2) = 14$$

Add, $$2a(1 + r^2) = 20$$
$$a(1 + r^2) = 10 \tag{5}$$

Subtract, $$2ar = 8 \tag{6}$$
$$ar = 4$$

Divide (5) by (6), $$\frac{1 + r^2}{r} = \frac{5}{2}$$
$$2r^2 - 5r + 2 = 0$$
$$\therefore r = 2, \text{ or } \tfrac{1}{2}$$
$$a = 2, \text{ or } 8$$

$\therefore$ the numbers are 2, 4, and 8.

Sum to infinity:

12. $1, \frac{1}{2}, \frac{1}{4}, \cdots$.

$$s = \frac{1}{1 - \frac{1}{2}} = 2$$

13. $8, 2, \frac{1}{2}, \cdots$.

$$s = \frac{8}{1 - \frac{1}{4}} = \frac{32}{3}$$

14. $2, \frac{2}{7}, \frac{2}{49}, \cdots$.

$$s = \frac{2}{1 - \frac{1}{7}} = \frac{7}{3}$$

15. $5, 3, \frac{9}{5}, \cdots$.

$$s = \frac{5}{1 - \frac{3}{5}} = \frac{25}{2}$$

16. $1, \frac{1}{4}, \frac{1}{16}, \cdots$.

$$s = \frac{1}{1 - \frac{1}{4}} = \frac{4}{3}$$

17. $3, -2, \frac{4}{3}, \cdots$.

$$s = \frac{3}{1 + \frac{2}{3}} = \frac{9}{5}$$

Find the value of:

18. $$0.1\dot{6} = \frac{1}{10} + \frac{\frac{6}{100}}{1-\frac{1}{10}} = \frac{1}{10} + \frac{6}{90} = \frac{5}{30} = \frac{1}{6}$$

19. $$0.3\dot{7}\dot{8} = \frac{3}{10} + \frac{\frac{78}{1000}}{1-\frac{1}{100}} = \frac{3}{10} + \frac{78}{990} = \frac{3}{10} + \frac{13}{165} = \frac{125}{330} = \frac{25}{66}$$

20. $$0.\dot{8}\dot{6} = \frac{\frac{86}{100}}{1-\frac{1}{100}} = \frac{86}{99}$$

21. $$0.5\dot{4} = \frac{5}{10} + \frac{\frac{4}{100}}{1-\frac{1}{10}} = \frac{5}{10} + \frac{4}{90} = \frac{49}{90}$$

22. $$0.7\dot{3}\dot{6} = \frac{7}{10} + \frac{\frac{36}{1000}}{1-\frac{1}{100}} = \frac{7}{10} + \frac{4}{110} = \frac{81}{110}$$

23. $$0.3\dot{6}\dot{3} = \frac{3}{10} + \frac{\frac{63}{1000}}{1-\frac{1}{100}} = \frac{3}{10} + \frac{7}{110} = \frac{40}{110} = \frac{4}{11}$$

Exercise 109. Page 322.

1. If a, b, c are in harmonical progression, show that $a-b:b-c = a:c$.

$$\frac{1}{a} + \frac{1}{c} = \frac{2}{b}$$

$$\frac{1}{a} - \frac{1}{b} = \frac{1}{b} - \frac{1}{c}$$

$$\frac{a-b}{ab} = \frac{b-c}{bc}$$

$$\frac{a-b}{a} = \frac{b-c}{c}$$

$$\therefore\ a-b:b-c = a:c$$

2. Show that if the terms of a harmonical series are all multiplied by the same number, the products will form a harmonical progression.

Let $\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}, \cdots$ be the given series,

and $\frac{k}{a}, \frac{k}{a+d}, \frac{k}{a+2d}, \cdots$ be the second series.

The last series may be written

$\frac{1}{\frac{a}{k}}, \frac{1}{\frac{a}{k}+\frac{d}{k}}, \frac{1}{\frac{a}{k}+\frac{2d}{k}}, \cdots$ which is again a harmonical series.

3. The second term of a harmonical series is 2, and the fourth term 6. Find the series.

In the corresponding arithmetical series $a = \frac{1}{2}$, $n = 3$, $l = \frac{1}{6}$.

$$\therefore \frac{1}{6} = \frac{1}{2} + 2d$$
$$d = -\frac{1}{6}$$

$\therefore$ the arithmetical series is $\frac{2}{3}, \frac{1}{2}, \frac{1}{3}, \frac{1}{6}, \cdots$, and the harmonical series is $\frac{3}{2}, 2, 3, 6, \cdots$

4. Insert the harmonical mean between 2 and 3.

The arithmetical mean between $\frac{1}{2}$ and $\frac{1}{3}$ is $\frac{1}{2}(\frac{1}{2} + \frac{1}{3})$, or $\frac{5}{12}$.
Therefore the harmonical mean between 2 and 3 is $\frac{12}{5}$.

5. Insert 2 harmonical means between 1 and $\frac{1}{4}$.

The two arithmetical means between 1 and 4 are 2 and 3.
Therefore the two harmonical means between 1 and $\frac{1}{4}$ are $\frac{1}{2}$ and $\frac{1}{3}$.

6. Insert 5 harmonical means between 1 and $\frac{1}{7}$.

The 5 arithmetical means between 1 and 7 are 2, 3, 4, 5, 6.
Therefore the 5 harmonical means between 1 and $\frac{1}{7}$ are $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}$.

7. The first term of a harmonical progression is 1, and the third term $\frac{1}{3}$. Find the 8th term.

In the corresponding arithmetical progression $a = 1$, $l = 3$, $n = 3$.

$$\therefore 3 = 1 + 2d$$
$$d = 1$$

Therefore the arithmetical series is $1, 2, 3, \cdots$, and the harmonical series is $1, \frac{1}{2}, \frac{1}{3}, \cdots$.

Therefore the 8th term is $\frac{1}{8}$.

8. The first term of a harmonical progression is 1, and the sum of the first three terms is $1\frac{5}{6}$. Find the series.

Let d be the common difference of the corresponding arithmetical series.

Then the harmonical series is

$$1, \frac{1}{1+d}, \frac{1}{1+2d}, \cdots$$

$$\therefore 1 + \frac{1}{1+d} + \frac{1}{1+2d} = 1\tfrac{5}{6}.$$

$$6(1+d)(1+2d) + 6(1+2d) + 6(1+d) = 11(1+d)(1+2d)$$
$$5(1+d)(1+2d) = 6(1+2d+1+d)$$
$$5 + 15d + 10d^2 = 12 + 18d$$
$$10d^2 - 3d = 7$$
$$\therefore d = 1, \text{ or } -\tfrac{7}{10}$$

Therefore the series is $1, \frac{1}{2}, \frac{1}{3}$, etc.

9. If a is the arithmetical mean between b and c, and b the geometrical mean between a and c, show that c is the harmonical mean between a and b.

$$a=\frac{b+c}{2},\quad b=\sqrt{ac}$$

$$\therefore\ 2a=b+c$$

Transpose, $$b=2a-c$$

Multiply by b, $$b^2=2ab-bc$$

But, $$b^2=ac$$

$$\therefore\ ac=2ab-bc$$

$$(a+b)c=2ab$$

$$\therefore\ c=\frac{2ab}{a+b}$$

10. The arithmetical mean between two numbers exceeds the harmonical mean by 1, and twice the square of the arithmetical mean exceeds the sum of the squares of the harmonical and geometrical means by 11. Find the numbers.

Let a and b denote the numbers.

Then, $$\frac{a+b}{2}=\frac{2ab}{a+b}+1 \tag{1}$$

$$2\left(\frac{a+b}{2}\right)^2=\left(\frac{2ab}{a+b}\right)^2+ab+11 \tag{2}$$

From (1), $$(a+b)^2=4ab+2a+2b$$

$$a^2-2ab+b^2=2(a+b) \tag{3}$$

From (2), $$(a+b)^4=8a^2b^2+2ab(a+b)^2+22(a+b)^2$$

$$a^4+2a^3b-6a^2b^2+2ab^3+b^4=22(a+b)^2 \tag{4}$$

Divide (4) by (3), $$a^2+4ab+b^2=11(a+b) \tag{5}$$

From (5) and (3),

$$2(a^2+4ab+b^2)=11(a^2-2ab+b^2)$$

$$9a^2-30ab+9b^2=0$$

$$3a^2-10ab+3b^2=0$$

$$(3a-b)(a-3b)=0$$

$$\therefore\ a=3b,\text{ or }\frac{b}{3}$$

Substitute in (3),

$$9b^2-6b^2+b^2=6b+2b \qquad\text{or,}\qquad \frac{b^2}{9}-\frac{2b^2}{3}+b^2=\frac{2b}{3}+2b$$

$$\therefore\ b=2 \qquad\qquad \therefore\ b=6$$

$$a=6 \qquad\qquad a=2$$

Therefore the numbers are 2 and 6.

Exercise 110. Page 327.

Expand to four terms:

1. $\dfrac{1}{1+2x}=A+Bx+Cx^2+Dx^3+\cdots$

$$\begin{aligned}1&=A+Bx+Cx^2+Dx^3+\cdots\\&\quad+2Ax+2Bx^2+2Cx^3+\cdots\\1&=A+(B+2A)x+(C+2B)x^2+(D+2C)x^3+\cdots\end{aligned}$$

$A=1,\ B+2A=0,\ C+2B=0,\ D+2C=0$, etc.

$A=1,\ B=-2,\ C=4,\ D=-8$, etc.

$\therefore \dfrac{1}{1+2x}=1-2x+4x^2-8x^3+\cdots$

2. $\dfrac{1}{2-3x}=A+Bx+Cx^2+Dx^3+\cdots$

$$\begin{aligned}1&=2A+2Bx+2Cx^2+2Dx^3+\cdots\\&\quad-3Ax-3Bx^2-3Cx^3\cdots\\1&=2A+(2B-3A)x+(2C-3B)x^2+(2D-3C)x^3+\cdots\end{aligned}$$

$2A=1,\ 2B-3A=0,\ 2C-3B=0,\ 2D-3C=0$, etc.

$A=\frac{1}{2},\ B=\frac{3}{4},\ C=\frac{9}{8},\ D=\frac{27}{16}$, etc.

$\therefore \dfrac{1}{2-3x}=\dfrac{1}{2}+\dfrac{3x}{4}+\dfrac{9x^2}{8}+\dfrac{27x^3}{16}+\cdots$

3. $\dfrac{1+x}{2+3x}=A+Bx+Cx^2+Dx^3+\cdots$

$$\begin{aligned}1+x&=2A+2Bx+2Cx^2+2Dx^3+\cdots\\&\quad+3Ax+3Bx^2+3Cx^3+\cdots\\1+x&=2A+(2B+3A)x+(2C+3B)x^2+(2D+3C)x^3+\cdots\end{aligned}$$

$2A=1,\ 2B+3A=1,\ 2C+3B=0,\ 2D+3C=0$, etc.

$A=\frac{1}{2},\ B=-\frac{1}{4},\ C=\frac{3}{8},\ D=-\frac{9}{16}$, etc.

$\therefore \dfrac{1+x}{2+3x}=\dfrac{1}{2}-\dfrac{x}{4}+\dfrac{3x^2}{8}-\dfrac{9x^3}{16}+\cdots$

4. $\dfrac{1-x}{1+x+x^2}=A+Bx+Cx^2+Dx^3+\cdots$

$$\begin{aligned}1-x&=A+Bx+Cx^2+Dx^3+\cdots\\&\quad+Ax+Bx^2+Cx^3+\cdots\\&\quad+Ax^2+Bx^3+\cdots\\1-x&=A+(B+A)x+(C+B+A)x^2+(D+C+B)x^3+\cdots\end{aligned}$$

$A=1,\ B+A=-1,\ C+B+A=0,\ D+C+B=0$, etc.

$A=1,\ B=-2,\ C=1,\ D=1$, etc.

$\therefore \dfrac{1-x}{1+x+x^2}=1-2x+x^2+x^3+\cdots$

5. $\dfrac{5-2x}{1+x-x^2}=A+Bx+Cx^2+Dx^3+\cdots$

$$\begin{aligned}5-2x &= A+Bx+Cx^2+Dx^3+\cdots\\ &\qquad\; +Ax+Bx^2+Cx^3+\cdots\\ &\qquad\qquad\; -Ax^2-Bx^3-\cdots\end{aligned}$$

$$5-2x=A+(B+A)x+(C+B-A)x^2+(D+C--B)x^3+\cdots$$

$$A=5,\ B+A=-2,\ C+B-A=0,\ D+C-B=0,\ \text{etc.}$$

$$A=5,\ B=-7,\ C=12,\ D=-19,\ \text{etc.}$$

$$\therefore\ \frac{5-2x}{1+x-x^2}=5-7x+12x^2-19x^3+\cdots$$

6. $\dfrac{2-3x}{1-2x+3x^2}=A+Bx+Cx^2+Dx^3+\cdots$

$$\begin{aligned}2-3x &= A+Bx+Cx^2+Dx^3+\cdots\\ &\qquad\; -2Ax-2Bx^2-2Cx^3-\cdots\\ &\qquad\qquad\; +3Ax^2+3Bx^3+\cdots\end{aligned}$$

$$2-3x=A+(B-2A)x+(C-2B+3A)x^2+(D-2C+3B)x^3+\cdots$$

$$A=2,\ B-2A=-3,\ C-2B+3A=0,\ D-2C+3B=0.\ \text{etc.}$$

$$A=2,\ B=1,\ C=-4,\ D=-11,\ \text{etc.}$$

$$\therefore\ \frac{2-3x}{1-2x+3x^2}=2+x-4x^2-11x^3+\cdots$$

7. $\dfrac{3+x}{1-x-x^2}=A+Bx+Cx^2+Dx^3+\cdots$

$$\begin{aligned}3+x &= A+Bx+Cx^2+Dx^3+\cdots\\ &\qquad\; -Ax-Bx^2-Cx^3\cdots\\ &\qquad\qquad\; -Ax^2-Bx^3-\cdots\end{aligned}$$

$$3+x=A+(B-A)x+(C-B-A)x^2+(D-C-B)x^3+\cdots$$

$$A=3,\ B-A=1,\ C-B-A=0,\ D-C-B=0,\ \text{etc.}$$

$$A=3,\ B=4,\ C=7,\ D=11,\ \text{etc.}$$

$$\therefore\ \frac{3+x}{1-x-x^2}=3+4x+7x^2+11x^3+\cdots$$

8. $\dfrac{1+x}{1+x+x^2}=A+Bx+Cx^2+Dx^3+\cdots$

$$\begin{aligned}1+x &= A+Bx+Cx^2+Dx^3+\cdots\\ &\qquad\; +Ax+Bx^2+Cx^3+\cdots\\ &\qquad\qquad\; +Ax^2+Bx^3+\cdots\end{aligned}$$

$$1+x=A+(B+A)x+(C+B+A)x^2+(D+C+B)x^3+\cdots$$

$$A=1,\ B+A=1,\ C+B+A=0,\ D+C+B=0,\ \text{etc.}$$

$$A=1,\ B=0,\ C=-1,\ D=1,\ \text{etc.}$$

$$\therefore\ \frac{1+x}{1+x+x^2}=1-x^2+x^3+\cdots$$

9. $\dfrac{1-8x}{1-x-6x^2}=A+Bx+Cx^2+Dx^3+\cdots$

$$\begin{aligned}1-8x&=A+Bx+Cx^2+Dx^3+\cdots\\&\qquad-Ax-Bx^2-Cx^3-\cdots\\&\qquad\qquad-6Ax^2-6Bx^3-\cdots\\1-8x&=A+(B-A)x+(C-B-6A)x^2+(D-C-6B)x^3+\cdots\end{aligned}$$

$$A=1,\ B-A=-8,\ C-B-6A=0,\ D-C-6B=0, \text{ etc.}$$

$$A=1,\ B=-7,\ C=-1,\ D=-43, \text{ etc.}$$

$$\therefore\ \frac{1-8x}{1-x-6x^2}=1-7x-x^2-43x^3-\cdots$$

Expand to five terms:

10. $\dfrac{4}{2+x}=A+Bx+Cx^2+Dx^3+Ex^4+\cdots$

$$\begin{aligned}4&=2A+2Bx+2Cx^2+2Dx^3+2Ex^4+\cdots\\&\qquad+\ \ Ax+Bx^2+Cx^3+Dx^4+\cdots\\4&=2A+(2B+A)x+(2C+B)x^2+(2D+C)x^3+(2E+D)x^4+\cdots\end{aligned}$$

$$2A=4,\ 2B+A=0,\ 2C+B=0,\ 2D+C=0,\ 2E+D=0, \text{ etc.}$$

$$A=2,\ B=-1,\ C=\tfrac{1}{2},\ D=-\tfrac{1}{4},\ E=\tfrac{1}{8}, \text{ etc.}$$

$$\therefore\ \frac{4}{2+x}=2-x+\frac{x^2}{2}-\frac{x^3}{4}+\frac{x^4}{8}-\cdots$$

11. $\dfrac{2-x}{3+x}=A+Bx+Cx^2+Dx^3+Ex^4+\cdots$

$$\begin{aligned}2-x&=3A+3Bx+3Cx^2+3Dx^3+3Ex^4+\cdots\\&\qquad+\ \ Ax+Bx^2+Cx^3+Dx^4+\cdots\\2-x&=3A+(3B+A)x+(3C+B)x^2+(3D+C)x^3+(3E+D)x^4+\cdots\end{aligned}$$

$$3A=2,\ 3B+A=-1,\ 3C+B=0,\ 3D+C=0,\ 3E+D=0, \text{ etc.}$$

$$A=\tfrac{2}{3},\ B=-\tfrac{5}{9},\ C=\tfrac{5}{27},\ D=-\tfrac{5}{81},\ E=\tfrac{5}{243}, \text{ etc.}$$

$$\therefore\ \frac{2-x}{3+x}=\frac{2}{3}-\frac{5x}{9}+\frac{5x^2}{27}-\frac{5x^3}{81}+\frac{5x^4}{243}-\cdots$$

12. $\dfrac{5-2x}{1+3x-x^2}=A+Bx+Cx^2+Dx^3+Ex^4+\cdots$

$$\begin{aligned}5-2x&=A+Bx+Cx^2+Dx^3+Ex^4+\cdots\\&\qquad+3Ax+3Bx^2+3Cx^3+3Dx^4+\cdots\\&\qquad\qquad-Ax^2-Bx^3-Cx^4-\cdots\\5-2x&=A+(B+3A)x^2+(C+3B-A)x^2+(D+3C-B)x^2\\&\qquad+(E+3D-C)x^4-\cdots\end{aligned}$$

$$A=5,\ B+3A=-2,\ C+3B-A=0,$$
$$D+3C-B=0,\ E+3D-C=0, \text{ etc.}$$

$$A=5,\ B=-17,\ C=56,\ D=-185,\ E=611, \text{ etc.}$$

$$\therefore\ \frac{5-2x}{1+3x-x^2}=5-17x+56x^2-185x^3+611x^4-\cdots$$

13. $$\frac{x^2-x+1}{(x-2)} = A + Bx + Cx^2 + Dx^3 + Ex^4 + \cdots$$

$$x^2 - x + 1 = -2A - 2Bx - 2Cx^2 - 2Dx^3 - 2Ex^4 - \cdots$$
$$+ Ax + Bx^2 + Cx^3 + Dx^4 + \cdots$$

$$x^2 - x + 1 = -2A + (A - 2B)x + (B - 2C)x^2 + (C - 2D)x^3$$
$$+ (D - 2E)x^4 + \cdots$$

$$-2A = 1,\ A - 2B = -1,\ B - 2C = 1,\ C - 2D = 0,$$
$$D - 2E = 0, \text{ etc.}$$

$$A = -\tfrac{1}{2},\ B = \tfrac{1}{4},\ C = -\tfrac{3}{8},\ D = -\tfrac{3}{16},\ E = -\tfrac{3}{32}, \text{ etc.}$$

$$\frac{x^2-x+1}{x-2} = -\frac{1}{2} + \frac{x}{4} - \frac{3x^2}{8} - \frac{3x^3}{16} - \frac{3x^4}{32} - \cdots$$

$$\therefore\ \frac{x^2-x+1}{x(x-2)} = -\frac{1}{2x} + \frac{1}{4} - \frac{3x}{8} - \frac{3x^2}{16} - \frac{3x^3}{32} - \cdots$$

14. $$\frac{3x-2}{(x-1)^2} = \frac{3x-2}{x^2-2x+1} = A + Bx + Cx^2 + Dx^3 + Ex^4 + \cdots$$

$$3x - 2 = A + Bx + Cx^2 + Dx^3 + Ex^4 + \cdots$$
$$-2Ax - 2Bx^2 - 2Cx^3 - 2Dx^4 + \cdots$$
$$+ Ax^2 + Bx^3 + Cx^4 + \cdots$$

$$3x - 2 = A + (B - 2A)x + (A - 2B + C)x^2$$
$$+ (B - 2C + D)x^3 + (C - 2D + E)x^4$$

$$A = -2,\ B - 2A = 3,\ A - 2B + C = 0,\ B - 2C + D = 0,$$
$$C - 2D + E = 0$$

$$A = -2,\ B = -1,\ C = 0,\ D = 1,\ E = 2, \text{ etc.}$$

$$\frac{3x-2}{(x-1)^2} = -2 - x + x^3 + 2x^4 + \cdots$$

$$\therefore\ \frac{3x-2}{x(x-1)^2} = -\frac{2}{x} - 1 + x^2 + 2x^3 + \cdots$$

15. $$\frac{x^2-x+1}{(x-1)(x^2+1)} = \frac{x^2-x+1}{x^3-x^2+x-1} = A + Bx + Cx^2 + Dx^3 + Ex^4 + \cdots$$

$$x^2 - x + 1 = -A - Bx - Cx^2 - Dx^3 - Ex^4 - \cdots$$
$$+ Ax + Bx^2 + Cx^3 + Dx^4 + \cdots$$
$$- Ax^2 - Bx^3 - Cx^4 + \cdots$$
$$+ Ax^3 + Bx^4 + \cdots$$

$$x^2 - x + 1 = -A + (A - B)x - (A - B + C)x^2$$
$$+ (A - B + C - D)x^3 + (B - C + D - E)x^4 + \cdots$$

$$-A = 1,\ A - B = -1,\ A - B + C = -1,\ A - B + C - D = 0,$$
$$B - C + D - E = 0, \text{ etc.}$$

$$A = -1,\ B = 0,\ C = 0,\ D = -1,\ E = -1, \text{ etc.}$$

$$\therefore\ \frac{x^2-x+1}{(x-1)(x^2+1)} = -1 - x^3 - x^4 - \cdots$$

Exercise 111. Page 329

Resolve into partial fractions:

1. $$\frac{7x+1}{(x+4)(x-5)}=\frac{A}{x+4}+\frac{B}{x-5}=\frac{Ax-5A+Bx+4B}{(x+4)(x-5)}$$

$$A+B=7 \qquad -5A+4B=1$$

$$A=3 \qquad B=4$$

$$\therefore\ \frac{7x+1}{(x+4)(x-5)}=\frac{3}{x+4}+\frac{4}{x-5}$$

2. $$\frac{7x-1}{(1-2x)(1-3x)}=\frac{A}{1-2x}+\frac{B}{1-3x}=\frac{A-3Ax+B-2Bx}{(1-2x)(1-3x)}$$

$$A+B=-1 \qquad -3A-2B=7$$

$$A=-5, \qquad B=4$$

$$\therefore\ \frac{7x-1}{(1-2x)(1-3x)}=\frac{5}{2x-1}-\frac{4}{3x-1}$$

3. $$\frac{5x-1}{(2x-1)(x-5)}=\frac{A}{2x-1}+\frac{B}{x-5}=\frac{Ax-5A+2Bx-B}{(2x-1)(x-5)}$$

$$A+2B=5, \qquad -5A-B=-1$$

$$A=-\tfrac{1}{3}, \qquad B=\tfrac{8}{3}$$

$$\therefore\ \frac{5x-1}{(2x-1)(x-5)}=-\frac{1}{3(2x-1)}+\frac{8}{3(x-5)}$$

4. $$\frac{x-2}{(x-5)(x+2)}=\frac{A}{x-5}+\frac{B}{x+2}=\frac{Ax+2A+Bx-5B}{(x-5)(x+2)}$$

$$A+B=1, \qquad 2A-5B=-2$$

$$A=\tfrac{3}{7}, \qquad B=\tfrac{4}{7}$$

$$\therefore\ \frac{x-2}{(x-5)(x+2)}=\frac{3}{7(x-5)}+\frac{4}{7(x+2)}$$

5. $$\frac{3}{x^3-1}=\frac{A}{x-1}+\frac{Bx+C}{x^2+x+1}=\frac{Ax^2+Ax+A+Bx^2-Bx+Cx-C}{x^3-1}$$

$$A+B=0, \quad A-B+C=0, \quad A-C=3$$

$$A=1, \qquad B=-1, \qquad C=-2$$

$$\therefore\ \frac{3}{x^3-1}=\frac{1}{x-1}-\frac{x+2}{x^2+x+1}$$

6. $$\frac{x^2-x-3}{x(x^2-4)}=\frac{A}{x}+\frac{B}{x-2}+\frac{C}{x+2}=\frac{Ax^2-4A+Bx^2+2Bx+Cx^2-2Cx}{x(x^2-4)}$$

$$A+B+C=1, \ 2B-2C=-1, \ -4A=-3$$

$$A=\frac{3}{4}, \ B=\frac{-1}{8}, \ C=\frac{3}{8}$$

$$\therefore\ \frac{x^2-x-3}{x(x^2-4)}=\frac{3}{4x}-\frac{1}{8(x-2)}+\frac{3}{8(x+2)}$$

7. $$\frac{3x^2-4}{x^2(x+5)}=\frac{A}{x^2}+\frac{B}{x}+\frac{C}{x+5}=\frac{Ax+5A+Bx^2+5Bx+Cx^2}{x^2(x+5)}$$

$$B+C=3,\ A+5B=0,\ 5A=-4$$

$$A=-\tfrac{4}{5},\ B=\tfrac{4}{25},\ C=\tfrac{71}{25}$$

$$\therefore\ \frac{3x^2-4}{x^2(x+5)}=-\frac{4}{5x^2}+\frac{4}{25x}+\frac{71}{25(x+5)}$$

8. $$\frac{7x^2-x}{(x-1)^2(x+2)}=\frac{A}{(x-1)^2}+\frac{B}{x-1}+\frac{C}{x+2}$$

$$=\frac{Ax+2A+Bx^2+Bx-2B+Cx^2-2Cx+C}{(x-1)^2(x+2)}$$

$$B+C=7,\ A+B-2C=-1,\ 2A-2B+C=0$$

$$A=2,\ B=\tfrac{11}{3},\ C=\tfrac{10}{3}$$

$$\therefore\ \frac{7x^2-x}{(x-1)^2(x+2)}=\frac{2}{(x-1)^2}+\frac{11}{3(x-1)}+\frac{10}{3(x+2)}$$

9. $$\frac{2x^2-7x+1}{x^3-1}=\frac{A}{x-1}+\frac{Bx+C}{x^2+x+1}$$

$$=\frac{Ax^2+Ax+A+Bx^2-Bx+Cx-C}{x^3-1}$$

$$A+B=2,\ A-B+C=-7,\ A-C=1$$

$$A=-\tfrac{4}{3},\ B=\tfrac{10}{3},\ C=-\tfrac{7}{3}$$

$$\therefore\ \frac{2x^2-7x+1}{x^3-1}=-\frac{4}{3(x-1)}+\frac{10x-7}{3(x^2+x+1)}$$

10. $$\frac{7x-1}{(6x+1)(x-1)}=\frac{A}{6x+1}+\frac{B}{x-1}=\frac{Ax-A+6Bx+B}{(6x+1)(x-1)}$$

$$A+6B=7,\ -A+B=-1$$

$$A=\tfrac{13}{7},\qquad B=\tfrac{6}{7}$$

$$\therefore\ \frac{7x-1}{(6x+1)(x-1)}=\frac{13}{7(6x+1)}+\frac{6}{7(x-1)}$$

11. $$\frac{x^2-3}{(x^2+1)(x+2)}=\frac{Ax+B}{x^2+1}+\frac{C}{x+2}$$

$$=\frac{Ax^2+2Ax+Bx+2B+Cx^2+C}{(x^2+1)(x+2)}$$

$$A+C=1,\ 2A+B=0,\ 2B+C=-3$$

$$A=\tfrac{4}{5},\ B=-\tfrac{8}{5},\ C=\tfrac{1}{5}$$

$$\therefore\ \frac{x^2-3}{(x^2+1)(x+2)}=\frac{4x-8}{5(x^2+1)}+\frac{1}{5(x+2)}$$

12. $$\frac{x^2-x+1}{(x^2+1)(x-1)^2}=\frac{Ax+B}{x^2+1}+\frac{C}{(x-1)^2}+\frac{D}{x-1}$$
$$=\frac{Ax^3-2Ax^2+Ax+Bx^2-2Bx+B+Cx^2+C+Dx^3-Dx^2+Dx-D}{(x^2+1)(x-1)^2}$$
$$A+D=0,\ -2A+B+C-D=1$$
$$A-2B+D=-1,\ B+C-D=1$$
$$A=0,\ B=\tfrac{1}{2},\ C=\tfrac{1}{2},\ D=0$$
$$\therefore\ \frac{x^2-x+1}{(x^2+1)(x-1)^2}=\frac{1}{2(x^2+1)}+\frac{1}{2(x-1)^2}$$

Exercise 112. Page 335.

Expand:

1. $(a+b)^7=a^7+7a^6b+21a^5b^2+35a^4b^3+35a^3b^4+21a^2b^5+7ab^6+b^7.$

2. $(x-2)^5=x^5-5\times 2x^4+10\times 2^2x^3-10\times 2^3x^2+5\times 2^4x-2^5$
$=x^5-10x^4+40x^3-80x^2+80x-32$

3. $(3x-2y)^4$
$=(3x)^4-4(3x)^3(2y)+6(3x)^2(2y)^2-4(3x)(2y)^3+(2y)^4$
$=81x^4-216x^3y+216x^2y^2-96xy^3+16y^4.$

4. $$\left(\frac{x}{y}-\frac{y}{x}\right)^6$$
$$=\left(\frac{x}{y}\right)^6-6\left(\frac{x}{y}\right)^5\left(\frac{y}{x}\right)+15\left(\frac{x}{y}\right)^4\left(\frac{y}{x}\right)^2-20\left(\frac{x}{y}\right)^3\left(\frac{y}{x}\right)^3+15\left(\frac{x}{y}\right)^2\left(\frac{y}{x}\right)^4$$
$$-6\left(\frac{x}{y}\right)\left(\frac{y}{x}\right)^5+\left(\frac{y}{x}\right)^6$$
$$=\frac{x^6}{y^6}-\frac{6x^4}{y^4}+\frac{15x^2}{y^2}-20+\frac{15y^2}{x^2}-\frac{6y^4}{x^4}+\frac{y^6}{x^6}.$$

5. $(4+3y)^4$
$=4^4+4(4^3)(3y)+6(4^2)(3y)^2+4(4)(3y)^3+(3y)^4$
$=256+768y+864y^2+432y^3+81y^4.$

6. $(a^3+b)^7$
$=(a^3)^7+7(a^3)^6b+21(a^3)^5b^2+35(a^3)^4b^3+35(a^3)^3b^4+21(a^3)^2b^5$
$+7(a^3)b^6+b^7$
$=a^{21}+7a^{18}b+21a^{15}b^2+35a^{12}b^3+35a^9b^4+21a^6b^5+7a^3b^6+b^7.$

7. $(m^2+n^3)^8$

$$\begin{aligned}&= (m^2)^8 + 8(m^2)^7(n^3) + 28(m^2)^6(n^3)^2 + 56(m^2)^5(n^3)^3\\&\qquad + 70(m^2)^4(n^3)^4 + 56(m^2)^3(n^3)^5 + 28(m^2)^2(n^3)^6\\&\qquad + 8(m^2)(n^3)^7 + (n^3)^8\\&= m^{16} + 8\,m^{14}n^3 + 28\,m^{12}n^6 + 56\,m^{10}n^9 + 70\,m^8n^{12} + 56\,m^6n^{15}\\&\qquad + 28\,m^4n^{18} + 8\,m^2n^{21} + n^{24}.\end{aligned}$$

8. $(a-b^3)^7$

$$= a^7 - 7\,a^6b^3 + 21\,a^5b^6 - 35\,a^4b^9 + 35\,a^3b^{12} - 21\,a^2b^{15} + 7\,ab^{18} - b^{21}.$$

9. $(a^{\frac{1}{2}}+b^{\frac{2}{3}})^5$

$$\begin{aligned}&= (a^{\frac{1}{2}})^5 + 5(a^{\frac{1}{2}})^4(b^{\frac{2}{3}}) + 10(a^{\frac{1}{2}})^3(b^{\frac{2}{3}})^2 + 10(a^{\frac{1}{2}})^2(b^{\frac{2}{3}})^3\\&\qquad + 5(a^{\frac{1}{2}})(b^{\frac{2}{3}})^4 + (b^{\frac{2}{3}})^5\\&= a^{\frac{5}{2}} + 5\,a^2b^{\frac{2}{3}} + 10\,a^{\frac{3}{2}}b^{\frac{4}{3}} + 10\,ab^2 + 5\,a^{\frac{1}{2}}b^{\frac{8}{3}} + b^{\frac{10}{3}}.\end{aligned}$$

10. $(a^{-1}+b^{-2})^3 = a^{-3} + 3\,a^{-2}b^{-2} + 3\,a^{-1}b^{-4} + b^{-6}.$

11. $(m^{-\frac{1}{2}}+n^2)^4$

$$\begin{aligned}&= (m^{-\frac{1}{2}})^4 + 4(m^{-\frac{1}{2}})^3(n^2) + 6(m^{-\frac{1}{2}})^2(n^2)^2 + 4(m^{-\frac{1}{2}})(n^2)^3 + (n^2)^4\\&= m^{-2} + 4\,m^{-\frac{3}{2}}n^2 + 6\,m^{-1}n^4 + 4\,m^{-\frac{1}{2}}n^6 + n^8.\end{aligned}$$

12. $(x^{-2}+z^{\frac{2}{3}})^6$

$$\begin{aligned}&= (x^{-2})^6 + 6(x^{-2})^5(z^{\frac{2}{3}}) + 15(x^{-2})^4(z^{\frac{2}{3}})^2 + 20(x^{-2})^3(z^{\frac{2}{3}})^3\\&\qquad + 15(x^{-2})^2(z^{\frac{2}{3}})^4 + 6(x^{-2})(z^{\frac{2}{3}})^5 + (z^{\frac{2}{3}})^6\\&= x^{-12} + 6\,x^{-10}z^{\frac{2}{3}} + 15\,x^{-8}z^{\frac{4}{3}} + 20\,x^{-6}z^2 + 15\,x^{-4}z^{\frac{8}{3}} + 6\,x^{-2}z^{\frac{10}{3}} + z^4.\end{aligned}$$

13. $(2\,x^2+y^{\frac{1}{2}})^5$

$$\begin{aligned}&= (2\,x^2)^5 + 5(2\,x^2)^4(y^{\frac{1}{2}}) + 10(2\,x^2)^3(y^{\frac{1}{2}})^2 + 10(2\,x^2)^2(y^{\frac{1}{2}})^3\\&\qquad + 5(2\,x^2)(y^{\frac{1}{2}})^4 + (y^{\frac{1}{2}})^5\\&= 32\,x^{10} + 80\,x^8y^{\frac{1}{2}} + 80\,x^6y + 40\,x^4y^{\frac{3}{2}} + 10\,x^2y^2 + y^{\frac{5}{2}}.\end{aligned}$$

14. $(a^{\frac{1}{2}}-c^{\frac{2}{3}})^4$

$$\begin{aligned}&= (a^{\frac{1}{2}})^4 - 4(a^{\frac{1}{2}})^3(c^{\frac{2}{3}}) + 6(a^{\frac{1}{2}})^2(c^{\frac{2}{3}})^2 - 4(a^{\frac{1}{2}})(c^{\frac{2}{3}})^3 + (c^{\frac{2}{3}})^4\\&= a^2 - 4\,a^{\frac{3}{2}}c^{\frac{2}{3}} + 6\,ac^{\frac{4}{3}} - 4\,a^{\frac{1}{2}}c^2 + c^{\frac{8}{3}}.\end{aligned}$$

15. $(2a^2-\frac{1}{2}\sqrt{a})^5$

$$\begin{aligned}&=(2a^2)^5-5(2a^2)^4(\tfrac{1}{2}\sqrt{a})+10(2a^2)^3(\tfrac{1}{2}\sqrt{a})^2\\&\qquad-10(2a^2)^2(\tfrac{1}{2}\sqrt{a})^3+5(2a^2)(\tfrac{1}{2}\sqrt{a})^4-(\tfrac{1}{2}\sqrt{a})^5\\&=32a^{10}-40a^{\frac{17}{2}}+20a^7-5a^{\frac{11}{2}}+\tfrac{5}{8}a^4-\tfrac{1}{32}a^{\frac{5}{2}}.\end{aligned}$$

16. $\left(\frac{a^2}{b}-\frac{\sqrt{b}}{2a}\right)^7$

$$\begin{aligned}&=\left(\frac{a^2}{b}\right)^7-7\left(\frac{a^2}{b}\right)^6\left(\frac{\sqrt{b}}{2a}\right)+21\left(\frac{a^2}{b}\right)^5\left(\frac{\sqrt{b}}{2a}\right)^2-35\left(\frac{a^2}{b}\right)^4\left(\frac{\sqrt{b}}{2a}\right)^3\\&\qquad+35\left(\frac{a^2}{b}\right)^3\left(\frac{\sqrt{b}}{2a}\right)^4-21\left(\frac{a^2}{b}\right)^2\left(\frac{\sqrt{b}}{2a}\right)^5\\&\qquad+7\left(\frac{a^2}{b}\right)\left(\frac{\sqrt{b}}{2a}\right)^6-\left(\frac{\sqrt{b}}{2a}\right)^7\\&=\frac{a^{14}}{b^7}-\frac{7a^{11}}{2b^{\frac{11}{2}}}+\frac{21a^8}{4b^4}-\frac{35a^5}{8b^{\frac{5}{2}}}+\frac{35a^2}{16b}-\frac{21b^{\frac{1}{2}}}{32a}+\frac{7b^2}{64a^4}-\frac{b^{\frac{7}{2}}}{128a^7}\end{aligned}$$

17. $\left(\frac{2a}{b^2}+\frac{1}{2}b\sqrt{a}\right)^3$

$$\begin{aligned}&=\left(\frac{2a}{b^2}\right)^3+3\left(\frac{2a}{b^2}\right)^2(\tfrac{1}{2}b\sqrt{a})+3\left(\frac{2a}{b^2}\right)(\tfrac{1}{2}b\sqrt{a})^2+(\tfrac{1}{2}b\sqrt{a})^3\\&=\frac{8a^3}{b^6}+\frac{6a^{\frac{5}{2}}}{b^3}+\frac{3a^2}{2}+\frac{a^{\frac{3}{2}}b^3}{8}.\end{aligned}$$

18. $(2x^2y^{-1}-y\sqrt{y})^4$

$$\begin{aligned}&=(2x^2y^{-1})^4-4(2x^2y^{-1})^3(y\sqrt{y})+6(2x^2y^{-1})^2(y\sqrt{y})^2\\&\qquad-4(2x^2y^{-1})(y\sqrt{y})^3+(y\sqrt{y})^4\\&=16x^8y^{-4}-32x^6y^{-\frac{3}{2}}+24x^4y-8x^2y^{\frac{7}{2}}+y^6.\end{aligned}$$

19. $\left(\frac{a\sqrt[3]{a}}{\sqrt[7]{b^5}}+\frac{\sqrt[3]{b}}{a}\right)^7$

$$\begin{aligned}&=\left(\frac{a\sqrt[3]{a}}{\sqrt[7]{b^5}}\right)^7+7\left(\frac{a\sqrt[3]{a}}{\sqrt[7]{b^5}}\right)^6\left(\frac{\sqrt[3]{b}}{a}\right)+21\left(\frac{a\sqrt[3]{a}}{\sqrt[7]{b^5}}\right)^5\left(\frac{\sqrt[3]{b}}{a}\right)^2\\&\qquad+35\left(\frac{a\sqrt[3]{a}}{\sqrt[7]{b^5}}\right)^4\left(\frac{\sqrt[3]{b}}{a}\right)^3+35\left(\frac{a\sqrt[3]{a}}{\sqrt[7]{b^5}}\right)^3\left(\frac{\sqrt[3]{b}}{a}\right)^4\\&\qquad+21\left(\frac{a\sqrt[3]{a}}{\sqrt[7]{b^5}}\right)^2\left(\frac{\sqrt[3]{b}}{a}\right)^5+7\left(\frac{a\sqrt[3]{a}}{\sqrt[7]{b^5}}\right)\left(\frac{\sqrt[3]{b}}{a}\right)^6+\left(\frac{\sqrt[3]{b}}{a}\right)^7\\&=\frac{a^{\frac{28}{3}}}{b^5}+\frac{7a^7}{b^{\frac{83}{21}}}+\frac{21a^{\frac{14}{3}}}{b^{\frac{61}{21}}}+\frac{35a^{\frac{7}{3}}}{b^{\frac{18}{7}}}+\frac{35}{b^{\frac{17}{21}}}+\frac{21b^{\frac{5}{21}}}{a^{\frac{7}{3}}}+\frac{7b^{\frac{9}{7}}}{a^{\frac{14}{3}}}+\frac{b^{\frac{7}{3}}}{a^7}.\end{aligned}$$

20. $\left(\frac{\sqrt{a}}{\sqrt[3]{b^2}}-\frac{1}{2}\sqrt{\frac{b}{a}}\right)^5$

$$=\left(\frac{\sqrt{a}}{\sqrt[3]{b^2}}\right)^5-5\left(\frac{\sqrt{a}}{\sqrt[3]{b^2}}\right)^4\left(\frac{1}{2}\sqrt{\frac{b}{a}}\right)+10\left(\frac{\sqrt{a}}{\sqrt[3]{b^2}}\right)^3\left(\frac{1}{2}\sqrt{\frac{b}{a}}\right)^2$$

$$-10\left(\frac{\sqrt{a}}{\sqrt[3]{b^2}}\right)^2\left(\frac{1}{2}\sqrt{\frac{b}{a}}\right)^3+5\left(\frac{\sqrt{a}}{\sqrt[3]{b^2}}\right)\left(\frac{1}{2}\sqrt{\frac{b}{a}}\right)^4-\left(\frac{1}{2}\sqrt{\frac{b}{a}}\right)^5$$

$$=\frac{a^{\frac{5}{2}}}{b^{\frac{10}{3}}}-\frac{5a^{\frac{3}{2}}}{2b^{\frac{13}{6}}}+\frac{5a^{\frac{1}{2}}}{2b}-\frac{5b^{\frac{1}{6}}}{4a^{\frac{1}{2}}}+\frac{5b^{\frac{4}{3}}}{16a^{\frac{3}{2}}}-\frac{b^{\frac{5}{2}}}{32a^{\frac{5}{2}}}.$$

21. $(2ab^{-2}-ba^{\frac{1}{2}})^7$

$$=(2ab^{-2})^7-7(2ab^{-2})^6(ba^{\frac{1}{2}})+21(2ab^{-2})^5(ba^{\frac{1}{2}})^2$$

$$-35(2ab^{-2})^4(ba^{\frac{1}{2}})^3+35(2ab^{-2})^3(ba^{\frac{1}{2}})^4$$

$$-21(2ab^{-2})^2(ba^{\frac{1}{2}})^5+7(2ab^{-2})(ba^{\frac{1}{2}})^6-(ba^{\frac{1}{2}})^7$$

$$=128a^7b^{-14}-448a^{\frac{13}{2}}b^{-11}+672a^6b^{-8}-560a^{\frac{11}{2}}b^{-5}+280a^5b^{-2}$$

$$-84a^{\frac{9}{2}}b+14a^4b^4-a^{\frac{7}{2}}b^7.$$

22. $\left(\frac{a}{b}\sqrt{\frac{c}{d}}-\frac{c}{d}\right)^6$

$$=\left(\frac{a}{b}\sqrt{\frac{c}{d}}\right)^6-6\left(\frac{a}{b}\sqrt{\frac{c}{d}}\right)^5\left(\frac{c}{d}\right)+15\left(\frac{a}{b}\sqrt{\frac{c}{d}}\right)^4\left(\frac{c}{d}\right)^2-20\left(\frac{a}{b}\sqrt{\frac{c}{d}}\right)^3\left(\frac{c}{d}\right)^3$$

$$+15\left(\frac{a}{b}\sqrt{\frac{c}{d}}\right)^2\left(\frac{c}{d}\right)^4-6\left(\frac{a}{b}\sqrt{\frac{c}{d}}\right)\left(\frac{c}{d}\right)^5+\left(\frac{c}{d}\right)^6$$

$$=\frac{a^6c^3}{b^6d^3}-\frac{6a^5c^{\frac{7}{2}}}{b^5d^{\frac{7}{2}}}+\frac{15a^4c^4}{b^4d^4}-\frac{20a^3c^{\frac{9}{2}}}{b^3d^{\frac{9}{2}}}+\frac{15a^2c^5}{b^2d^5}-\frac{6ac^{\frac{11}{2}}}{bd^{\frac{11}{2}}}+\frac{c^6}{d^6}.$$

23. $\left(\frac{a}{b}\sqrt{\frac{b}{a}}-\frac{b^{\frac{1}{2}}}{\sqrt{ac}}\right)^5$

$$=\left(\frac{a}{b}\sqrt{\frac{b}{a}}\right)^5-5\left(\frac{a}{b}\sqrt{\frac{b}{a}}\right)^4\left(\frac{b^{\frac{1}{2}}}{\sqrt{ac}}\right)+10\left(\frac{a}{b}\sqrt{\frac{b}{a}}\right)^3\left(\frac{b^{\frac{1}{2}}}{\sqrt{ac}}\right)^2$$

$$-10\left(\frac{a}{b}\sqrt{\frac{b}{a}}\right)^2\left(\frac{b^{\frac{1}{2}}}{\sqrt{ac}}\right)^3+5\left(\frac{a}{b}\sqrt{\frac{b}{a}}\right)\left(\frac{b^{\frac{1}{2}}}{\sqrt{ac}}\right)^4-\left(\frac{b^{\frac{1}{2}}}{\sqrt{ac}}\right)^5$$

$$=\frac{a^{\frac{5}{2}}}{b^{\frac{5}{2}}}-\frac{5a^{\frac{3}{2}}}{b^{\frac{3}{2}}c^{\frac{1}{2}}}+\frac{10a^{\frac{1}{2}}}{b^{\frac{1}{2}}c}-\frac{10b^{\frac{1}{2}}}{a^{\frac{1}{2}}c^{\frac{3}{2}}}+\frac{5b^{\frac{3}{2}}}{a^{\frac{3}{2}}c^2}-\frac{b^{\frac{5}{2}}}{a^{\frac{5}{2}}c^{\frac{5}{2}}}.$$

24. Find the fourth term of $(2x-3y)^7$.

$$\frac{7\cdot 6\cdot 5}{1\cdot 2\cdot 3}(2x)^4(-3y)^3=-35\times 2^4\times 3^3x^4y^3=-15120x^4y^3.$$

25. Find the ninety-seventh term of $(2a-b)^{100}$.

$$\frac{100\cdot 99\cdot 98\cdot 97}{1\cdot 2\cdot 3\cdot 4}(2a)^4(-b)^{96}$$
$$=25\cdot 33\cdot 49\cdot 97\cdot 2^4a^4b^{96}=400\cdot 33\cdot 49\cdot 97\,a^4b^{96}.$$

26. Find the eighth term of $(3x-y)^{11}$.

$$\frac{11\cdot 10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}(3x)^4(-y)^7$$
$$=-11\times 5\times 3\times 2\times 3^4x^4y^7=-26730x^4y^7.$$

27. Find the tenth term of $(2a^2-\frac{1}{2}a)^{20}$.

$$\frac{20\cdot 19\cdot 18\cdot 17\cdot 16\cdot 15\cdot 14\cdot 13\cdot 12}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8\cdot 9}(2a^2)^{11}(-\tfrac{1}{2}a)^9$$
$$=-20\cdot 19\cdot 2\cdot 17\cdot 13\cdot 2^{11}\frac{1}{2^9}a^{31}=-5\cdot 19\cdot 17\cdot 13\cdot 2^5a^{31}.$$

28. Find the fifth term of $(a-2\sqrt{b})^{25}$.

$$\frac{25\cdot 24\cdot 23\cdot 22}{1\cdot 2\cdot 3\cdot 4}a^{21}(-2\sqrt{b})^4$$
$$=25\cdot 23\cdot 22\cdot 2^4a^{21}b^2=25\cdot 23\cdot 11\cdot 2^5a^{21}b^2.$$

29. Find the eleventh term of $(2-a)^{16}$.

$$\frac{16\cdot 15\cdot 14\cdot 13\cdot 12\cdot 11\cdot 10\cdot 9\cdot 8\cdot 7}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8\cdot 9\cdot 10}2^6(-a)^{10}$$
$$=4\cdot 14\cdot 13\cdot 11\cdot 2^6a^{10}=7\cdot 11\cdot 13\cdot 2^9a^{10}.$$

30. Find the fifteenth term of $(x+y)^{20}$.

$$\frac{20\cdot 19\cdot 18\cdot 17\cdot 16\cdot 15\cdot 14\cdot 13\cdot 12\cdot 11\cdot 10\cdot 9\cdot 8\cdot 7}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8\cdot 9\cdot 10\cdot 11\cdot 12\cdot 13\cdot 14}x^6y^{14}$$
$$=20\cdot 19\cdot 3\cdot 17\cdot 2x^6y^{14}=3\cdot 5\cdot 17\cdot 19\cdot 2^3x^6y^{14}.$$

31. Find the fourth term of $(3-2x^2)^9$.

$$\frac{9\cdot 8\cdot 7}{1\cdot 2\cdot 3}(3^6)(-2x^2)^3=-84\cdot 3^6\cdot 2^3x^6=-3^7\cdot 2^5\cdot 7x^6.$$

32. Find the twelfth term of $(a^2-a\sqrt{x})^{17}$.

$$\frac{17\cdot16\cdot15\cdot14\cdot13\cdot12\cdot11\cdot10\cdot9\cdot8\cdot7}{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7\cdot8\cdot9\cdot10\cdot11}(a^2)^6(-a\sqrt{x})^{11}$$

$$=-17\cdot14\cdot13\cdot4\,a^{12}a^{11}x^{\frac{11}{2}}=-17\cdot13\cdot7\cdot2^3a^{23}x^{\frac{11}{2}}.$$

33. Find the seventh term of $(y^2-1)^{38}$.

$$\frac{38\cdot37\cdot36\cdot35\cdot34\cdot33}{1\cdot2\cdot3\cdot4\cdot5\cdot6}(y^2)^{32}(-1)^6=19\cdot37\cdot7\cdot17\cdot33\,y^{64}.$$

34. Find the fifth term of $(\frac{1}{2}a-b\sqrt{b})^{21}$.

$$\frac{21\cdot20\cdot19\cdot18}{1\cdot2\cdot3\cdot4}(\tfrac{1}{2}a)^{17}(-b\sqrt{b})^4$$

$$=21\cdot5\cdot19\cdot3\frac{a^{17}}{2^{17}}b^6=\frac{3^2\cdot5\cdot7\cdot19}{2^{17}}a^{17}b^6.$$

35. Find the fourth term of $(\sqrt{a}-\sqrt[3]{b^2})^{20}$.

$$\frac{20\cdot19\cdot18}{1\cdot2\cdot3}(\sqrt{a})^{17}(-\sqrt[3]{b^2})^3=-1140\,a^{\frac{17}{2}}b^2.$$

36. Find the third term of $(\sqrt{a}-\sqrt{-b})^7$.

$$\frac{7\cdot6}{1\cdot2}(\sqrt{a})^5(-\sqrt{-b})^2=-21\,a^{\frac{5}{2}}b.$$

37. Find the sixth term of $(\sqrt[3]{a^2}-\sqrt{-1})^9$.

$$\frac{9\cdot8\cdot7\cdot6\cdot5}{1\cdot2\cdot3\cdot4\cdot5}(\sqrt[3]{a^2})^4(-\sqrt{-1})^5=-126\,a^{\frac{8}{3}}\sqrt{-1}.$$

38. Find the eighth term of $(\sqrt{\frac{2}{3}a}+\sqrt{\frac{3}{4}x})^{20}$.

$$\frac{20\cdot19\cdot18\cdot17\cdot16\cdot15\cdot14}{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7}(\sqrt{\tfrac{2}{3}a})^{13}(\sqrt{\tfrac{3}{4}x})^7$$

$$=19\cdot17\cdot16\cdot15\frac{2^{\frac{13}{2}}}{3^{\frac{13}{2}}}\,\frac{3^{\frac{7}{2}}}{4^{\frac{7}{2}}}a^{\frac{13}{2}}x^{\frac{7}{2}}=\frac{19\cdot17\cdot16\cdot15}{3^3\cdot2^{\frac{1}{2}}}a^{\frac{13}{2}}x^{\frac{7}{2}}$$

$$=\frac{19\cdot17\cdot8\cdot5\cdot2^{\frac{1}{2}}}{9}a^{\frac{13}{2}}x^{\frac{7}{2}}.$$

39. Find the ninth term of $(x\sqrt{-1}+y\sqrt{-1})^{16}$.

$$\frac{16\cdot15\cdot14\cdot13\cdot12\cdot11\cdot10\cdot9}{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7\cdot8}(x\sqrt{-1})^8(y\sqrt{-1})^8$$

$$=13\cdot11\cdot10\cdot9\,x^8y^8.$$

40. Find the fifth term of $\left(a^3b - \frac{3b^{-2}}{\sqrt{a^5}}\right)^{31}$.

$$\frac{31\cdot30\cdot29\cdot28}{1\cdot2\cdot3\cdot4}(a^3b)^{27}\left(-\frac{3b^{-2}}{\sqrt{a^5}}\right)^4$$

$$= 31\cdot5\cdot29\cdot7\,a^{81}b^{27}\frac{3^4b^{-8}}{a^{10}} = 5\cdot7\cdot29\cdot31\cdot3^4a^{71}b^{19}.$$

41. Find the seventh term of $(x + x^{-1})^{2n}$.

$$\frac{2n(2n-1)(2n-2)(2n-3)(2n-4)(2n-5)}{1\cdot2\cdot3\cdot4\cdot5\cdot6}x^{2n-6}\left(\frac{1}{x}\right)^6$$

$$= \frac{n(2n-1)(n-1)(2n-3)(n-2)(2n-5)}{3\cdot5\cdot6}x^{2n-12}.$$

Exercise 113. Page 341.

Expand to four terms:

1. $$(1+x)^{\frac{1}{2}} = 1 + \tfrac{1}{2}x + \frac{\frac{1}{2}\cdot-\frac{1}{2}}{2}x^2 + \frac{\frac{1}{2}\cdot-\frac{1}{2}\cdot-\frac{3}{2}}{2\cdot3}x^3 + \cdots$$
$$= 1 + \tfrac{1}{2}x - \tfrac{1}{8}x^2 + \tfrac{1}{16}x^3 - \cdots$$

2. $$(1+x)^{-2} = 1 - 2x + \frac{-2\cdot-3}{2}x^2 + \frac{-2\cdot-3\cdot-4}{2\cdot3}x^3 + \cdots$$
$$= 1 - 2x + 3x^2 - 4x^3 + \cdots$$

3. $$(1+x)^{-\frac{1}{2}} = 1 - \tfrac{1}{2}x + \frac{-\frac{1}{2}\cdot-\frac{3}{2}}{2}x^2 + \frac{-\frac{1}{2}\cdot-\frac{3}{2}\cdot-\frac{5}{2}}{2\cdot3}x^3 + \cdots$$
$$= 1 - \tfrac{1}{2}x + \tfrac{3}{8}x^2 - \tfrac{5}{16}x^3 + \cdots$$

4. $$(1-x)^{\frac{2}{3}} = 1 - \tfrac{2}{3}x + \frac{\frac{2}{3}\cdot-\frac{1}{3}}{2}x^2 - \frac{\frac{2}{3}\cdot-\frac{1}{3}\cdot-\frac{4}{3}}{2\cdot3}x^3 + \cdots$$
$$= 1 - \tfrac{2}{3}x - \tfrac{1}{9}x^2 - \tfrac{4}{81}x^3 - \cdots$$

5. $$(1-x)^{-\frac{2}{3}} = 1 + \tfrac{2}{3}x + \frac{-\frac{2}{3}\cdot-\frac{5}{3}}{2}x^2 - \frac{-\frac{2}{3}\cdot-\frac{5}{3}\cdot-\frac{8}{3}}{2\cdot3}x^3 + \cdots$$
$$= 1 + \tfrac{2}{3}x + \tfrac{5}{9}x^2 + \tfrac{40}{81}x^3 + \cdots$$

6. $$(1+x)^{\frac{5}{3}} = 1 + \tfrac{5}{3}x + \frac{\frac{5}{3}\cdot\frac{2}{3}}{2}x^2 + \frac{\frac{5}{3}\cdot\frac{2}{3}\cdot-\frac{1}{3}}{2\cdot3}x^3 + \cdots$$
$$= 1 + \tfrac{5}{3}x + \tfrac{5}{9}x^2 - \tfrac{5}{81}x^3 + \cdots$$

7. $$\begin{aligned}(1+x)^{-\frac{5}{3}} &= 1-\tfrac{5}{3}x+\frac{-\frac{5}{3}\cdot-\frac{8}{3}}{2}x^2+\frac{-\frac{5}{3}\cdot-\frac{8}{3}\cdot-\frac{11}{3}}{2\cdot3}x^3+\cdots\\ &= 1-\tfrac{5}{3}x+\tfrac{20}{9}x^2-\tfrac{220}{81}x^3+\cdots\end{aligned}$$

8. $$\begin{aligned}(1+x)^{-3} &= 1-3x+\frac{-3\cdot-4}{2}x^2+\frac{-3\cdot-4\cdot-5}{2\cdot3}x^3+\cdots\\ &= 1-3x+6x^2-10x^3+\cdots\end{aligned}$$

9. $$\begin{aligned}(1+5x)^{-5} &= 1-5(5x)+\frac{-5\cdot-6}{2}(5x)^2+\frac{-5\cdot-6\cdot-7}{2\cdot3}(5x)^3+\cdots\\ &= 1-25x+375x^2-4375x^3+\cdots\end{aligned}$$

10. $$\begin{aligned}(1+5x)^{\frac{5}{3}} &= 1+\tfrac{5}{3}(5x)+\frac{\frac{5}{3}\cdot\frac{2}{3}}{2}(5x)^2+\frac{\frac{5}{3}\cdot\frac{2}{3}\cdot-\frac{1}{3}}{2\cdot3}(5x)^3+\cdots\\ &= 1+\tfrac{25}{3}x+\tfrac{125}{9}x^2-\tfrac{625}{81}x^3+\cdots\end{aligned}$$

11. $$\begin{aligned}&(2x+3y)^{\frac{3}{4}}\\ &= (2x)^{\frac{3}{4}}+\tfrac{3}{4}(2x)^{-\frac{1}{4}}(3y)+\frac{\frac{3}{4}\cdot-\frac{1}{4}}{2}(2x)^{-\frac{5}{4}}(3y)^2\\ &\qquad+\frac{\frac{3}{4}\cdot-\frac{1}{4}\cdot-\frac{5}{4}}{2\cdot3}(2x)^{-\frac{9}{4}}(3y)^3+\cdots\\ &= 2^{\frac{3}{4}}x^{\frac{3}{4}}+\frac{9y}{2^{\frac{9}{4}}x^{\frac{1}{4}}}-\frac{27y^2}{2^{\frac{25}{4}}x^{\frac{5}{4}}}+\frac{135y^3}{2^{\frac{37}{4}}x^{\frac{9}{4}}}-\cdots\end{aligned}$$

12. $$\begin{aligned}&(2x+3y)^{-\frac{3}{4}}\\ &= (2x)^{-\frac{3}{4}}-\tfrac{3}{4}(2x)^{-\frac{7}{4}}(3y)+\frac{-\frac{3}{4}\cdot-\frac{7}{4}}{2}(2x)^{-\frac{11}{4}}(3y)^2\\ &\qquad+\frac{-\frac{3}{4}\cdot-\frac{7}{4}\cdot-\frac{11}{4}}{2\cdot3}(2x)^{-\frac{15}{4}}(3y)^3+\cdots\\ &= \frac{1}{2^{\frac{3}{4}}x^{\frac{3}{4}}}-\frac{9y}{2^{\frac{15}{4}}x^{\frac{7}{4}}}+\frac{189y^2}{2^{\frac{31}{4}}x^{\frac{11}{4}}}-\frac{7\cdot11\cdot27y^3}{2^{\frac{43}{4}}x^{\frac{15}{4}}}+\cdots\end{aligned}$$

13. $$\begin{aligned}&\frac{1}{\sqrt[4]{a^2-x}}\\ &= (a^2-x)^{-\frac{1}{4}}\\ &= (a^2)^{-\frac{1}{4}}+\tfrac{1}{4}(a^2)^{-\frac{5}{4}}x+\frac{-\frac{1}{4}\cdot-\frac{5}{4}}{2}(a^2)^{-\frac{9}{4}}x^2\\ &\qquad-\frac{-\frac{1}{4}\cdot-\frac{5}{4}\cdot-\frac{9}{4}}{2\cdot3}(a^2)^{-\frac{13}{4}}x^3+\cdots\\ &= \frac{1}{a^{\frac{1}{2}}}+\frac{x}{4a^{\frac{5}{2}}}+\frac{5x^2}{32a^{\frac{9}{2}}}+\frac{15x^3}{128a^{\frac{13}{2}}}+\cdots\end{aligned}$$

14. $\dfrac{1}{\sqrt[5]{(a-x)^2}}$

$$= (a-x)^{-\frac{2}{5}}$$

$$= a^{-\frac{2}{5}} + \tfrac{2}{5}a^{-\frac{7}{5}}x + \frac{-\frac{2}{5}\cdot -\frac{7}{5}}{2}a^{-\frac{12}{5}}x^2 - \frac{-\frac{2}{5}\cdot -\frac{7}{5}\cdot -\frac{12}{5}}{2\cdot 3}a^{-\frac{17}{5}}x^3 + \cdots$$

$$= a^{-\frac{2}{5}} + \tfrac{2}{5}a^{-\frac{7}{5}}x + \tfrac{7}{25}a^{-\frac{12}{5}}x^2 + \tfrac{28}{125}a^{-\frac{17}{5}}x^3 + \cdots$$

15. Find the fourth term of $\left(a - \dfrac{3}{2\sqrt{x}}\right)^{\frac{1}{2}}$.

$$\frac{\frac{1}{2}\cdot -\frac{1}{2}\cdot -\frac{3}{2}}{2\cdot 3}a^{-\frac{5}{2}}\left(-\frac{3}{2\sqrt{x}}\right)^3 = \tfrac{1}{16}a^{-\frac{5}{2}}\left(-\frac{27}{8x^{\frac{3}{2}}}\right) = -\tfrac{27}{128}a^{-\frac{5}{2}}x^{-\frac{3}{2}}.$$

16. Find the fifth term of $\dfrac{1}{\sqrt[3]{(a-2x)^2}}$.

$$\frac{-\frac{2}{3}\cdot -\frac{5}{3}\cdot -\frac{8}{3}\cdot -\frac{11}{3}}{2\cdot 3\cdot 4}a^{-\frac{14}{3}}(-2x)^4 = \tfrac{110}{243}a^{-\frac{14}{3}}(16x^4) = \tfrac{1760}{243}a^{-\frac{14}{3}}x^4.$$

17. Find the third term of $(4 - 7x)^{\frac{2}{7}}$.

$$\frac{\frac{2}{7}\cdot -\frac{5}{7}}{2}(4)^{-\frac{12}{7}}(-7x)^2 = -\tfrac{5}{49}(4)^{-\frac{12}{7}}(49x^2) = -\frac{5x^2}{4^{\frac{12}{7}}}.$$

18. Find the sixth term of $(a^2 - 2ax)^{\frac{5}{3}}$.

$$\frac{\frac{5}{3}\cdot \frac{2}{3}\cdot -\frac{1}{3}\cdot -\frac{4}{3}\cdot -\frac{7}{3}}{2\cdot 3\cdot 4\cdot 5}(a^2)^{-\frac{10}{3}}(-2ax)^5 = -\frac{7}{3^6}a^{-\frac{20}{3}}(-32a^5x^5)$$

$$= \tfrac{224}{729}a^{-\frac{5}{3}}x^5.$$

19. Find the fifth term of $(1 - 2x)^{-\frac{1}{2}}$.

$$\frac{-\frac{1}{2}\cdot -\frac{3}{2}\cdot -\frac{5}{2}\cdot -\frac{7}{2}}{2\cdot 3\cdot 4}(-2x)^4 = \frac{35}{2^7}2^4x^4 = \tfrac{35}{8}x^4.$$

20. Find the fifth term of $(1 - x)^{-3}$.

$$\frac{-3\cdot -4\cdot -5\cdot -6}{2\cdot 3\cdot 4}(-x)^4 = 15x^4.$$

21. Find the seventh term of $(1 - x)^{\frac{1}{2}}$.

$$\frac{\frac{1}{2}\cdot -\frac{1}{2}\cdot -\frac{3}{2}\cdot -\frac{5}{2}\cdot -\frac{7}{2}\cdot -\frac{9}{2}}{2\cdot 3\cdot 4\cdot 5\cdot 6}(-x)^6 = -\frac{21}{2^{10}}x^6.$$

22. Find the third term of $(1+x)^{-\frac{1}{2n}}$.

$$\frac{-\frac{1}{2n}\cdot\left(-\frac{1}{2n}-1\right)}{2}x^2=\frac{2n+1}{8n^2}x^2.$$

23. Find the fourth term of $(1+x)^{-\frac{5}{3}}$.

$$\frac{-\frac{5}{3}\cdot-\frac{8}{3}\cdot-\frac{11}{3}}{2\cdot3}x^3=-\frac{220}{81}x^3.$$

24. Find the sixth term of $\left(2-\frac{1}{x}\right)^{-\frac{2}{7}}$.

$$\frac{-\frac{2}{7}\cdot-\frac{9}{7}\cdot-\frac{16}{7}\cdot-\frac{23}{7}\cdot-\frac{30}{7}}{2\cdot3\cdot4\cdot5}(2)^{-\frac{37}{7}}\left(-\frac{1}{x}\right)^5=-\frac{1656}{7^5}2^{-\frac{37}{7}}\left(-\frac{1}{x^5}\right)$$
$$=\frac{207}{7^5\cdot2^{\frac{16}{7}}x^5}.$$

25. Find the fifth term of $(2x-3y)^{-\frac{3}{4}}$.

$$\frac{-\frac{3}{4}\cdot-\frac{7}{4}\cdot-\frac{11}{4}\cdot-\frac{15}{4}}{2\cdot3\cdot4}(2x)^{-\frac{19}{4}}(-3y)^4=\frac{7\cdot11\cdot15}{2^{11}}2^{-\frac{19}{4}}x^{-\frac{19}{4}}(81y^4)$$
$$=\frac{7\cdot11\cdot15\cdot81}{2^{\frac{63}{4}}}x^{-\frac{19}{4}}y^4.$$

26. Find the fourth term of $(1-5x)^{-\frac{5}{3}}$.

$$\frac{-\frac{5}{3}\cdot-\frac{8}{3}\cdot-\frac{11}{3}}{2\cdot3}(-5x)^3=-\frac{220}{81}(-125x^3)=\frac{27500x^3}{81}.$$

Exercise 114. Page 346.

Given: $\log 2=0.3010$; $\log 3=0.4771$; $\log 5=0.6990$; $\log 7=0.8451$.

Find the common logarithms of the following numbers by resolving the numbers into factors, and taking the sum of the logarithms of the factors:

1. $\log 6$.

$$\begin{aligned}\log\ 2&=0.3010\\ \log\ 3&=0.4771\\ \hline \therefore \log\ 6&=0.7781\end{aligned}$$

2. $\log 15$.

$$\begin{aligned}\log\ 3&=0.4771\\ \log\ 5&=0.6990\\ \hline \therefore \log 15&=1.1761\end{aligned}$$

3. $\log 21.$

$$\begin{aligned} \log\ 3 &= 0.4771 \\ \log\ 7 &= \underline{0.8451} \\ \therefore\ \log 21 &= 1.3222 \end{aligned}$$

4. $\log 14.$

$$\begin{aligned} \log\ 2 &= 0.3010 \\ \log\ 7 &= \underline{0.8451} \\ \therefore\ \log 14 &= 1.1461 \end{aligned}$$

5. $\log 25.$

$$\begin{aligned} \log\ 5 &= 0.6990 \\ \log\ 5 &= \underline{0.6990} \\ \therefore\ \log 25 &= 1.3980 \end{aligned}$$

6. $\log 30.$

$$\begin{aligned} \log 10 &= 1.0000 \\ \log\ 3 &= \underline{0.4771} \\ \therefore\ \log 30 &= 1.4771 \end{aligned}$$

7. $\log 42.$

$$\begin{aligned} \log\ 2 &= 0.3010 \\ \log\ 3 &= 0.4771 \\ \log\ 7 &= \underline{0.8451} \\ \therefore\ \log 42 &= 1.6232 \end{aligned}$$

8. $\log 420.$

$$\begin{aligned} \log\ 42 &= 1.6232 \\ \log\ 10 &= \underline{1.0000} \\ \therefore\ \log 420 &= 2.6232 \end{aligned}$$

9. $\log 0.021.$

$$\begin{aligned} \log\ 21 &= 1.3222 \\ \log 0.021 &= 8.3222 - 10 \end{aligned}$$

10. $\log 0.35.$

$$\begin{aligned} \log\ 5 &= 0.6990 \\ \log\ 7 &= \underline{0.8451} \\ \therefore\ \log\ 35 &= 1.5441 \\ \therefore\ \log 0.35 &= 9.5441 - 10 \end{aligned}$$

11. $\log 0.0035.$

$$\log 0.0035 = 7.5441 - 10$$

12. $\log 0.004.$

$$\begin{aligned} \log 2 &= 0.3010 \\ \log 2 &= \underline{0.3010} \\ \therefore\ \log 4 &= 0.6020 \\ \therefore\ 0.004 &= 7.6020 - 10 \end{aligned}$$

13. $\log 2.1.$

$$\begin{aligned} \log\ 21 &= 1.3222 \\ \log 2.1 &= 0.3222 \end{aligned}$$

14. $\log 16.$

$$\begin{aligned} \log\ 2 &= 0.3010 \\ \log\ 2 &= 0.3010 \\ \log\ 2 &= 0.3010 \\ \log\ 2 &= \underline{0.3010} \\ \therefore\ \log 16 &= 1.2040 \end{aligned}$$

15. $\log 0.056.$

$$\begin{aligned} \log\ 2 &= 0.3010 \\ \log\ 2 &= 0.3010 \\ \log\ 2 &= \underline{0.3010} \\ \therefore\ \log\ 8 &= 0.9030 \\ \log\ 7 &= \underline{0.8451} \\ \therefore\ \log 56 &= 1.7481 \\ \therefore\ \log 0.056 &= 8.7481 - 10 \end{aligned}$$

16. $\log 0.63$.

$$\begin{array}{rl} \log\ 3 = & 0.4771 \\ \log\ 3 = & 0.4771 \\ \cline{2-2} \therefore \log\ 9 = & 0.9542 \\ \log\ 7 = & 0.8451 \\ \cline{2-2} \therefore \log 63 = & 1.7993 \\ \therefore \log 0.63 = & 9.7993 - 10 \end{array}$$

Find the common logarithms of the following:

17. 2^3.

$$\begin{array}{rr} \log\ 2 = & 0.3010 \\ & 3 \\ \cline{2-2} \therefore \log 2^3 = & 0.9030 \end{array}$$

18. 5^2.

$$\begin{array}{rr} \log 5\ = & 0.6990 \\ & 2 \\ \cline{2-2} \therefore \log 5^2 = & 1.3980 \end{array}$$

19. 7^4.

$$\begin{array}{rr} \log 7\ = & 0.8451 \\ & 4 \\ \cline{2-2} \therefore \log 7^4 = & 3.3804 \end{array}$$

20. 5^5.

$$\begin{array}{rr} \log 5 = & 0.6990 \\ & 5 \\ \cline{2-2} \therefore \log 5^5 = & 3.4950 \end{array}$$

21. $2^{\frac{1}{3}}$.

$$\begin{array}{rrl} \log\ 2 = & 0.3010 & \underline{|3} \\ \cline{2-2} \therefore \log 2^{\frac{1}{3}} = & 0.1003 & \end{array}$$

22. $5^{\frac{1}{2}}$.

$$\begin{array}{rrl} \log\ 5 = & 0.6990 & \underline{|2} \\ \cline{2-2} \therefore \log 5^{\frac{1}{2}} = & 0.3495 & \end{array}$$

23. $5^{\frac{1}{5}}$.

$$\begin{array}{rrl} \log\ 5 = & 0.6990 & \underline{|5} \\ \cline{2-2} \therefore \log 5^{\frac{1}{5}} = & 0.1398 & \end{array}$$

24. $7^{\frac{1}{11}}$.

$$\begin{array}{rrl} \log\ 7 = & 0.8451 & \underline{|11} \\ \cline{2-2} \therefore \log 7^{\frac{1}{11}} = & 0.0768 & \end{array}$$

25. $2^{\frac{3}{4}}$.

$$\begin{array}{rrl} \log\ 2 = & 0.3010 & \\ & 3 & \\ \cline{2-2} & 0.9030 & \underline{|4} \\ \cline{2-2} \therefore \log 2^{\frac{3}{4}} = & 0.2258 & \end{array}$$

26. $7^{\frac{2}{7}}$.

$$\begin{array}{rrl} \log\ 7 = & 0.8451 & \\ & 2 & \\ \cline{2-2} & 1.6902 & \underline{|7} \\ \cline{2-2} \therefore \log 7^{\frac{2}{7}} = & 0.2415 & \end{array}$$

27. $5^{\frac{5}{3}}$.

$$\begin{array}{rrl} \log\ 5 = & 0.6990 & \\ & 5 & \\ \cline{2-2} & 3.4950 & \underline{|3} \\ \cline{2-2} \therefore \log 5^{\frac{5}{3}} = & 1.1650 & \end{array}$$

28. $3^{\frac{9}{11}}$.

$$\begin{array}{rr}\log\ 3 = & 0.4771\\ & 9\\ \cline{2-2} & 4.2939\ \underline{|11}\\ \cline{2-2} \therefore \log 3^{\frac{9}{11}} = & 0.3904\end{array}$$

29. $5^{\frac{7}{2}}$.

$$\begin{array}{rr}\log\ 5 = & 0.6990\\ & 7\\ \cline{2-2} & 4.8930\ \underline{|2}\\ \cline{2-2} \therefore \log 5^{\frac{7}{2}} = & 2.4465\end{array}$$

30. $2^{\frac{11}{7}}$.

$$\begin{array}{rr}\log\ 2 = & 0.3010\\ & 11\\ \cline{2-2} & 3.3110\ \underline{|7}\\ \cline{2-2} \therefore \log 2^{\frac{11}{7}} = & 0.4730\end{array}$$

31. $5^{\frac{3}{4}}$.

$$\begin{array}{rr}\log\ 5 = & 0.6990\\ & 3\\ \cline{2-2} & 2.0970\ \underline{|4}\\ \cline{2-2} \therefore \log 5^{\frac{3}{4}} = & 0.5243\end{array}$$

Exercise 115. Page 348.

Given: $\log 2 = 0.3010$; $\log 3 = 0.4771$; $\log 5 = 0.6990$; $\log 7 = 0.8451$; $\log 11 = 1.0414$.

Find the logarithms of the following quotients:

1. $\frac{2}{7}$.

$$\begin{array}{rl}\log 2 = & 0.3010\\ \text{colog } 7 = & 9.1549 - 10\\ \cline{2-2} \therefore \log \frac{2}{7} = & 9.4559 - 10\end{array}$$

2. $\frac{4}{5}$.

$$\begin{array}{rl}\log 4 = & 0.6020\\ \text{colog } 5 = & 9.3010 - 10\\ \cline{2-2} \therefore \log \frac{4}{5} = & 9.9030 - 10\end{array}$$

3. $\frac{5}{6}$.

$$\begin{array}{rl}\log 5 = & 0.6990\\ \text{colog } 6 = & 9.2219 - 10\\ \cline{2-2} \therefore \log \frac{5}{6} = & 9.9209 - 10\end{array}$$

4. $\frac{7}{5}$.

$$\begin{array}{rl}\log 7 = & 0.8451\\ \text{colog } 5 = & 9.3010 - 10\\ \cline{2-2} \therefore \log \frac{7}{5} = & 10.1461 - 10\\ = & 0.1461.\end{array}$$

5. $\frac{7}{11}$.

$$\begin{array}{rl}\log\ 7 = & 0.8451\\ \text{colog } 11 = & 8.9586 - 10\\ \cline{2-2} \therefore \log \frac{7}{11} = & 9.8037 - 10\end{array}$$

6. $\frac{11}{7}$.

$$\begin{array}{rl}\log 11 = & 1.0414\\ \text{colog }\ 7 = & 9.1549 - 10\\ \cline{2-2} \therefore \log \frac{11}{7} = & 10.1963 - 10\\ = & 0.1963\end{array}$$

7. $\frac{5}{7}$.

$$\begin{array}{rl}\log 5 = & 0.6990\\ \text{colog } 7 = & 9.1549 - 10\\ \cline{2-2} \therefore \log \frac{5}{7} = & 9.8539 - 10\end{array}$$

8. $\frac{5}{11}$.

$$\begin{array}{rl}\log\ 5 = & 0.6990\\ \text{colog } 11 = & 8.9586 - 10\\ \cline{2-2} \therefore \log \frac{5}{11} = & 9.6576 - 10\end{array}$$

9. $\frac{7}{11}$.

$$\begin{aligned} \log\ 7 &= 0.8451 \\ \text{colog}\ 11 &= \underline{8.9586 - 10} \\ \therefore \log \tfrac{7}{11} &= 9.8037 - 10 \end{aligned}$$

10. $\frac{3}{8}$.

$$\begin{aligned} \log 3 &= 0.4771 \\ \text{colog}\ 8 &= \underline{9.0970 - 10} \\ \therefore \log \tfrac{3}{8} &= 9.5741 - 10 \end{aligned}$$

11. $\frac{8}{11}$.

$$\begin{aligned} \log\ 8 &= 0.9030 \\ \text{colog}\ 11 &= \underline{8.9586 - 10} \\ \therefore \log \tfrac{8}{11} &= 9.8616 - 10 \end{aligned}$$

12. $\frac{5}{22}$.

$$\begin{aligned} \log\ 5 &= 0.6990 \\ \text{colog}\ 22 &= \underline{8.6576 - 10} \\ \therefore \log \tfrac{5}{22} &= 9.3566 - 10 \end{aligned}$$

13. $\frac{0.7}{11}$.

$$\begin{aligned} \log 0.7 &= 9.8451 - 10 \\ \text{colog}\ 11 &= \underline{8.9586 - 10} \\ \therefore \log \frac{0.7}{11} &= 8.8037 - 10 \end{aligned}$$

14. $\frac{6.05}{4}$.

$$6.05 = 5 \times 1.21 = 5 \times 1.1^2$$

$$\begin{aligned} \log 6.05 &= 0.6990 + 0.0828 \\ &= 0.7818 \\ \text{colog}\quad 4 &= \underline{9.3080 - 10} \\ \therefore \log.\frac{6.05}{4} &= 10.0898 - 10 \\ &= 0.0898 \end{aligned}$$

15. $\frac{0.03}{44}$.

$$\begin{aligned} \log 0.03 &= 8.4771 - 10 \\ \text{colog}\quad 44 &= \underline{8.3566 - 10} \\ \therefore \log \frac{0.03}{44} &= 6.8337 - 10 \end{aligned}$$

16. $\frac{44}{0.05}$.

$$\begin{aligned} \log\quad 44 &= 1.6434 \\ \text{colog}\ 0.05 &= \underline{1.3010} \\ \therefore \log \frac{44}{0.05} &= 2.9444 \end{aligned}$$

17. $\frac{35}{0.11}$.

$$\begin{aligned} \log\quad 35 &= 1.5441 \\ \text{colog}\ 0.11 &= \underline{0.9586} \\ \therefore \log \frac{35}{0.11} &= 2.5027 \end{aligned}$$

18. $\frac{24}{55}$.

$$\begin{aligned} \log 24 &= 1.3801 \\ \text{colog}\ 55 &= \underline{8.2596 - 10} \\ \therefore \log \tfrac{24}{55} &= 9.6397 - 10 \end{aligned}$$

19. $\frac{11^3}{12^2}$.

$$\begin{aligned} \log 11^3 &= 3.1242 \\ \text{colog}\ 12^2 &= \underline{7.8418 - 10} \\ \therefore \log \frac{11^3}{12^2} &= 10.9660 - 10 \\ &= 0.9660 \end{aligned}$$

20. $\frac{5^5}{6^4}$.

$$\begin{aligned} \log 5^5 &= 3.4950 \\ \text{colog}\ 6^4 &= \underline{6.8876 - 10} \\ \therefore \log \frac{5^5}{6^4} &= 10.3826 - 10 \\ &= 0.3826 \end{aligned}$$

21. $\frac{7^3}{2^5}$.

$$\begin{aligned} \log 7^3 &= 2.5353 \\ \text{colog}\, 2^5 &= \underline{8.4950 - 10} \\ \therefore \log \frac{7^3}{2^5} &= 11.0303 - 10 \\ &= 1.0303 \end{aligned}$$

22. $\frac{8^5}{3^7}$.

$$\begin{aligned} \log 8^5 &= 4.5150 \\ \text{colog}\, 3^7 &= \underline{6.6603 - 10} \\ \therefore \log \frac{8^5}{3^7} &= 11.1753 - 10 \\ &= 1.1753 \end{aligned}$$

23. $\frac{55^3}{28^2}$.

$$\begin{aligned} \log 55^3 &= 5.2212 \\ \text{colog}\, 28^2 &= \underline{7.1058 - 10} \\ \therefore \log \frac{55^3}{28^2} &= 12.3270 - 10 \\ &= 2.3270 \end{aligned}$$

24. $\frac{77^2}{15^7}$.

$$\begin{aligned} \log 77^2 &= 3.7730 \\ \text{colog}\, 15^7 &= \underline{1.7673 - 10} \\ \therefore \log \frac{77^2}{15^7} &= 5.5403 - 10 \end{aligned}$$

25. $\frac{33^2}{0.005}$.

$$\begin{aligned} \log 33^2 &= 3.0370 \\ \text{colog}\, 0.005 &= \underline{2.3010} \\ \therefore \log \frac{33^2}{0.005} &= 5.3380 \end{aligned}$$

26. $\frac{0.007}{15^3}$.

$$\begin{aligned} \log 0.007 &= 7.8451 - 10 \\ \text{colog}\, 15^3 &= \underline{6.4717 - 10} \\ \therefore \log \frac{0.007}{15^3} &= 4.3168 - 10 \end{aligned}$$

27. $\frac{0.011}{0.05^4}$.

$$\begin{aligned} \log 0.011 &= 8.0414 - 10 \\ \text{colog}\, 0.05^4 &= \underline{5.2040} \\ \therefore \log \frac{0.011}{0.05^4} &= 3.2454 \end{aligned}$$

28. $\frac{0.05^5}{0.011^3}$.

$$\begin{aligned} \log 0.05^5 &= 3.4950 - 10 \\ \text{colog}\, 0.011^3 &= \underline{5.8758} \\ \therefore \log \frac{0.05^5}{0.011^3} &= 9.3708 - 10 \end{aligned}$$

Exercise 116. Page 351.

Find, from the table, the commom logarithms of:

1. 50.
   1.6990.
2. 201.
   2.3032.
3. 888.
   2.9484.
4. 7803.
   3.8923.
5. 4325.
   3.6360.
6. 8109.
   3.9090.

7. 7063.
3.8490.

8. 1202.
3.0799.

9. 0.00789.
7.8971 – 10.

10. 000.5234.
9.7188 – 10.

11. 0.01423.
8.1532 – 10.

12. 0.1987.
9.2983 – 10.

Find antilogarithms to the following common logarithms:

13. 4.1432.
13910.

14. 3.5317.
3402.

15. 2.3177.
207.8.

16. 1.3709.
23.49.

17. 9.0380 – 10.
0.1092.

18. 9.9204 – 10.
0.8326.

Exercise 117. Page 353.

Find by logarithms the following products:

1. 948.7×0.04387.

$$\begin{aligned} \log\ 948.7 &= 2.9771\\ \log 0.04387 &= \underline{8.6422-10}\\ \text{log of product} &= 1.6193\\ \therefore\ \text{product} &= 41.62 \end{aligned}$$

2. 3.409×0.008763.

$$\begin{aligned} \log\ 3.409 &= 0.5326\\ \log 0.008763 &= \underline{7.9426-10}\\ \text{log of product} &= 8.4752-10\\ \therefore\ \text{product} &= 0.02987 \end{aligned}$$

3. 830.7×0.0003769.

$$\begin{aligned} \log\ 830.7 &= 2.9195\\ \log 0.0003769 &= \underline{6.5762-10}\\ \text{log of product} &= 9.4957-10\\ \therefore\ \text{product} &= 0.3131 \end{aligned}$$

4. 8.439×0.9827.

$$\begin{aligned} \log\ 8.439 &= 0.9263\\ \log 0.9827 &= \underline{9.9924-10}\\ \text{log of product} &= 0.9187\\ \therefore\ \text{product} &= 8.292 \end{aligned}$$

5. $7564 \times (-0.003764)$.

$$\begin{aligned} \log\ 7564 &= 3.8787\\ \log 0.003764 &= \underline{7.5757-10}\\ \text{log of product} &= 1.4544\\ \therefore\ \text{product} &= -28.47 \end{aligned}$$

6. $3.764 \times (-0.08349)$.

$$\begin{aligned} \log\ 3.764 &= 0.5757\\ \log 0.08349 &= \underline{8.9217-10}\\ \text{log of product} &= 9.4974-10\\ \therefore\ \text{product} &= -0.3144 \end{aligned}$$

7. $-5.845 \times (-0.00178)$.

$$\begin{aligned}
\log\ 5.845 &= 0.7668 \\
\log 0.00178 &= \underline{7.2504 - 10} \\
\text{log of product} &= 8.0172 - 10 \\
\therefore\ \text{product} &= 0.0104
\end{aligned}$$

8. -8945.7×73.84.

$$\begin{aligned}
\log 8945.7 &= 3.9516 \\
\log\ 73.84 &= \underline{1.8683} \\
\text{log of product} &= 5.8199 \\
\therefore\ \text{product} &= 660600
\end{aligned}$$

Find by logarithms:

9. $\frac{7065}{5401}$.

$$\begin{aligned}
\log\ 7065 &= \ 3.8491 \\
\text{colog}\ 5401 &= \ \underline{6.2675 - 10} \\
\text{log of quotient} &= 10.1166 - 10 \\
&= 0.1166 \\
\therefore\ \text{quotient} &= 1.308
\end{aligned}$$

10. $\dfrac{7.652}{-0.06875}$.

$$\begin{aligned}
\log\ 7.652 &= 0.8838 \\
\text{colog}\, 0.06875 &= \underline{1.1627} \\
\text{log of quotient} &= 2.0465 \\
\therefore\ \text{quotient} &= -111.3
\end{aligned}$$

11. $\dfrac{0.07654}{83.94 \times 0.8395}$.

$$\begin{aligned}
\log 0.07654 &= 8.8839 - 10 \\
\text{colog}\ 83.94 &= 8.0760 - 10 \\
\text{colog}\ 0.8395 &= \underline{0.0759\qquad\ } \\
\text{log of quotient} &= 7.0358 - 10 \\
\therefore\ \text{fraction} &= 0.001086
\end{aligned}$$

12. $\dfrac{212 \times (-6.12) \times (-2008)}{365 \times (-531) \times 2.576}$.

$$\begin{aligned}
\log\ 212 &= 2.3263 \\
\log\ 6.12 &= 0.7868 \\
\log\ 2008 &= 3.3027 \\
\text{colog}\ 365 &= 7.4377 - 10 \\
\text{colog}\ 531 &= 7.2749 - 10 \\
\text{colog}\, 2.576 &= \underline{9.5891 - 10} \\
\text{log of quotient} &= 0.7175 \\
\therefore\ \text{fraction} &= -5.218
\end{aligned}$$

13. 0.1768^5.

$$\begin{aligned}
\log\ 0.1768 &= 9.2475 - 10 \\
&\qquad\qquad\ \ \underline{\quad 5} \\
\log 0.1768^5 &= 6.2375 - 10 \\
\therefore\ 0.1768^5 &= 0.0001728
\end{aligned}$$

14. 1.211^{10}.

$$\begin{aligned}
\log\ 1.211 &= 0.0831 \\
&\qquad\ \ \underline{\ 10} \\
\log 1.211^{10} &= 0.8310 \\
\therefore\ 1.211^{10} &= 6.776
\end{aligned}$$

15. $11^{\frac{1}{5}}$.

$$\begin{aligned}
\log\ 11 &= \underline{1.0414}\ \underline{|5} \\
\log 11^{\frac{1}{5}} &= 0.2081 \\
\therefore\ 11^{\frac{1}{5}} &= 1.615
\end{aligned}$$

16. $(\frac{73}{61})^{11}$.

$$\begin{aligned}
\log 73 &= 1.8633 \\
\text{colog}\, 61 &= \underline{8.2147 - 10} \\
\text{log of quotient} &= 0.0780 \\
&\quad\ 0.0780 \\
&\qquad\ \underline{\ 11} \\
&\quad\ 0.8580 \\
\therefore\ (\tfrac{73}{61})^{11} &= 7.212
\end{aligned}$$

17. $\left(\frac{14}{51}\right)^7$.

$$\begin{aligned} \log 14 &= 1.1461\\ \text{colog } 51 &= 8.2924-10\\ \hline \text{log of quotient} &= 9.4385-10 \end{aligned}$$

$$\begin{array}{r} 9.4385-10\\ 7\\ \hline 6.0695-10 \end{array}$$

$$\therefore \left(\tfrac{14}{51}\right)^7 = 0.0001174$$

18. $906.8^{\frac{1}{4}}$.

$$\begin{aligned} \log\ 906.8 &= 2.9575 \,\underline{|4}\\ \hline \log 906.8^{\frac{1}{4}} &= 0.7394 \end{aligned}$$

$$\therefore 906.8^{\frac{1}{4}} = 5.487$$

19. $\left(\frac{951}{823}\right)^6$.

$$\begin{aligned} \log 951 &= 2.9782\\ \text{colog } 823 &= 7.0846-10\\ \hline \log \tfrac{951}{823} &= 0.0628 \end{aligned}$$

$$\begin{array}{r} 0.0628\\ 6\\ \hline 0.3768 \end{array}$$

$$\therefore \left(\tfrac{951}{823}\right)^6 = 2.381$$

20. $\left(7\frac{6}{11}\right)^{0.38}$.

$$\begin{aligned} \log 83 &= 1.9191\\ \text{colog } 11 &= 8.9586-10\\ \hline \log \tfrac{83}{11} &= 0.8777 \end{aligned}$$

$$\begin{array}{r} 0.8777\\ 0.38\\ \hline 0.3335 \end{array}$$

$$\therefore \left(\tfrac{83}{11}\right)^{0.38} = 2.155$$

21. $2.563^{\frac{3}{11}}$.

$$\begin{aligned} \log 2.563 &= 0.4087\\ &3\\ \hline &\underline{1.2261}(11\\ &0.1115 \end{aligned}$$

$$\therefore 2.563^{\frac{3}{11}} = 1.293$$

22. $\left(8\frac{3}{4}\right)^{2.3}$.

$$\begin{aligned} \log 35 &= 1.5441\\ \text{colog}\ \ 4 &= 9.3979-10\\ \hline \log \tfrac{35}{4} &= 0.9420 \end{aligned}$$

$$\begin{array}{r} 0.9420\\ 2.3\\ \hline 2.1666 \end{array}$$

$$\therefore \left(8\tfrac{3}{4}\right)^{2.3} = 146.6$$

23. $\left(5\frac{31}{37}\right)^{0.375}$.

$$\begin{aligned} \log 216 &= 2.3345\\ \text{colog}\ \ 37 &= 8.4318-10\\ \hline \log \tfrac{216}{37} &= 0.7663 \end{aligned}$$

$$\begin{array}{r} 0.7663\\ 0.375\\ \hline 0.2874 \end{array}$$

$$\therefore \left(5\tfrac{31}{37}\right)^{0.375} = 1.938$$

24. $\left(9\frac{20}{43}\right)^{\frac{1}{5}}$.

$$\begin{aligned} \log 407 &= 2.6096\\ \text{colog}\ \ 43 &= 8.3665-10\\ \hline \log \tfrac{407}{43} &= 0.9761 \end{aligned}$$

$$\begin{array}{r} \underline{0.9761}(5\\ 0.1952 \end{array}$$

$$\therefore \left(\tfrac{407}{43}\right)^{\frac{1}{5}} = 1.568$$

25. $$\sqrt[3]{\frac{0.0075^2\times 78.34\times 172.4^{\frac{1}{3}}\times 0.00052}{4285^{\frac{1}{3}}\times 54.27^4\times 0.001\times 867.9^{\frac{1}{2}}}}.$$

$$\begin{array}{rll}
2\log 0.0075 & = 2(7.8751-10) & = 5.7502-10 \\
\log 78.34 & & = 1.8940 \\
\frac{1}{3}\log 172.4 & = \frac{1}{3}(2.2365) & = 0.7455 \\
\log 0.00052 & & = 6.7160-10 \\
\frac{1}{3}\operatorname{colog} 4285 & = \frac{1}{3}(6.3671-10) & = 8.7890-10 \\
4\operatorname{colog} 54.27 & = 4(8.2653-10) & = 3.0612-10 \\
\operatorname{colog} 0.001 & & = 3.0000 \\
\frac{1}{2}\operatorname{colog} 867.9 & = \frac{1}{2}(7.0616-10) & = \underline{8.5308-10} \\
 & & 3\underline{)18.4867-30} \\
 & & 6.1622-10
\end{array}$$

∴ the fraction = 0.0001453.

26. $$\sqrt[4]{\frac{0.03271^2\times 3.429\times 0.7752^3}{32.79\times 0.00371^4}}.$$

$$\begin{array}{rll}
2\log 0.3271 & = 2(9.5146-10) & = 9.0292-10 \\
\log 3.429 & & = 0.5351 \\
3\log 0.7752 & = 3(9.8894-10) & = 9.6682-10 \\
\operatorname{colog} 32.79 & & = 8.4843-10 \\
4\operatorname{colog} 0.00371 & = 4(2.4306) & = \underline{9.7224} \\
 & & 7.4392 \\
 & & 4\underline{)7.4392} \\
 & & 1.8598
\end{array}$$

∴ the fraction = 72.42.

27. $$\sqrt[3]{\frac{7.126\times\sqrt{0.1327}\times 0.05738}{\sqrt{0.4346}\times 17.38\times\sqrt{0.006372}}}.$$

$$\begin{array}{rll}
\log 7.126 & & = 0.8529 \\
\frac{1}{2}\log 0.1327 & = \frac{1}{2}(9.1228-10) & = 9.5614-10 \\
\log 0.05738 & & = 8.7588-10 \\
\frac{1}{2}\operatorname{colog} 0.4346 & = \frac{1}{2}(0.3619) & = 0.1809 \\
\operatorname{colog} 17.38 & & = 8.7600-10 \\
\frac{1}{2}\operatorname{colog} 0.006372 & = \frac{1}{2}(2.1958) & = \underline{1.0979} \\
 & & 9.2119-10 \\
 & & 3\underline{)29.2119-30} \\
 & & 9.7373-10
\end{array}$$

∴ the fraction = 0.5461.

Find x from the equations.

28. $5^x = 10$.

$$\begin{aligned} 5^x &= 10 \\ x \log 5 &= \log 10 \\ x &= \frac{\log 10}{\log 5} \\ &= \frac{1}{0.6990} \end{aligned}$$

$$\begin{aligned} \log 1 \qquad &= 0.0000 \\ \text{colog}\, 0.6990 &= \underline{0.1555} \\ \log x &= 0.1555 \\ \therefore\ x &= 1.431 \end{aligned}$$

29. $4^x = 20$.

$$\begin{aligned} 4^x &= 20 \\ x \log 4 &= \log 20 \\ x &= \frac{\log 20}{\log 4} \\ &= \frac{1.3010}{0.6020} \end{aligned}$$

$$\begin{aligned} \log\ \ 1.301 &= 0.1142 \\ \text{colog}\, 0.6020 &= \underline{0.2204} \\ \log x &= 0.3346 \\ \therefore\ x &= 2.160 \end{aligned}$$

30. $7^x = 40$.

$$\begin{aligned} 7^x &= 40 \\ x \log 7 &= \log 40 \\ x &= \frac{\log 40}{\log 7} \\ &= \frac{1.6020}{0.8451} \end{aligned}$$

$$\begin{aligned} \log 1.6020 &= 0.2046 \\ \text{colog}\, 0.8451 &= \underline{0.0731} \\ \log x &= 0.2777 \\ \therefore\ x &= 1.895 \end{aligned}$$

31. $(1.3)^x = 4.2$.

$$\begin{aligned} (1.3)^x &= 4.2 \\ x \log 1.3 &= \log 4.2 \\ x &= \frac{\log 4.2}{\log 1.3} \\ &= \frac{0.6232}{0.1139} \end{aligned}$$

$$\begin{aligned} \log 0.6232 &= 9.7946 - 10 \\ \text{colog}\, 0.1139 &= \underline{0.9435 \qquad\quad} \\ \log x &= 0.7371 \\ \therefore\ x &= 5.459 \end{aligned}$$

32. $(0.4)^{-x} = 3$.

$$\begin{aligned} (0.4)^{-x} &= 3 \\ -x \log 0.4 &= \log 3 \\ x &= -\frac{\log 3}{\log 0.4} \\ &= -\frac{0.4771}{9.6021 - 10} \\ &= \frac{4771}{3979} \end{aligned}$$

$$\begin{aligned} \log 4771 &= 3.6786 \\ \text{colog}\, 3779 &= \underline{6.4227 - 10} \\ \log x &= 0.1013 \\ \therefore\ x &= 1.263 \end{aligned}$$

33. $(0.9)^{-x} = 2$.

$$\begin{aligned} (0.9)^{-x} &= 2 \\ -x \log 0.9 &= \log 2 \\ x &= -\frac{\log 2}{\log 0.9} \\ &= -\frac{0.3010}{9.9542 - 10} \\ &= \frac{3010}{458} \end{aligned}$$

$$\begin{aligned} \log 3010 &= 3.4786 \\ \text{colog}\ \ 458 &= \underline{7.3391 - 10} \\ \log x &= 0.8177 \\ \therefore\ x &= 6.571 \end{aligned}$$

General Review Exercise. Page 356.

If $a=6$, $b=5$, $c=-4$, $d=-3$, **find the value of**

1. $\sqrt{b^2+ac}+\sqrt{c^2-2ac}$.

$$\sqrt{b^2+ac}+\sqrt{c^2-2ac}=\sqrt{25-24}+\sqrt{16+48}=1+8=9.$$

2. $\dfrac{a^2-\sqrt{b^2+ac}}{2a-\sqrt{b^2-ac}}$.

$$\frac{a^2-\sqrt{b^2+ac}}{2a-\sqrt{b^2-ac}}=\frac{36-\sqrt{25-24}}{12-\sqrt{25+24}}=\frac{36-1}{12-7}=\frac{35}{5}=7.$$

3. $\sqrt{b^2+ac+\sqrt{c^2-2ac}}$.

$$\sqrt{b^2+ac+\sqrt{c^2-2ac}}=\sqrt{25-24+\sqrt{16+48}}$$
$$=\sqrt{1+\sqrt{64}}=\sqrt{9}=3.$$

4. $\dfrac{c+\sqrt{d^2+c^2}}{c^3+2d(d^2-c^2)}$.

$$\frac{c+\sqrt{d^2+c^2}}{c^3+2d(d^2-c^2)}=\frac{-4+\sqrt{9+16}}{-64-6(9-16)}=\frac{-4+5}{-64+42}=-\frac{1}{22}.$$

Find the value of

5. $\dfrac{x}{a}+\dfrac{x}{b}$, when $x=\dfrac{abc}{a+b}$.

If $x=\dfrac{abc}{a+b}$,

$$\frac{x}{a}+\frac{x}{b}=\frac{abc}{a(a+b)}+\frac{abc}{b(a+b)}$$
$$=\frac{bc}{a+b}+\frac{ac}{a+b}$$
$$=\frac{bc+ac}{a+b}$$
$$=\frac{c(a+b)}{a+b}$$
$$=c.$$

6. $\frac{1}{a}(b-x)+\frac{1}{b}(c-x)+\frac{1}{a}(x-c)$, when $x=\frac{ab-b^2+bc}{a}$.

If $x=\frac{ab-b^2+bc}{a}$,

$$\frac{1}{a}(b-x)+\frac{1}{b}(c-x)+\frac{1}{a}(x-c)$$

$$=\frac{1}{a}\left(b-\frac{ab-b^2+bc}{a}\right)+\frac{1}{b}\left(c-\frac{ab-b^2+bc}{a}\right)+\frac{1}{a}\left(\frac{ab-b^2+bc}{a}-c\right)$$

$$=\frac{1}{a}\left(\frac{ab-(ab-b^2+bc)}{a}\right)+\frac{1}{b}\left(\frac{ac-(ab-b^2+bc)}{a}\right)+\frac{1}{a}\left(\frac{ab-b^2+bc-ac}{a}\right)$$

$$=\frac{1}{a}\left(\frac{b^2-bc}{a}\right)+\frac{1}{b}\left(\frac{ac-ab+b^2-bc}{a}\right)+\frac{1}{a}\left(\frac{ab-b^2+bc-ac}{a}\right)$$

$$=\frac{b^2-bc}{a^2}+\frac{ac-ab+b^2-bc}{ab}+\frac{ab-b^2+bc-ac}{a^2}$$

$$=\frac{ab-ac}{a^2}+\frac{ac-ab+b^2-bc}{ab}$$

$$=\frac{b-c}{a}+\frac{ac-ab+b^2-bc}{ab}$$

$$=\frac{b^2-bc}{ab}+\frac{ac-ab+b^2-bc}{ab}$$

$$=\frac{2b^2-2bc+ac-ab}{ab}$$

$$=\frac{(a-2b)(c-b)}{ab}.$$

7. $\frac{x}{a}+\frac{x}{b-a}$, when $x=\frac{a^2(b-a)}{b(b+a)}$.

If $x=\frac{a^2(b-a)}{b(b+a)}$,

$$\frac{x}{a}+\frac{x}{b-a}=\frac{a^2(b-a)}{ab(b+a)}+\frac{a^2(b-a)}{b(b-a)(b+a)}$$

$$=\frac{a(b-a)}{b(b+a)}+\frac{a^2}{b(b+a)}$$

$$=\frac{ab-a^2+a^2}{b(b+a)}$$

$$=\frac{ab}{b(b+a)}=\frac{a}{b+a}$$

8. $(a+x)(b+x)-a(b+c)+x^2$, when $x=\frac{ac}{b}$.

If $x=\frac{ac}{b}$,

$$\begin{aligned}
(a+x)(b+x)-a(b+c)+x^2\\
&=\left(a+\frac{ac}{b}\right)\left(b+\frac{ac}{b}\right)-a(b+c)+\frac{a^2c^2}{b^2}\\
&=\frac{ab+ac}{b}\times\frac{b^2+ac}{b}-a(b+c)+\frac{a^2c^2}{b^2}\\
&=\frac{(ab+ac)(b^2+ac)}{b^2}-a(b+c)+\frac{a^2c^2}{b^2}\\
&=\frac{(ab+ac)(b^2+ac)-ab^{\text{[illegible]}}(b+c)+a^2c^2}{b^2}\\
&=\frac{ab^3+a^2bc+ab^2c+a^2c^2-ab^3-ab^2c+a^2c^2}{b^2}\\
&=\frac{a^2bc+2a^2c^2}{b^2}.
\end{aligned}$$

9. $\frac{a(1+b)+bx}{a(1+b)-bx}-\frac{a}{a-2bx}$, when $x=-a$.

If $x=-a$,

$$\begin{aligned}
\frac{a(1+b)+bx}{a(1+b)-bx}-\frac{a}{a-2bx}&=\frac{a(1+b)-ab}{a(1+b)+ab}-\frac{a}{a+2ab}\\
&=\frac{a}{a+2ab}-\frac{a}{a+2ab}\\
&=0.
\end{aligned}$$

10. Add $(a-b)x^2+(b-c)y^2+(c-a)z^2$, $(b-c)x^2+(c-a)y^2+(a-b)z^2$, and $(c-a)x^2+(a-b)y^2+(b-c)z^2$.

$$\begin{array}{r}
(a-b)x^2+(b-c)y^2+(c-a)z^2\\
(b-c)x^2+(c-a)y^2+(a-b)z^2\\
(c-a)x^2+(a-b)y^2+(b-c)z^2\\
\hline
0\qquad\qquad\qquad
\end{array}$$

11. Add $(a+b)x+(b+c)y-(c+a)z$, $(b+c)z+(c+a)x-(a+b)y$, and $(a+c)y+(a+b)z-(b+c)x$.

$$\begin{array}{rrr}
(a+b)x & +\,(b+c)y & -\,(c+a)z\\
(c+a)x & -\,(a+b)y & +\,(b+c)z\\
-\,(b+c)x & +\,(a+c)y & +\,(a+b)z\\
\hline
2ax & +\,2cy & +\,2bz
\end{array}$$

12. Show that $x^3 + y^3 + z^3 - 3xyz = 0$, if $x + y + z = 0$.

If $x + y + z = 0$, then $x = -y - z$.

$$\begin{aligned} x^3 &= -y^3 - 3y^2z - 3xy^2 - z^3 \\ xyz &= -y^2z - yz^2 \\ \therefore\ x^3 + y^3 + z^3 - 3xyz &= -y^3 - 3y^2z - 3yz^2 - z^3 + y^3 + z^3 + 3y^2z + 3yz^2 \\ &= 0. \end{aligned}$$

13. Show that $x^3 - 8y^3 - 27z^3 - 18xyz = 0$, if $x = 2y + 3z$.

If

$$\begin{aligned} x &= 2y + 3z, \\ x^3 &= 8y^3 + 36y^2z + 54yz^2 + 27z^3 \\ xyz &= 2y^2z + 3yz^2 \\ \therefore\ x^3 - 8y^3 - 27z^3 - 18xyz &= 8y^3 + 36y^2z + 54yz^2 + 27z^3 - 8y^3 - 27z^3 \\ &\qquad - 36y^2z - 54yz^2 \\ &= 0. \end{aligned}$$

Simplify by removing parenthesis and collecting terms.

14. $3a - 2(b - c) - [2(a - b) - 3(c + a)] - [9c - 4(c - a)]$

$$\begin{aligned} &= 3a - 2b + 2c - [2a - 2b - 3c - 3a] - [9c - 4c + 4a] \\ &= 3a - 2b + 2c - [-2b - 3c - a] - [5c + 4a] \\ &= 3a - 2b + 2c + 2b + 3c + a - 5c - 4a \\ &= 0. \end{aligned}$$

15. $7(2a + b) - \{19b - [13(c - a) + 12(b - c)]\}$

$$\begin{aligned} &= 14a + 7b - \{19b - [13c - 13a + 12b - 12c]\} \\ &= 14a + 7b - \{19b - [-13a + 12b + c]\} \\ &= 14a + 7b - \{19b + 13a - 12b - c\} \\ &= 14a + 7b - \{13a + 7b - c\} \\ &= 14a + 7b - 13a - 7b + c \\ &= a + c. \end{aligned}$$

16. $x - \{4y + [3(z - x) - (x + 2y)] - (2y + z - 2x)\}$

$$\begin{aligned} &= x - \{4y + [3z - 3x - x - 2y] - 2y - z + 2x\} \\ &= x - \{4y + [3z - 4x - 2y] - 2y - z + 2x\} \\ &= x - \{4y + 3z - 4x - 2y - 2y - z + 2x\} \\ &= x - \{-2x + 2z\} \\ &= x + 2x - 2z \\ &= 3x - 2z. \end{aligned}$$

17. $1 + 2\{x + 4 - 3[x + 5 - 4(x + 1)]\}$

$$\begin{aligned}
&= 1 + 2\{x + 4 - 3[x + 5 - 4x - 4]\}\\
&= 1 + 2\{x + 4 - 3[1 - 3x]\}\\
&= 1 + 2\{x + 4 - 3 + 9x\}\\
&= 1 + 2\{1 + 10x\}\\
&= 1 + 2 + 20x\\
&= 3 + 20x.
\end{aligned}$$

18. $10x - \{4[5x - 3(x - 1)] - 3[4x - 3(x + 1)]\}$

$$\begin{aligned}
&= 10x - \{4[5x - 3x + 3] - 3[4x - 3x - 3]\}\\
&= 10x - \{4[2x + 3] - 3[x - 3]\}\\
&= 10x - \{8x + 12 - 3x + 9\}\\
&= 10x - \{5x + 21\}\\
&= 10x - 5x - 21\\
&= 5x - 21.
\end{aligned}$$

19. $3x^2 - \{2x^2 - (3x - 7) - [2x^2 - (3x - x^2)] - [5 - (2x^2 - 4x)]\}$

$$\begin{aligned}
&= 3x^2 - \{2x^2 - 3x + 7 - [2x^2 - 3x + x^2] - [5 - 2x^2 + 4x]\}\\
&= 3x^2 - \{2x^2 - 3x + 7 - [3x^2 - 3x] - [5 - 2x^2 + 4x]\}\\
&= 3x^2 - \{2x^2 - 3x + 7 - 3x^2 + 3x - 5 + 2x^2 - 4x\}\\
&= 3x^2 - \{x^2 - 4x + 2\}\\
&= 3x^2 - x^2 + 4x - 2\\
&= 2x^2 + 4x - 2.
\end{aligned}$$

20. $(x - 2)(x - 3) - (x - 7)(x - 1) + (x - 1)(x - 2)$

$$\begin{aligned}
&= x^2 - 5x + 6 - x^2 + 8x - 7 + x^2 - 3x + 2\\
&= x^2 + 1.
\end{aligned}$$

21. $(x + y)^2 - 2x(3x + 2y) - (y - x)(-x + y)$

$$\begin{aligned}
&= x^2 + 2xy + y^2 - 6x^2 - 4xy + yx - y^2 - x^2 + xy\\
&= -6x^2.
\end{aligned}$$

22. $3a - [2a - (3a - b)^2] + 3a\left(2b - 3a - \dfrac{b^2}{3a}\right)$

$$\begin{aligned}
&= 3a - [2a - 9a^2 + 6ab - b^2] + 6ab - 9a^2 - b^2\\
&= 3a - 2a + 9a^2 - 6ab + b^2 + 6ab - 9a^2 - b^2\\
&= a.
\end{aligned}$$

Resolve into lowest factors:

23. $(x + y)^2 - 4z^2$

$$= (x + y + 2z)(x + y - 2z).$$

24. $(x^2 + y^2)^2 - 4x^2y^2$

$$= (x^2 + y^2 + 2xy)(x^2 + y^2 - 2xy) = (x + y)^2(x - y)^2.$$

$-$ $-$

25. $a^2-b^2-c^2+2bc=a^2-(b^2-2bc+c^2)$
$=[a-(b-c)][a+(b-c)]$
$=(a-b+c)(a+b-c).$

26. $(x^2-y^2-z^2)^2-4y^2z^2=(x^2-y^2-z^2+2yz)(x^2-y^2-z^2-2yz)$
$=[x^2-(y^2-2yz+z^2)][x^2-(y^2+2yz+z^2)]$
$=(x+y-z)(x-y+z)(x+y+z)(x-y-z).$

27. $9x^8-\frac{3}{2}x^4y^2+\frac{1}{16}y^4=(3x^4-\frac{1}{4}y^2)^2$
$=\left(\sqrt{3}x^2-\frac{y}{2}\right)^2\left(\sqrt{3}x^2+\frac{y}{2}\right)^2.$

28. $\left(\frac{a}{b}\right)^{2m}+\left(\frac{b}{a}\right)^{2m}+2=\left\{\left(\frac{a}{b}\right)^m+\left(\frac{b}{a}\right)^m\right\}^2$

29. $81a^4-1=(9a^2+1)(9a^2-1)=(9a^2+1)(3a+1)(3a-1).$

30. $a^{12}-b^{12}=(a^6+b^6)(a^6-b^6)$
$=(a^2+b^2)(a^4-a^2b^2+b^4)(a^3+b^3)(a^3-b^3)$
$=(a^2+b^2)(a^4-a^2b^2+b^4)(a+b)(a^2-ab+b^2)(a-b)(a^2+ab+b^2).$

31. $(a^2-b^2+c^2-d^2)^2-(2ac-2bd)^2$
$=(a^2-b^2+c^2-d^2+2ac-2bd)(a^2-b^2+c^2-d^2-2ac+2bd$
$=(a^2+2ac+c^2-b^2-2bd-d^2)(a^2-2ac+c^2-b^2+2bd-d^2)$
$=[(a+c)^2-(b+d)^2][(a-c)^2-(b-d)^2]$
$=(a+c+b+d)(a+c-b-d)(a-c+b-d)(a-c-b+d)$
$=(a+b+c+d)(a-b+c-d)(a+b-c-d)(a-b-c+d).$

32. $x^2-19x+84=(x-7)(x-12).$

33. $\frac{1}{4}x^2+2\frac{1}{2}x-36=(\frac{1}{2}x+9)(\frac{1}{2}x-4).$

34. $x^2-8x+15=(x-3)(x-5).$

35. $9x^2-150x+600=3(x-10)(3x-20).$

36. $56x^2+3xy-20y^2=(7x-4y)(8x+5y).$

37. $12x^2+37x+21=(4x+3)(3x+7).$

38. $33-14x-40x^2=(11+10x)(3-4x).$

39. $6x^2+5x-4=(2x-1)(3x+4).$

40. $x^3-y^6=(x-y^2)(x^2+xy^2+y^4).$

41. $8+a^3x^3=(2+ax)(4-2ax+a^2x^2).$

42. $x^5-a^{10}=(x-a^2)(x^4+a^2x^3+a^4x^2+a^6x+a^8).$

43. $27a^3-64=(3a-4)(9a^2+12a+16).$

44. $x^{10} - 32y^5 = (x^2 - 2y)(x^8 + 2x^6y + 4x^4y^2 + 8x^2y^3 + 16y^4).$

45. $a^{12} - b^8 = (a^6 + b^4)(a^6 - b^4) = (a^6 - b^4)(a^3 + b^2)(a^3 - b^2).$

46. $x^{15} + 1024y^{10} = (x^3 + 4y^2)(x^{12} + 4x^9y^2 + 16x^6y^4 + 64x^3y^6 + 256y^8).$

47. $$\begin{aligned} a^3 - (b+c)^3 &= [a - (b+c)][a^2 + a(b+c) + (b+c)^2] \\ &= (a - b - c)(a^2 + b^2 + c^2 + ab + ac + 2bc). \end{aligned}$$

48. $$\begin{aligned} 8x^3 - 6xy(2x + 3y) + 27y^3 &= 8x^3 + 27y^3 - 6xy(2x + 3y) \\ &= (2x + 3y)(4x^2 - 6xy + 9y^2 - 6xy) \\ &= (2x + 3y)(4x^2 - 12xy + 9y^2) \\ &= (2x + 3y)(2x - 3y)^2. \end{aligned}$$

Find the H. C. F. of

49. $6x^4 - 2x^3 + 9x^2 + 9x - 4$, and $9x^4 + 80x^2 - 9$.

$9x^4 + 80x^2 - 9 = (9x^2 - 1)(x^2 + 9) = (3x - 1)(3x + 1)(x^2 + 9).$

By trial we find that of these factors only $3x - 1$ will divide $6x^4 - 2x^3 + 9x^2 + 9x - 4$.

$\therefore$ the H. C. F. is $3x - 1$.

50. $3x^5 - 5x^3 + 2$, and $2x^5 - 5x^2 + 3$.

$$\begin{array}{r|r|l}
2x^5 - 5x^2 + 3 & 3x^5 - 5x^3 + 2 & \\
\underline{2x^5 - 3x^4 + x^2} & \underline{2} & \\
3)\,\underline{3x^4 - 6x^2 + 3} & 6x^5 - 10x^3 + 4 & 3 \\
x^4 - 2x^2 + 1 & \underline{6x^5 - 15x^2 + 9} & \\
\underline{2} & -5)\,\underline{-10x^3 + 15x^2 - 5} & \\
2x^4 - 4x^2 + 2 & 2x^3 - 3x^2 + 1 & x^2 + x + 3 \\
\underline{2x^4 - 3x^3 + x} & \underline{2x^3 - 4x^2 + 2x} & 2x + 1 \\
3x^3 - 4x^2 - x + 2 & x^2 - 2x + 1 & \\
\underline{2} & \underline{x^2 - 2x + 1} & \\
6x^3 - 8x^2 - 2x + 4 & & \\
\underline{6x^3 - 9x^2 \quad + 3} & & \\
x^2 - 2x + 1 & &
\end{array}$$

$\therefore$ the H. C. F. is $x^2 - 2x + 1$.

51. $x^3 - 93x - 308$, and $x^3 - 21x^2 + 131x - 231$.

$$\begin{array}{r|r|l}
x^3 - 93x - 308 & x^3 - 21x^2 + 131x - 231 & 1 \\
3 & x^3 \qquad - 93x - 308 & \\
\hline
3x^3 - 279x - 924 & -7)\underline{-21x^2 + 224x + 77} & \\
3x^3 - 32x^2 - 11x & 3x^2 - 32x - 11 & x + 8 \\
4)\underline{32x^2 - 268x - 924} & 3x^2 - 33x & 3x + 1 \\
8x^2 - 67x - 231 & x - 11 & \\
3 & x - 11 & \\
24x^2 - 201x - 693 & & \\
24x^2 - 256x - 88 & & \\
55)\underline{55x - 605} & & \\
x - 11 & &
\end{array}$$

$\therefore$ the H. C. F. is $x - 11$.

52. $x^4 - 2x^3 + 4x^2 - 6x + 3$, and $x^4 - 2x^3 - 2x^2 + 6x - 3$.

$$\begin{array}{r|r|l}
x^4 - 2x^3 + 4x^2 - 6x + 3 & x^4 - 2x^3 - 2x^2 + 6x - 3 & 1 \\
x^4 - 2x^3 + x^2 \qquad & x^4 - 2x^3 + 4x^2 - 6x + 3 & \\
\hline
3x^2 - 6x + 3 & -6)\underline{-6x^2 + 12x - 6} & \\
3x^2 - 6x + 3 & x^2 - 2x + 1 & x^2 + 3
\end{array}$$

$\therefore$ the H. C. F. is $x^2 - 2x + 1$.

53. $x^5 - 4x^3 - x^2 + 2x + 2$, and $x^3 - x^2 - 2x + 2$.

$$x^3 - x^2 - 2x + 2 = (x^2 - 2)(x - 1)$$

By trial we find that of these two factors only $x - 1$ will divide $x^5 - 4x^3 - x^2 + 2x + 2$.

$\therefore$ the H. C. F. is $x - 1$.

54. $3x^3 + 10x^2 + 7x - 2$, and $3x^3 + 13x^2 + 17x + 6$.

$$\begin{array}{r|r|l}
3x^3 + 10x^2 + 7x - 2 & 3x^3 + 13x^2 + 17x + 6 & 1 \\
3x^3 + 10x^2 + 8x \qquad & 3x^3 + 10x^2 + 7x - 2 & \\
\hline
- x - 2 & 3x^2 + 10x + 8 & x \\
& 3x^2 + 6x \qquad & -3x - 4 \\
& 4x + 8 & \\
& 4x + 8 &
\end{array}$$

$\therefore$ the H. C. F. is $x + 2$.

55. $4x^4 - 9x^2 + 6x - 1$, and $6x^3 - 7x^2 + 1$.

$$\begin{array}{r|r|l}
6x^3 - 7x^2 \qquad + 1 & 4x^4 \qquad - 9x^2 + 6x - 1 & \\
6x^3 - 9x^2 + 3x \quad & 3 & \\
\cline{1-2}
2x^2 - 3x + 1 & 12x^4 \qquad - 27x^2 + 18x - 3 & 2x \\
2x^2 - 3x + 1 & 12x^4 - 14x^3 \qquad + 2x \quad & \\
\cline{1-2}
 & 14x^3 - 27x^2 + 16x - 3 & \\
 & 3 & \\
\cline{2-2}
 & 42x^3 - 81x^2 + 48x - 9 & 7 \\
 & 42x^3 - 49x^2 \qquad + 7 & \\
\cline{2-2}
 & -16\underline{)-32x^2 + 48x - 16} & \\
 & 2x^2 - 3x + 1 & 3x + 1
\end{array}$$

$\therefore$ the H. C. F. is $2x^2 - 3x + 1$.

56. $x^5 + 11x - 12$, and $x^5 + 11x^3 + 54$.

$$\begin{array}{r|r|l}
x^5 \qquad + 11x - 12 & x^5 + 11x^3 \qquad + 54 & 1 \\
x^5 - x^3 + 6x^2 \qquad & x^5 \qquad + 11x - 12 & \\
\hline
x^3 - 6x^2 + 11x - 12 & 11\underline{)11x^3 \qquad - 11x + 66} & \\
x^3 \qquad - x + 6 & x^3 \qquad - x + 6 & x^2 + 1 \\
\cline{1-1}
-6\underline{)-6x^2 + 12x - 18} & x^3 - 2x^2 + 3x \quad & x + 2 \\
\cline{2-2}
x^2 - 2x + 3 & 2x^2 - 4x + 6 & \\
 & 2x^2 - 4x + 6 & \\
\cline{2-2}
\end{array}$$

$\therefore$ the H. C. F. is $x^2 - 2x + 3$.

Find the L. C. M. of

57. $4x^2 + 4x - 3$, and $4x^2 + 2x - 6$.

$$4x^2 + 4x - 3 = (2x - 1)(2x + 3)$$
$$4x^2 + 2x - 6 = (2x - 2)(2x + 3)$$

$\therefore$ the L. C. M. is $(2x - 1)(2x - 2)(2x + 3) = 8x^3 - 14x + 6$.

58. $x^2 - 4y^2$, and $x^2 + xy - 6y^2$.

$$x^2 - 4y^2 = (x + 2y)(x - 2y)$$
$$x^2 + xy - 6y^2 = (x - 2y)(x + 3y)$$

$\therefore$ the L. C. M. is $(x - 2y)(x + 2y)(x + 3y) = x^3 + 3x^2y - 4xy^2 - 12y^3$.

59. $7a^2x(a-x)$, $21ax(a^2-x^2)$, and $12ax^2(a+x)$.

$$\begin{aligned}7a^2x(a-x) &= 7aax(a-x)\\ 21ax(a^2-x^2) &= 3\times 7aax(a+x)(a-x)\\ 12ax^2(a+x) &= 3\times 2\times 2axx(a+x)\end{aligned}$$

$\therefore$ the L. C. M. is $3\times 2\times 2\times 7aaxx(a+x)(a-x) = 84a^2x^2(a^2-x^2)$.

60. $9x^3-x-2$, and $3x^3-10x^2-7x-4$.

$$\begin{array}{r|r|l}
9x^3 \qquad - x - 2 & 3x^3 - 10x^2 - 7x - 4 & \\
9x^3 + 6x^2 + 3x & 3 & \\
\cline{1-2}
-6x^2 - 4x - 2 & 9x^3 - 30x^2 - 21x - 12 & 1 \\
-6x^2 - 4x - 2 & 9x^3 \qquad - x - 2 & \\
\cline{1-2}
 & -10)\underline{-30x^2 - 20x - 10} & \\
 & 3x^2 + 2x + 1 & 3x-2
\end{array}$$

$\therefore$ the H. C. F. is $3x^2+2x+1$.

$$\begin{aligned}9x^3-x-2 &= (3x^2+2x+1)(3x-2)\\ 3x^3-10x^2-7x-4 &= (3x^2+2x+1)(x-4)\end{aligned}$$

$\therefore$ the L. C. M. is $(3x^2+2x+1)(3x-2)(x-4)$.

61. x^2-5x+6, x^2-4x+3, and x^2-3x+2.

$$\begin{aligned}x^2-5x+6 &= (x-2)(x-3)\\ x^2-4x+3 &= (x-1)(x-3)\\ x^2-3x+2 &= (x-1)(x-2)\end{aligned}$$

$\therefore$ the L. C. M. is $(x-1)(x-2)(x-3) = x^3-6x^2+11x-6$.

62. $2x^3+5x^2y-5xy^2+y^3$, and $2x^3-7x^2y+5xy^2-y^3$.

$$\begin{array}{r|r|l}
2x^3 + 5x^2y - 5xy^2 + y^3 & 2x^3 - 7x^2y + 5xy^2 - y^3 & 1 \\
3 & 2x^3 + 5x^2y - 5xy^2 + y^3 & \\
\cline{1-2}
6x^3 + 15x^2y - 15xy^2 + 3y^3 & -2y)\underline{-12x^2y + 10xy^2 - 2y^3} & \\
6x^3 - 5x^2y + xy^2 & 6x^2 - 5xy - y^2 & x+10y \\
\cline{1-1}
20x^2y - 16xy^2 + 3y^3 & 6x^2 - 3xy & \\
\cline{2-2}
3 & -2xy + y^2 & 3x-y \\
\cline{1-1}
60x^2y - 48xy^2 + 9y^3 & -2xy + y^2 & \\
\cline{2-2}
60x^2y - 50xy^2 + 10y^3 & & \\
\cline{1-1}
y^2)\underline{2xy^2 - y^3} & & \\
2x - y & &
\end{array}$$

$\therefore$ the H. C. F. is $2x-y$.

$$\begin{aligned}2x^3+5x^2y-5xy^2+y^3 &= (2x-y)(x^2+3xy-y^2)\\ 2x^3-7x^2y+5xy^2-y^3 &= (2x-y)(x^2-3xy+y^2)\end{aligned}$$

$\therefore$ the L. C. M. is $(2x-y)(x^2+3xy-y^2)(x^2-3xy+y^2)$

Simplify:

63. $\frac{x+1}{x^2-2x}+\frac{2x-1}{x^2-x-2}-\frac{3x+2}{x^2+x}$.

$$\begin{aligned}x^2-2x&=x(x-2)\\x^2-x-2&=(x+1)(x-2)\\x^2+x&=x(x+1)\end{aligned}$$

$\therefore$ the L.C.D. is $x(x+1)(x-2)$

$$\begin{aligned}(x+1)(x+1)&=x^2\qquad+1=\text{1st numerator.}\\x(2x-1)&=2x^2-\ x\qquad=\text{2d numerator.}\\-(3x+2)(x-2)&=\underline{-3x^2+4x+4}=\text{3d numerator.}\\&\qquad\qquad 3x+5=\text{sum of numerators.}\end{aligned}$$

$\therefore$ the sum of the fractions is $\frac{3x+5}{x(x+1)(x-2)}=\frac{3x+5}{x^3-x^2-2x}$.

64. $\frac{1+x}{1-x}+\frac{1-x}{1+x}-\frac{1+x^2}{1-x^2}-\frac{1-x^2}{1+x^2}$.

The L. C. D. is $(1-x^2)(1+x^2)$.

$$\begin{aligned}(1+x)(1+x)(1+x^2)&=x^4+2x^3+2x^2+2x+1=\text{1st numerator.}\\(1-x)(1-x)(1+x^2)&=x^4-2x^3+2x^2-2x+1=\text{2d numerator.}\\-(1+x^2)^2&=-x^4\qquad-2x^2\qquad-1=\text{3d numerator.}\\-(1-x^2)^2&=\underline{-x^4\qquad+2x^2\qquad-1}=\text{4th numerator.}\\&\qquad\qquad 4x^2\qquad\qquad=\text{sum of numerators.}\end{aligned}$$

$\therefore$ the sum of the fractions is $\frac{4x^2}{(1-x^2)(1+x^2)}=\frac{4x^2}{1-x^4}$.

65. $\frac{x-1}{(x+2)(x+5)}-\frac{2(x+2)}{(x+5)(x-1)}+\frac{x+5}{(x-1)(x+2)}$.

The L. C. D. is $(x-1)(x+2)(x+5)$.

$$\begin{aligned}(x-1)^2&=x^2-\ 2x+\ 1=\text{1st numerator.}\\-2(x+2)^2&=-2x^2-\ 8x-\ 8=\text{2d numerator.}\\(x+5)^2&=\underline{x^2+10x+25}=\text{3d numerator.}\\&\qquad\qquad 18=\text{sum of the numerators.}\end{aligned}$$

$\therefore$ the sum of the fractions is $\frac{18}{(x-1)(x+2)(x+5)}$.

66. $\dfrac{1}{ax-a^2}+\dfrac{1}{ax+2a^2}-\dfrac{3}{x^2+ax-2a^2}.$

$$\begin{aligned} ax-a^2&=a(x-a)\\ ax+2a^2&=a(x+2a)\\ x^2+ax-2a^2&=(x-a)(x+2a)\end{aligned}$$

$\therefore$ the L. C. D. is $a(x-a)(x+2a)$.

$$\begin{aligned} x+2a&=\text{1st numerator.}\\ x-a&=\text{2d numerator.}\\ -3a&=\text{3d numerator.}\\ \hline 2x-2a&=\text{sum of the numerators.}\end{aligned}$$

$\therefore$ the sum of the fractions is $\dfrac{2x-2a}{a(x-a)(x+2a)}=\dfrac{2}{a(x+2a)}.$

67. $\dfrac{x}{(x+3)(x-1)}+\dfrac{x-1}{(x+3)(2-x)}-\dfrac{x-3}{(2-x)(1-x)}.$

The L. C. D. is $(x-1)(2-x)(x+3)$.

$$\begin{aligned} x(2-x)&=-x^2+2x &&=\text{1st numerator.}\\ (x-1)(x-1)&=x^2-2x+1&&=\text{2d numerator.}\\ (x-3)(x+3)&=x^2-9&&=\text{3d numerator.}\\ &\overline{x^2-8}&&=\text{sum of the numerators.}\end{aligned}$$

$\therefore$ the sum of the fractions is $\dfrac{x^2-8}{(x-1)(2-x)(x+3)}.$

68. $\left(1+\dfrac{4}{x-1}+\dfrac{12}{x-3}\right)\left(1+\dfrac{4}{x+1}-\dfrac{12}{x+3}\right).$

$$\begin{aligned} 1+\frac{4}{x-1}+\frac{12}{x-3}&=\frac{(x-1)(x-3)+4(x-3)+12(x-1)}{(x-1)(x-3)}\\ &=\frac{x^2-4x+3+4x-12+12x-12}{(x-1)(x-3)}\\ &=\frac{x^2+12x-21}{(x-1)(x-3)}.\\ 1+\frac{4}{x+1}-\frac{12}{x+3}&=\frac{(x+1)(x+3)+4(x+3)-12(x+1)}{(x+1)(x+3)}\\ &=\frac{x^2+4x+3+4x+12-12x-12}{(x+1)(x+3)}\\ &=\frac{x^2-4x+3}{(x+1)(x+3)}\\ &=\frac{(x-1)(x-3)}{(x+1)(x+3)}.\end{aligned}$$

$$\therefore \left(1+\frac{4}{x-1}+\frac{12}{x-3}\right)\left(1+\frac{4}{x+1}-\frac{12}{x+3}\right)$$

$$=\frac{x^2+12x-21}{(x-1)(x-3)}\times\frac{(x-1)(x-3)}{(x+1)(x+3)}$$

$$=\frac{x^2+12x-21}{(x+1)(x+3)}=\frac{x^2+12x-21}{x^2+4x+3}.$$

69. $\left(\frac{x+2y}{x+y}+\frac{x}{y}\right)\div\left(\frac{x}{x+y}-\frac{x+2y}{y}\right).$

$$\frac{x+2y}{x+y}+\frac{x}{y}=\frac{x^2+xy+xy+2y^2}{xy+y^2}=\frac{x^2+2xy+2y^2}{xy+y^2}$$

$$\frac{x}{x+y}-\frac{x+2y}{y}=\frac{xy-x^2-2xy-xy-2y^2}{xy+y^2}=\frac{-x^2-2xy-2y^2}{xy+y^2}.$$

$$\therefore \left(\frac{x+2y}{x+y}+\frac{x}{y}\right)\div\left(\frac{x}{x+y}-\frac{x+2y}{y}\right)$$

$$=\frac{x^2+2xy+2y^2}{xy+y^2}\div\frac{-x^2-2xy-2y^2}{xy+y^2}=-1.$$

70. $\frac{a^3-b^3}{a^4-b^4}-\frac{a-b}{2(a^2-b^2)}-\frac{1}{2}\left(\frac{a+b}{a^2+b^2}-\frac{1}{a+b}\right)$

$$=\frac{a^3-b^3}{(a^2+b^2)(a^2-b^2)}-\frac{1}{2(a+b)}-\frac{a+b}{2(a^2+b^2)}+\frac{1}{2(a+b)}$$

$$=\frac{a^3-b^3}{(a^2+b^2)(a^2-b^2)}-\frac{a+b}{2(a^2+b^2)}$$

$$=\frac{a^2+ab+b^2}{(a^2+b^2)(a+b)}-\frac{a+b}{2(a^2+b^2)}$$

$$=\frac{2a^2+2ab+2b^2-a^2-2ab-b^2}{2(a^2+b^2)(a+b)}$$

$$=\frac{a^2+b^2}{2(a^2+b^2)(a+b)}$$

$$=\frac{1}{2(a+b)}.$$

71. $\frac{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}{\frac{1}{a^2}+\frac{1}{b^2}-\frac{1}{c^2}+\frac{2}{ab}}=\frac{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}{\frac{1}{a^2}+\frac{2}{ab}+\frac{1}{b^2}-\frac{1}{c^2}}=\frac{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}{\left(\frac{1}{a}+\frac{1}{b}\right)^2-\frac{1}{c^2}}$

$$=\frac{1}{\frac{1}{a}+\frac{1}{b}-\frac{1}{c}}=\frac{abc}{bc+ac-ab}.$$

72. $$\frac{\dfrac{1}{1-a}-\dfrac{1}{1+a}}{\dfrac{a}{1-a}+\dfrac{1}{1+a}}=\frac{\dfrac{1+a-(1-a)}{1-a^2}}{\dfrac{a(1+a)+1-a}{1-a^2}}=\frac{1+a-1+a}{a+a^2+1-a}=\frac{2a}{a^2+1}.$$

73. $$\frac{1-\frac{1}{2}[1-\frac{1}{3}(1-x)]}{1-\frac{1}{3}[1-\frac{1}{2}(1-x)]}=\frac{1-\dfrac{1}{2}\left[\dfrac{2+x}{3}\right]}{1-\dfrac{1}{3}\left[\dfrac{1+x}{2}\right]}=\frac{1-\dfrac{2+x}{6}}{1-\dfrac{1+x}{6}}$$
$$=\frac{6-2-x}{6-1-x}=\frac{4-x}{5-x}.$$

74. $$\frac{\dfrac{1}{a}+\dfrac{1}{ab^3}}{b-1+\dfrac{1}{b}}=\frac{\dfrac{b^3+1}{ab^3}}{\dfrac{b^2-b+1}{b}}=\frac{b^3+1}{ab^3}\times\frac{b}{b^2-b+1}=\frac{b+1}{ab^2}.$$

75. $$\frac{1}{x-1+\dfrac{1}{1+\dfrac{x}{4-x}}}=\frac{1}{x-1+\dfrac{1}{\dfrac{4-x+x}{4-x}}}=\frac{1}{x-1+\dfrac{4-x}{4}}$$
$$=\frac{4}{4x-4+4-x}=\frac{4}{3x}.$$

76. $$\frac{1}{1+\dfrac{a}{1+a+\dfrac{2a^2}{1+a}}}=\frac{1}{1+\dfrac{a}{\dfrac{(1+a)^2+2a^2}{1+a}}}=\frac{1}{1+\dfrac{a(1+a)}{1+2a+3a^2}}$$
$$=\frac{1}{\dfrac{1+2a+3a^2+a+a^2}{1+2a+3a^2}}=\frac{1+2a+3a^2}{1+3a+4a^2}.$$

Solve:

77. $$\frac{6x+13}{15}-\frac{9x+15}{5x-25}+3=\frac{2x+15}{5}.$$
$$\frac{6x+13}{15}-\frac{2x+15}{5}-\frac{9x+15}{5x-25}+3=0$$
$$\frac{6x+13}{15}-\frac{6x+45}{15}-\frac{9x+15}{5x-25}+\frac{45}{15}=0$$
$$\frac{13}{15}-\frac{9x+15}{5x-25}=0$$

$$\begin{aligned}13(5x-25)-15(9x+15)&=0\\65x-325-135x-225&=0\\-70x&=550\\\therefore\ x&=-\tfrac{55}{7}.\end{aligned}$$

78. $\dfrac{2x+a}{3(x-a)}+\dfrac{3x-a}{2(x+a)}=2\tfrac{1}{6}.$

$$\begin{aligned}2(2x+a)(x+a)+3(3x-a)(x-a)&=13(x-a)(x+a)\\4x^2+6ax+2a^2+9x^2-12ax+3a^2&=13x^2-13a^2\\-6ax&=-18a^2\\\therefore\ x&=3a.\end{aligned}$$

79. $\dfrac{x}{x-2}-\dfrac{x+1}{x-1}=\dfrac{x-8}{x-6}-\dfrac{x-9}{x-7}.$

$$\begin{aligned}\frac{x(x-1)-(x+1)(x-2)}{(x-2)(x-1)}&=\frac{(x-8)(x-7)-(x-9)(x-6)}{(x-6)(x-7)}\\\frac{x^2-x-x^2+x+2}{(x-2)(x-1)}&=\frac{x^2-15x+56-x^2+15x-54}{(x-6)(x-7)}\\\frac{2}{(x-2)(x-1)}&=\frac{2}{(x-6)(x-7)}\\\therefore\ (x-2)(x-1)&=(x-6)(x-7)\\x^2-3x+2&=x^2-13x+42\\10x&=40\\\therefore\ x&=4.\end{aligned}$$

80. $\begin{cases}\dfrac{x}{3}+\dfrac{5}{y}=4\tfrac{1}{3} & (1)\\[2ex]\dfrac{x}{6}+\dfrac{10}{y}=2\tfrac{2}{3} & (2)\end{cases}$

2 × (1) is $\dfrac{2x}{3}+\dfrac{10}{y}=8\tfrac{2}{3}$

(2) is $\dfrac{x}{6}+\dfrac{10}{y}=2\tfrac{2}{3}$

Subtract $\dfrac{x}{2}=6$

$$\begin{aligned}\therefore\ x&=12\\y&=15\end{aligned}$$

81. $\begin{cases}\dfrac{5x}{0.7}+\dfrac{0.3}{y}=6 & (1)\\[2ex]\dfrac{10x}{7}+\dfrac{9}{y}=31 & (2)\end{cases}$

30 × (1) is $\dfrac{150x}{0.7}+\dfrac{9}{y}=180$

(2) is $\dfrac{10x}{7}+\dfrac{9}{y}=31$

Subtract $\dfrac{1490}{7}x=149$

$$\begin{aligned}\therefore\ x&=\tfrac{7}{10}\\y&=\tfrac{3}{10}\end{aligned}$$

82. $$\begin{cases} \dfrac{2x}{a}+\dfrac{3y}{b}-\dfrac{4z}{c}=1 & (1) \\ \dfrac{4x}{a}+\dfrac{2y}{b}-\dfrac{3z}{c}=3 & (2) \\ \dfrac{3x}{a}+\dfrac{4y}{b}-\dfrac{2z}{c}=5 & (3) \end{cases}$$

(2) is $\dfrac{4x}{a}+\dfrac{2y}{b}-\dfrac{3z}{c}=3$

2×(1) is $\dfrac{4x}{a}+\dfrac{6y}{b}-\dfrac{8z}{c}=2$

Subtract $\dfrac{4y}{b}-\dfrac{5z}{c}=-1$ (4)

2×(3) is $\dfrac{6x}{a}+\dfrac{8y}{b}-\dfrac{4z}{c}=10$

3×(1) is $\dfrac{6x}{a}+\dfrac{9y}{b}-\dfrac{12z}{c}=3$

Subtract $\dfrac{y}{b}-\dfrac{8z}{c}=-7$ (5)

4×(5) is $\dfrac{4y}{b}-\dfrac{32z}{c}=-28$

(4) is $\dfrac{4y}{b}-\dfrac{5z}{c}=-1$

Subtract $\dfrac{27z}{c}=27$

$$\begin{aligned} \therefore\ z&=c \\ y&=b \\ x&=a \end{aligned}$$

83. $$\begin{cases} \dfrac{a}{x}+\dfrac{b}{y}+\dfrac{c}{z}=3 & (1) \\ \dfrac{a}{x}+\dfrac{b}{y}-\dfrac{c}{z}=1 & (2) \\ \dfrac{2a}{x}-\dfrac{b}{y}-\dfrac{c}{z}=0 & (3) \end{cases}$$

Subtract (2) from (1).

$$\frac{2c}{z}=2$$

$$\therefore\ z=c$$

Substitute in (2) and (3).

$$\frac{a}{x}+\frac{b}{y}=2 \qquad (4)$$

$$\frac{2a}{x}-\frac{b}{y}=1 \qquad (5)$$

2 × (4) is $\dfrac{2a}{x}+\dfrac{2b}{y}=4$

(5) is $\dfrac{2a}{x}-\dfrac{b}{y}=1$

Subtract $\dfrac{3b}{y}=3$

$$\begin{aligned} \therefore\ y&=b \\ x&=a \end{aligned}$$

Find the arithmetical value of

84. $36^{\frac{1}{2}}$; $27^{\frac{1}{3}}$; $16^{\frac{1}{4}}$; $32^{\frac{1}{5}}$; $4^{\frac{5}{2}}$; $8^{\frac{2}{3}}$; $27^{\frac{5}{3}}$; $64^{\frac{2}{3}}$.
6; 3; 2; 2; 32; 4; 243; 16.

85. $32^{\frac{3}{5}}$; $64^{\frac{5}{6}}$; $81^{\frac{3}{4}}$; $(3\frac{3}{8})^{\frac{1}{3}}$; $(5\frac{1}{16})^{\frac{1}{4}}$; $(1\frac{9}{16})^{\frac{3}{2}}$.
8; 32; 27; $\frac{3}{2}$; $\frac{3}{2}$; $\frac{125}{64}$.

86. $(0.25)^{\frac{1}{2}}$; $(0.027)^{\frac{2}{3}}$; $49^{0.5}$; $32^{0.2}$; $81^{0.75}$.
0.5; 0.9; 7; 2; 27.

87. $36^{-\frac{1}{2}};\ 27^{-\frac{1}{3}};\ (\frac{9}{16})^{-\frac{3}{2}};\ (0.16)^{-\frac{3}{2}};\ (0.0016)^{-\frac{3}{4}}.$

$\frac{1}{6};\ \frac{1}{3};\ \frac{64}{27};\ \frac{125}{8};\ 125.$

88. Interpret $a^{-2};\ a^0;\ a^{\frac{1}{2}};\ a^{-\frac{1}{3}};\ (a^{-\frac{1}{2}})^{-\frac{1}{3}}.$

$$a^{-2}=\frac{1}{a^2};\ a^0=\frac{a}{a}=1;\ a^{\frac{1}{2}}=\sqrt{a};\ a^{-\frac{1}{3}}=\frac{1}{\sqrt[3]{a}};\ (a^{-\frac{1}{2}})^{-\frac{1}{3}}=a^{\frac{1}{6}}=\sqrt[6]{a}.$$

Simplify:

89. $a^{\frac{1}{2}}\times a^{\frac{1}{3}};\ c^{\frac{1}{3}}\times c^{\frac{1}{6}};\ m^{\frac{1}{2}}\times m^{-\frac{1}{6}};\ n^{\frac{3}{4}}\times n^{-\frac{1}{12}}.$

$$a^{\frac{1}{2}}\times a^{\frac{1}{3}}=a^{\frac{5}{6}};\ c^{\frac{1}{3}}\times c^{\frac{1}{6}}=c^{\frac{1}{2}};\ m^{\frac{1}{2}}\times m^{-\frac{1}{6}}=m^{\frac{1}{3}};\ n^{\frac{3}{4}}\times n^{-\frac{1}{12}}=n^{\frac{2}{3}}$$

90. $a^{\frac{1}{3}}b^{\frac{2}{3}}c^{\frac{1}{6}}\times a^{-\frac{2}{3}}b^{-\frac{1}{3}}c^{-\frac{1}{6}};\ a^{\frac{2}{3}}b^{\frac{1}{2}}c^{-\frac{1}{4}}\times a^{\frac{1}{3}}b^{-\frac{1}{2}}c^{\frac{1}{2}}d.$

$$a^{\frac{1}{3}}b^{\frac{2}{3}}c^{\frac{1}{6}}\times a^{-\frac{2}{3}}b^{-\frac{1}{3}}c^{-\frac{1}{6}}=a^{-\frac{1}{3}}b^{\frac{1}{3}}=\sqrt[3]{\frac{b}{a}};\ a^{\frac{2}{3}}b^{\frac{1}{2}}c^{-\frac{1}{4}}\times a^{\frac{1}{3}}b^{-\frac{1}{2}}c^{\frac{1}{2}}d=a^{\frac{1}{3}}c^{\frac{1}{4}}d.$$

91. $(2ab+2bc+2ac-a^2-b^2-c^2)\div(a^{\frac{1}{2}}+b^{\frac{1}{2}}+c^{\frac{1}{2}}).$

$$\begin{array}{l|l}
-a^2+2ab+2ac-b^2+2bc-c^2 & a^{\frac{1}{2}}+b^{\frac{1}{2}}+c^{\frac{1}{2}} \\ \cline{2-2}
\underline{-a^2-a^{\frac{3}{2}}b^{\frac{1}{2}}-a^{\frac{3}{2}}c^{\frac{1}{2}}} & -a^{\frac{3}{2}}+a(b^{\frac{1}{2}}+c^{\frac{1}{2}})+a^{\frac{1}{2}}(b-2b^{\frac{1}{2}}c^{\frac{1}{2}}+c)-b^{\frac{3}{2}}+bc^{\frac{1}{2}} \\
 & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad [+b^{\frac{1}{2}}c-c^{\frac{3}{2}} \\
\qquad a^{\frac{3}{2}}b^{\frac{1}{2}}+a^{\frac{3}{2}}c^{\frac{1}{2}}+2ab+2ac-b^2+2bc-c^2 & \\
\qquad \underline{a^{\frac{3}{2}}b^{\frac{1}{2}}+a^{\frac{3}{2}}c^{\frac{1}{2}}+ab+ac+2ab^{\frac{1}{2}}c^{\frac{1}{2}}} & \\
\qquad\qquad\qquad ab+ac-2ab^{\frac{1}{2}}c^{\frac{1}{2}}-b^2+2bc-c^2 & \\
\qquad\qquad\qquad \underline{ab+ac-2ab^{\frac{1}{2}}c^{\frac{1}{2}}+a^{\frac{1}{2}}(b^{\frac{3}{2}}-bc^{\frac{1}{2}}-b^{\frac{1}{2}}c+c^{\frac{3}{2}})} & \\
\qquad\qquad\qquad\qquad\qquad\qquad -a^{\frac{1}{2}}(b^{\frac{3}{2}}-bc^{\frac{1}{2}}-b^{\frac{1}{2}}c+c^{\frac{3}{2}})-b^2+2bc-c^2 & \\
\qquad\qquad\qquad\qquad\qquad\qquad \underline{-a^{\frac{1}{2}}(b^{\frac{3}{2}}-bc^{\frac{1}{2}}-b^{\frac{1}{2}}c+c^{\frac{3}{2}})-b^2+2bc-c^2} &
\end{array}$$

Therefore the quotient is

$$-a^{\frac{3}{2}}-b^{\frac{3}{2}}-c^{\frac{3}{2}}+ab^{\frac{1}{2}}+a^{\frac{1}{2}}b+ac^{\frac{1}{2}}+a^{\frac{1}{2}}c+bc^{\frac{1}{2}}+b^{\frac{1}{2}}c-2a^{\frac{1}{2}}b^{\frac{1}{2}}c^{\frac{1}{2}}.$$

92. $\sqrt{12};\ \sqrt{8};\ \sqrt{50};\ \sqrt[3]{16};\ 4\sqrt[3]{250};\ \sqrt{\frac{1}{2}};\ \sqrt[3]{\frac{1}{4}};\ \sqrt{\frac{8}{27}}.$

$2\sqrt{3};\ 2\sqrt{2};\ 5\sqrt{2};\ 2\sqrt[3]{2};\ 20\sqrt[3]{2};\ \frac{1}{2}\sqrt{2};\ \frac{1}{2}\sqrt[3]{2};\ \frac{2}{9}\sqrt{6}.$

93. $5\sqrt[3]{-320};\ \sqrt{a^3b^7};\ \sqrt[3]{a^5};\ \sqrt{a^2x+a^3};\ 3\frac{1}{3}\sqrt[3]{54x^9}.$

$$5\sqrt[3]{-320}=5\sqrt[3]{-5\times 64}=-20\sqrt[3]{5};\ \sqrt{a^3b^7}=ab^3\sqrt{ab};\ \sqrt[3]{a^5}=a\sqrt[3]{a^2};$$
$$\sqrt{a^2x+a^3}=\sqrt{a^2(a+x)}=a\sqrt{a+x};\ 3\tfrac{1}{3}\sqrt[3]{54x^9}=\tfrac{10}{3}\sqrt[3]{2\times 27x^9}$$
$$=\tfrac{10}{3}\times 3x^3\sqrt[3]{2}=10x^3\sqrt[3]{2}.$$

94. $2\sqrt{18}-3\sqrt{8}+2\sqrt{50};\ \sqrt[3]{81}+\sqrt[3]{24}-\sqrt[3]{192}.$

$$\begin{aligned}2\sqrt{18}-3\sqrt{8}+2\sqrt{50}&=2\sqrt{2\times 9}-3\sqrt{2\times 4}-2\sqrt{2\times 25}\\&=6\sqrt{2}-6\sqrt{2}-10\sqrt{2}\\&=-10\sqrt{2}\\\sqrt[3]{81}+\sqrt[3]{24}-\sqrt[3]{192}&=\sqrt[3]{3\times 27}+\sqrt[3]{3\times 8}-\sqrt[3]{3\times 64}\\&=3\sqrt[3]{3}+2\sqrt[3]{3}-4\sqrt[3]{3}\\&=\sqrt[3]{3}.\end{aligned}$$

95. $\frac{3}{2}\sqrt{\frac{5}{9}}+\sqrt{80}-\frac{1}{4}\sqrt{20};\ 8\sqrt{\frac{5}{16}}+10\sqrt{\frac{20}{25}}-2\sqrt{\frac{5}{4}}.$

$$\begin{aligned}\tfrac{3}{2}\sqrt{\tfrac{5}{9}}+\sqrt{80}-\tfrac{1}{4}\sqrt{20}&=\tfrac{1}{2}\sqrt{5}+\sqrt{5\times 16}-\tfrac{1}{4}\sqrt{5\times 4}\\&=\tfrac{1}{2}\sqrt{5}+4\sqrt{5}-\tfrac{1}{2}\sqrt{5}\\&=4\sqrt{5}\\8\sqrt{\tfrac{5}{16}}+10\sqrt{\tfrac{20}{25}}-2\sqrt{\tfrac{5}{4}}&=2\sqrt{5}+2\sqrt{20}-\sqrt{5}\\&=2\sqrt{5}+4\sqrt{5}-\sqrt{5}\\&=5\sqrt{5}.\end{aligned}$$

Rationalize the divisor, and find the value of

96. $$\frac{1}{2-\sqrt{3}}=\frac{2+\sqrt{3}}{(2-\sqrt{3})(2+\sqrt{3})}=\frac{2+\sqrt{3}}{4-3}=2+\sqrt{3}.$$

97. $$\frac{3}{3+\sqrt{6}}=\frac{3(3-\sqrt{6})}{(3+\sqrt{6})(3-\sqrt{6})}=\frac{3(3-\sqrt{6})}{9-6}=3-\sqrt{6}.$$

98. $$\frac{5}{\sqrt{2}+\sqrt{7}}=\frac{5(\sqrt{2}-\sqrt{7})}{(\sqrt{2}+\sqrt{7})(\sqrt{2}-\sqrt{7})}=\frac{5(\sqrt{2}-\sqrt{7})}{2-7}=\sqrt{7}-\sqrt{2}.$$

99. $$\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}=\frac{(\sqrt{3}+\sqrt{2})^2}{(\sqrt{3}-\sqrt{2})(\sqrt{3}+\sqrt{2})}=\frac{3+2\sqrt{6}+2}{3-2}=5+2\sqrt{6}.$$

100. $$\frac{7\sqrt{5}}{\sqrt{5}+\sqrt{7}}=\frac{7\sqrt{5}(\sqrt{5}-\sqrt{7})}{(\sqrt{5}+\sqrt{7})(\sqrt{5}-\sqrt{7})}=\frac{35-7\sqrt{35}}{5-7}$$
$$=\frac{35-7\sqrt{35}}{-2}=\frac{7\sqrt{35}-35}{2}.$$

101. $$\frac{2\sqrt{6}}{\sqrt{3}-\sqrt{5}}=\frac{2\sqrt{6}(\sqrt{3}+\sqrt{5})}{(\sqrt{3}-\sqrt{5})(\sqrt{3}+\sqrt{5})}=\frac{2\sqrt{18}+2\sqrt{30}}{3-5}$$
$$=-\sqrt{18}-\sqrt{30}=-3\sqrt{2}-\sqrt{30}.$$

Solve:

102. $$\frac{5}{x-2}-\frac{4}{x}=\frac{3}{x+6}.$$
$$5x(x+6)-4(x-2)(x+6)=3x(x-2)$$
$$5x^2+30x-4x^2-16x+48=3x^2-6x$$
$$-2x^2+20x+48=0$$
$$x^2-10x-24=0$$
$$\therefore\ x=12,\text{ or }-2.$$

103. $$\frac{x+3}{2x-7}-\frac{2x-1}{x-3}=0.$$
$$(x+3)(x-3)-(2x-7)(2x-1)=0$$
$$x^2-9-4x^2+16x-7=0$$
$$-3x^2+16x-16=0$$
$$3x^2-16x+16=0$$
$$\therefore\ x=4,\text{ or }\tfrac{4}{3}.$$

104. $$\frac{x+4}{x-4}+\frac{x-2}{x-3}=6\tfrac{1}{3}.$$
$$3(x+4)(x-3)+3(x-2)(x-4)=19(x-4)(x-3)$$
$$3x^2+3x-36+3x^2-18x+24=19x^2-133x+228$$
$$-13x^2+118x-240=0$$
$$13x^2-118x+240=0$$
$$\therefore\ x=6,\text{ or }\tfrac{40}{13}.$$

105. $$\frac{x-a}{x-b}+\frac{x-b}{x-a}=\frac{a^2+b^2}{ab}.$$
$$ab(x-a)^2+ab(x-b)^2=(a^2+b^2)(x-a)(x-b)$$
$$abx^2-2a^2bx+a^3b+abx^2-2ab^2x+ab^3=(a^2+b^2)x^2-(a+b)(a^2+b^2)x$$
$$(a^2-2ab+b^2)x^2-(a^3-a^2b-ab^2+b^3)x=0 \qquad [+ab(a^2+b^2)$$
$$\therefore\ x=0,\text{ or }\frac{a^3-a^2b-ab^2+b^3}{a^2-2ab+b^2}$$
$$=0,\text{ or }a+b.$$

106. $\dfrac{ax+b}{a+bx}=\dfrac{cx+d}{c+dx}.$

$$\begin{aligned}(ax+b)(c+dx)&=(a+bx)(cx+d)\\ adx^2+bdx+acx+bc&=bcx^2+bdx+acx+ad\\ (ad-bc)x^2+bc-ad&=0\\ x^2&=1\\ \therefore\ x&=\pm 1.\end{aligned}$$

107. $\dfrac{c}{x-d}-\dfrac{d}{x+c}=\dfrac{c-d}{x}.$

$$\begin{aligned}cx(x+c)-dx(x-d)&=(c-d)(x-d)(x+c)\\ cx^2+c^2x-dx^2+d^2x&=(c-d)x^2+(c-d)^2x-cd(c-d)\\ 2\,cdx&=-cd(c-d)\\ \therefore\ x&=\frac{d-c}{2}.\end{aligned}$$

108. $\dfrac{ax}{x-b}+\dfrac{bx}{x-a}=a+b.$

$$\begin{aligned}ax(x-a)+bx(x-b)&=(a+b)(x-a)(x-b)\\ ax^2-a^2x+bx^2-b^2x&=(a+b)x^2-(a+b)^2x+ab(a+b)\\ 2\,abx&=ab(a+b)\\ \therefore\ x&=\frac{a+b}{2}\end{aligned}$$

109. $\dfrac{1}{a+x}+\dfrac{1}{b+x}=\dfrac{a+b}{ab}.$

$$\begin{aligned}ab(b+x)+ab(a+x)&=(a+b)(a+x)(b+x)\\ ab^2+abx+a^2b+abx&=(a+b)x^2+(a+b)^2x+ab(a+b)\\ (a^2+b^2)x&=-(a+b)x^2\\ \therefore\ x&=-\frac{a^2+b^2}{a+b}.\end{aligned}$$

110. $ax^2-\dfrac{6\,c^2}{a+b}=cx-bx^2.$

Transpose, $\qquad (a+b)x^2-cx=\dfrac{6\,c^2}{a+b}$

$$\begin{aligned}(a+b)^2x^2-(a+b)cx+\frac{c^2}{4}&=\frac{25\,c^2}{4}\\ (a+b)x-\frac{c}{2}&=\pm\frac{5\,c}{2}\\ (a+b)x&=3\,c,\ \text{or}\ -2\,c\\ \therefore\ x&=\frac{3\,c}{a+b},\ \text{or}\ -\frac{2\,c}{a+b}.\end{aligned}$$

111. $\dfrac{x+a}{x+b}+\dfrac{x+b}{x+c}=2.$

$$(x+a)(x+c)+(x+b)^2=2(x+b)(x+c)$$
$$x^2+ax+cx+ac+x^2+2bx+b^2=2x^2+2bx+2cx+2bc$$
$$(a-c)x=2bc-ac-b^2$$
$$\therefore\ x=\frac{2bc-ac-b^2}{a-c}.$$

112.
$$\begin{cases}3xy-5y^2=\ 1 & (1)\\ 5xy+3x^2=22 & (2)\end{cases}$$

Subtract $22\times(1)$ from (2).

$$3x^2-61xy+110y^2=0$$
$$(3x-55y)(x-2y)=0$$
$$\therefore\ x=\tfrac{55}{3}y,\text{ or }2y$$

Substitute these values in (1).

$$55y^2-5y^2=1$$
$$50y^2=1$$
$$\therefore\ y=\pm\tfrac{1}{10}\sqrt{2},\ x=\pm\tfrac{11}{6}\sqrt{2},$$

or
$$6y^2-5y^2=1$$
$$\therefore\ y=\pm1,\ x=\pm2.$$

113.
$$\begin{cases}x^2+10xy=11 & (1)\\ 5xy-3y^2=\ 2 & (2)\end{cases}$$

$$\begin{array}{ll}2\times(1)\text{ is} & 2x^2+20xy=22\\ 11\times(2)\text{ is} & \underline{55xy-33y^2=22}\\ \text{Subtract,} & 2x^2-35xy+33y^2=0\end{array}$$

$$(2x-33y)(x-y)=0$$
$$\therefore\ x=\tfrac{33}{2}y,\text{ or }y$$

Substitute these values in (2).

$$\tfrac{165}{2}y^2-3y^2=2$$
$$y^2=\tfrac{4}{159}$$
$$\therefore\ y=\pm\frac{2}{\sqrt{159}},\ x=\pm\frac{33}{\sqrt{159}},$$

or
$$5y^2-3y^2=2$$
$$y^2=1$$
$$\therefore\ y=\pm1,\ x=\pm1.$$

114.
$$\begin{cases} \sqrt{x+y} = \sqrt{y} + 2 & (1) \\ x - y = 7 & (2) \end{cases}$$

From (1),
$$x + y = y + 4\sqrt{y} + 4$$
$$x = 4\sqrt{y} + 4$$

Substitute in (2),
$$4\sqrt{y} + 4 - y = 7$$
$$y - 4\sqrt{y} = -3$$
$$y - 4\sqrt{y} + 4 = 1$$
$$\sqrt{y} - 2 = \pm 1$$
$$\sqrt{y} = 3, \text{ or } 1$$
$$\therefore\ y = 9, \text{ or } 1$$
$$x = 16, \text{ or } 8$$

115.
$$\begin{cases} x^2 + xy + y^2 = 52 & (1) \\ xy - x^2 = 8 & (2) \end{cases}$$

$2 \times (1)$ is $2x^2 + 2xy + 2y^2 = 104$

$13 \times (2)$ is $-13x^2 + 13xy = 104$

Subtract,
$$15x^2 - 11xy + 2y^2 = 0$$
$$(5x - 2y)(3x - y) = 0$$
$$\therefore\ x = \tfrac{2}{5}y, \text{ or } \tfrac{1}{3}y$$

Substitute in (1),
$$\tfrac{4}{25}y^2 + \tfrac{2}{5}y^2 + y^2 = 52$$
$$\tfrac{39}{25}y^2 = 52$$
$$y^2 = \tfrac{100}{3}$$
$$\therefore\ y = \pm \tfrac{10}{3}\sqrt{3},\ x = \pm \tfrac{4}{3}\sqrt{3},$$

or
$$\tfrac{1}{9}y^2 + \tfrac{1}{3}y^2 + y^2 = 52$$
$$\tfrac{13}{9}y^2 = 52$$
$$\therefore\ y = \pm 6,\ x = \pm 2.$$

Form the equations of which the roots are:

116. $a - b,\ a + b.$

$x^2 - 2ax + a^2 - b^2 = 0$

117. $a - 2b,\ a + 3b.$

$x^2 - (2a + b)x + a^2 + ab - 6b^2 = 0.$

118. $a + 2b,\ 2a + b.$

$x^2 - 3(a + b)x + 2a^2 + 5ab + 2b^2 = 0$

119. $1 + \sqrt{3},\ 1 - \sqrt{3}.$

$x^2 - 2x - 2 = 0.$

120. $-1 + \sqrt{3},\ -1 - \sqrt{3}.$

$x^2 + 2x - 2 = 0.$

121. $1 + \sqrt{-3},\ 1 - \sqrt{-3}.$

$x^2 - 2x + 4 = 0.$

Solve:

122. $x^4 - 5x^2 + 4 = 0$
$(x^2 - 1)(x^2 - 4) = 0$
$\therefore\ x^2 = 1$, or 4
$x = \pm 1$, or ± 2.

123. $x^6 - 9x^3 + 8 = 0$
$(x^3 - 1)(x^3 - 8) = 0$
$\therefore\ x^3 = 1$, or 8
$x = 1$, or 2.

124. $9x^4 - 13x^2 + 4 = 0$
$(x^2 - 1)(9x^2 - 4) = 0$
$\therefore\ x^2 = 1$, or $\frac{4}{9}$
$x = \pm 1$, or $\pm \frac{2}{3}$.

125. $4x^4 - 17x^2 + 4 = 0$
$(x^2 - 4)(4x^2 - 1) = 0$
$\therefore\ x^2 = 4$, or $\frac{1}{4}$
$x = \pm 2$, or $\pm \frac{1}{2}$.

126. $2x^4 - 5x^2 + 2 = 0$
$(2x^2 - 1)(x^2 - 2) = 0$
$\therefore\ x^2 = \frac{1}{2}$, or 2
$x = \pm \sqrt{\frac{1}{2}}$, or $\pm\sqrt{2}$.

127. $2x^6 - 19x^3 + 24 = 0$
$(x^3 - 8)(2x^3 - 3) = 0$
$\therefore\ x^3 = 8$, or $\frac{3}{2}$
$x = 2$, or $\sqrt[3]{\frac{3}{2}}$.

128. $x^4 - 1 = 0$
$(x^2 + 1)(x^2 - 1) = 0$
$\therefore\ x^2 = 1$, or -1
$x = \pm 1$, or $\pm\sqrt{-1}$.

129. $x^6 - 1 = 0$
$(x^3 + 1)(x^3 - 1) = 0$
$\therefore\ x^3 = -1$, or 1
$x = -1$, or 1.

130. $x^{\frac{2}{3}} + 8x^{\frac{1}{3}} - 9 = 0$
$(x^{\frac{1}{3}} - 1)(x^{\frac{1}{3}} + 9) = 0$
$\therefore\ x^{\frac{1}{3}} = 1$, or -9
$x = 1$, or -729.

131. $16x^{\frac{4}{5}} - 17x^{\frac{2}{5}} + 1 = 0$
$(x^{\frac{2}{5}} - 1)(16x^{\frac{2}{5}} - 1) = 0$
$\therefore\ x^{\frac{2}{5}} = 1$, or $\frac{1}{16}$
$x = \pm 1$, or $\pm \frac{1}{1024}$.

132. $2x^{\frac{1}{2}} - 3x^{\frac{1}{4}} + 1 = 0$
$(2x^{\frac{1}{4}} - 1)(x^{\frac{1}{4}} - 1) = 0$
$\therefore\ x^{\frac{1}{4}} = \frac{1}{2}$, or 1
$x = \frac{1}{16}$, or 1.

133. $3\sqrt[3]{x} - 5\sqrt[6]{x} + 2 = 0$
$(3\sqrt[6]{x} - 2)(\sqrt[6]{x} - 1) = 0$
$\therefore\ \sqrt[6]{x} = \frac{2}{3}$, or 1
$x = \frac{64}{729}$, or 1.

134. $6\sqrt{x} - 3\sqrt[4]{x} - 45 = 0$
$(2\sqrt[4]{x} + 5)(3\sqrt[4]{x} - 9) = 0$
$\therefore\ \sqrt[4]{x} = -\frac{5}{2}$, or 3
$x = \frac{625}{16}$, or 81.

135. $21\sqrt[3]{x^2} - 5\sqrt[3]{x} - 74 = 0$
$21x^{\frac{2}{3}} - 5x^{\frac{1}{3}} - 74 = 0$
$(21x^{\frac{1}{3}} + 37)(x^{\frac{1}{3}} - 2) = 0$
$\therefore\ x^{\frac{1}{3}} = -\frac{37}{21}$, or 2
$x = (-\frac{37}{21})^3$, or 8.

136. $$3\sqrt{x^5}+4\sqrt[4]{x^5}-20=0$$
$$3x^{\frac{5}{2}}+4x^{\frac{5}{4}}-20=0$$
$$(3x^{\frac{5}{4}}+10)(x^{\frac{5}{4}}-2)=0$$
$$\therefore\ x^{\frac{5}{4}}=-\tfrac{10}{3},\text{ or } 2$$
$$x=\sqrt[5]{\tfrac{10000}{81}},\text{ or } \sqrt[5]{16}.$$

137. $$x^{-\frac{1}{2}}+5x^{-\frac{1}{4}}-14=0$$
$$(x^{-\frac{1}{4}}+7)(x^{-\frac{1}{4}}-2)=0$$
$$\therefore\ x^{-\frac{1}{4}}=2,\text{ or } -7$$
$$x=\tfrac{1}{16},\text{ or } \tfrac{1}{2401}.$$

138. $$4x^{-\frac{2}{3}}-3x^{-\frac{1}{3}}-27=0$$
$$(4x^{-\frac{1}{3}}+9)(x^{-\frac{1}{3}}-3)=0$$
$$\therefore\ x^{-\frac{1}{3}}=-\tfrac{9}{4},\text{ or } 3$$
$$x=-\tfrac{64}{729},\text{ or } \tfrac{1}{27}.$$

139. $$x^{2n}+3x^n-4=0$$
$$(x^n-1)(x^n+4)=0$$
$$\therefore\ x^n=1,\text{ or } -4$$
$$x=1,\text{ or } \sqrt[n]{-4}.$$

140. $$3x^{\frac{1}{3}}-2ax^{\frac{1}{6}}-a^2=0$$
$$(3x^{\frac{1}{6}}+a)(x^{\frac{1}{6}}-a)=0$$
$$\therefore\ x^{\frac{1}{6}}=a,\text{ or } -\frac{a}{3}$$
$$x=a^6,\text{ or } \frac{a^6}{729}.$$

141. $$\sqrt{2x}-\sqrt[4]{2x}-2=0$$
$$(2x)^{\frac{1}{2}}-(2x)^{\frac{1}{4}}-2=0$$
$$[(2x)^{\frac{1}{4}}+1][(2x)^{\frac{1}{4}}-2]=0$$
$$\therefore\ (2x)^{\frac{1}{4}}=-1,\text{ or } 2$$
$$2x=1,\text{ or } 16$$
$$x=\tfrac{1}{2},\text{ or } 8.$$

142. $$3\sqrt[3]{9x^2}+4\sqrt[3]{3x}-39=0$$
$$3(3x)^{\frac{2}{3}}+4(3x)^{\frac{1}{3}}-39=0$$
$$[3(3x)^{\frac{1}{3}}+13][(3x)^{\frac{1}{3}}-3]=0$$
$$\therefore\ (3x)^{\frac{1}{3}}=3,\text{ or } -\tfrac{13}{3}$$
$$3x=27,\text{ or } -\tfrac{2197}{27}$$
$$x=9,\text{ or } -\tfrac{2197}{81}.$$

143. $$\sqrt{3ax}+a\sqrt[4]{3ax}-2a^2=0$$
$$(\sqrt[4]{3ax}+2a)(\sqrt[4]{3ax}-a)=0$$
$$\therefore\ \sqrt[4]{3ax}=a,\text{ or } -2a$$
$$3ax=a^4,\text{ or } 16a^4$$
$$x=\frac{a^3}{3},\text{ or } \frac{16a^3}{3}.$$

144.
$$3\sqrt[3]{2bx} - 5b\sqrt[6]{2bx} - 2b^2 = 0$$
$$(3\sqrt[6]{2bx} + b)(\sqrt[6]{2bx} - 2b) = 0$$
$$\therefore\ \sqrt[6]{2bx} = -\frac{b}{3},\ \text{or } 2b$$
$$2bx = \frac{b^6}{729},\ \text{or } 64b^6$$
$$x = \frac{b^5}{1458},\ \text{or } 32b^5.$$

145.
$$\sqrt{x+4} + \sqrt{3x+1} = \sqrt{9x+4}$$
$$x + 4 + 2\sqrt{x+4}\sqrt{3x+1} + 3x + 1 = 9x + 4$$
$$2\sqrt{x+4}\sqrt{3x+1} = 5x - 1$$
$$12x^2 + 52x + 16 = 25x^2 - 10x + 1$$
$$13x^2 - 62x - 15 = 0$$
$$(13x + 3)(x - 5) = 0$$
$$\therefore\ x = 5,\ \text{or } -\tfrac{3}{13}.$$

146.
$$\sqrt{5x+1} + 2\sqrt{4x-3} = 10\sqrt{x-2}$$
$$5x+1+4\sqrt{5x+1}\sqrt{4x-3}+16x-12 = 100x-200$$
$$4\sqrt{5x+1}\sqrt{4x-3} = 79x - 189$$
$$320x^2 - 176x - 48 = 6241x^2 - 29862x + 35721$$
$$5921x^2 - 29686x + 35769 = 0$$
$$(x-3)(5921x - 11923) = 0$$
$$\therefore\ x = 3,\ \text{or } \tfrac{11923}{5921}.$$

147.
$$2\sqrt{x+2} - 3\sqrt{3x-5} + \sqrt{5x+1} = 0$$
$$4(x+2)-12\sqrt{x+2}\sqrt{3x-5}+9(3x-5) = 5x+1$$
$$-12\sqrt{x+2}\sqrt{3x-5} = -26x + 38$$
$$6\sqrt{x+2}\sqrt{3x-5} = 13x - 19$$
$$108x^2 + 36x - 360 = 169x^2 - 494x + 361$$
$$61x^2 - 530x + 721 = 0$$
$$(x-7)(61x - 103) = 0$$
$$\therefore\ x = 7,\ \text{or } \tfrac{103}{61}.$$

148.
$$\sqrt{11-x} + \sqrt{8-2x} - \sqrt{21+2x} = 0$$
$$11 - x + 2\sqrt{11-x}\sqrt{8-2x} + 8 - 2x = 21 + 2x$$
$$2\sqrt{11-x}\sqrt{8-2x} = 2 + 5x$$
$$352 - 120x + 8x^2 = 4 + 20x + 25x^2$$
$$17x^2 + 140x - 348 = 0$$
$$(x-2)(17x + 174) = 0$$
$$\therefore\ x = 2,\ \text{or } -\tfrac{174}{17}.$$

Expand:

149. $(a^{\frac{1}{2}}-x^{\frac{2}{3}})^9$; $(3a^{\frac{1}{3}}-2b^2)^5$; $(a^{\frac{2}{3}}-\frac{1}{2}b^{\frac{1}{2}})^7$; $\left(2-\frac{x^2}{3}\right)^6$; $\left(3-\frac{x^{\frac{1}{2}}}{2}\right)^7$.

$$\begin{aligned}(a^{\frac{1}{2}}-x^{\frac{2}{3}})^9 &= (a^{\frac{1}{2}})^9-9(a^{\frac{1}{2}})^8(x^{\frac{2}{3}})+36(a^{\frac{1}{2}})^7(x^{\frac{2}{3}})^2-84(a^{\frac{1}{2}})^6(x^{\frac{2}{3}})^3\\ &\quad+126(a^{\frac{1}{2}})^5(x^{\frac{2}{3}})^4-126(a^{\frac{1}{2}})^4(x^{\frac{2}{3}})^5+84(a^{\frac{1}{2}})^3(x^{\frac{2}{3}})^6\\ &\quad-36(a^{\frac{1}{2}})^2(x^{\frac{2}{3}})^7+9(a^{\frac{1}{2}})(x^{\frac{2}{3}})^8-(x^{\frac{2}{3}})^9\\ &= a^{\frac{9}{2}}-9a^4x^{\frac{2}{3}}+36a^{\frac{7}{2}}x^{\frac{4}{3}}-84a^3x^2+126a^{\frac{5}{2}}x^{\frac{8}{3}}-126a^2x^{\frac{10}{3}}\\ &\quad+84a^{\frac{3}{2}}x^4-36ax^{\frac{14}{3}}+9a^{\frac{1}{2}}x^{\frac{16}{3}}-x^6.\end{aligned}$$

$$\begin{aligned}(3a^{\frac{1}{3}}-2b^2)^5 &= (3a^{\frac{1}{3}})^5-5(3a^{\frac{1}{3}})^4(2b^2)+10(3a^{\frac{1}{3}})^3(2b^2)^2\\ &\quad-10(3a^{\frac{1}{3}})^2(2b^2)^3+5(3a^{\frac{1}{3}})(2b^2)^4-(2b^2)^5\\ &= 243a^{\frac{5}{3}}-810a^{\frac{4}{3}}b^2+1080ab^4-720a^{\frac{2}{3}}b^6+240a^{\frac{1}{3}}b^8-32b^{10}.\end{aligned}$$

$$\begin{aligned}(a^{\frac{2}{3}}-\tfrac{1}{2}b^{\frac{1}{2}})^7 &= (a^{\frac{2}{3}})^7-7(a^{\frac{2}{3}})^6(\tfrac{1}{2}b^{\frac{1}{2}})+21(a^{\frac{2}{3}})^5(\tfrac{1}{2}b^{\frac{1}{2}})^2-35(a^{\frac{2}{3}})^4(\tfrac{1}{2}b^{\frac{1}{2}})^3\\ &\quad+35(a^{\frac{2}{3}})^3(\tfrac{1}{2}b^{\frac{1}{2}})^4-21(a^{\frac{2}{3}})^2(\tfrac{1}{2}b^{\frac{1}{2}})^5\\ &\quad+7(a^{\frac{2}{3}})(\tfrac{1}{2}b^{\frac{1}{2}})^6-(\tfrac{1}{2}b^{\frac{1}{2}})^7\\ &= a^{\frac{14}{3}}-\tfrac{7}{2}a^4b^{\frac{1}{2}}+\tfrac{21}{4}a^{\frac{10}{3}}b-\tfrac{35}{8}a^{\frac{8}{3}}b^{\frac{3}{2}}+\tfrac{35}{16}a^2b^2-\tfrac{21}{32}a^{\frac{4}{3}}b^{\frac{5}{2}}\\ &\quad+\tfrac{7}{64}a^{\frac{2}{3}}b^3-\tfrac{1}{128}b^{\frac{7}{2}}.\end{aligned}$$

$$\begin{aligned}\left(2-\frac{x^2}{3}\right)^6 &= 2^6-6(2^5)\left(\frac{x^2}{3}\right)+15(2^4)\left(\frac{x^2}{3}\right)^2-20(2^3)\left(\frac{x^2}{3}\right)^3\\ &\quad+15(2^2)\left(\frac{x^2}{3}\right)^4-6(2)\left(\frac{x^2}{3}\right)^5+\left(\frac{x^2}{3}\right)^6\\ &= 64-64x^2+\tfrac{80}{3}x^4-\tfrac{160}{27}x^6+\tfrac{20}{27}x^8-\tfrac{4}{81}x^{10}+\tfrac{1}{729}x^{12}.\end{aligned}$$

$$\begin{aligned}\left(3-\frac{x^{\frac{1}{2}}}{2}\right)^7 &= 3^7-7(3^6)\left(\frac{x^{\frac{1}{2}}}{2}\right)+21(3^5)\left(\frac{x^{\frac{1}{2}}}{2}\right)^2-35(3^4)\left(\frac{x^{\frac{1}{2}}}{2}\right)^3\\ &\quad+35(3^3)\left(\frac{x^{\frac{1}{2}}}{2}\right)^4-21(3^2)\left(\frac{x^{\frac{1}{2}}}{2}\right)^5+7(3)\left(\frac{x^{\frac{1}{2}}}{2}\right)^6-\left(\frac{x^{\frac{1}{2}}}{2}\right)^7\\ &= 2187-\tfrac{5103}{2}x^{\frac{1}{2}}+\tfrac{5103}{4}x-\tfrac{2835}{8}x^{\frac{3}{2}}+\tfrac{945}{16}x^2-\tfrac{189}{32}x^{\frac{5}{2}}\\ &\quad+\tfrac{21}{64}x^3-\tfrac{1}{128}x^{\frac{7}{2}}.\end{aligned}$$

150. $\left(x^2-\frac{y^3}{x}\right)^5$; $(2x^2+\sqrt{3x})^4$; $(\sqrt{a^2+1}+a^3)^5$; $(1+2x-x^2-x^3)^3$.

$$\begin{aligned}\left(x^2-\frac{y^3}{x}\right)^5 &= (x^2)^5-5(x^2)^4\left(\frac{y^3}{x}\right)+10(x^2)^3\left(\frac{y^3}{x}\right)^2-10(x^2)^2\left(\frac{y^3}{x}\right)^3\\ &\qquad +5(x^2)\left(\frac{y^3}{x}\right)^4-\left(\frac{y^3}{x}\right)^5\\ &= x^{10}-5x^7y^3+10x^4y^6-10xy^9+\frac{5y^{12}}{x^2}-\frac{y^{15}}{x^5}.\end{aligned}$$

$$\begin{aligned}(2x^2+\sqrt{3x})^4 &= (2x^2)^4+4(2x^2)^3(\sqrt{3x})+6(2x^2)^2(\sqrt{3x})^2\\ &\qquad +4(2x^2)(\sqrt{3x})^3+(\sqrt{3x})^4\\ &= 16x^8+32x^6\sqrt{3x}+72x^5+24x^3\sqrt{3x}+9x^2.\end{aligned}$$

$$\begin{aligned}(\sqrt{a^2+1}+a^3)^5 &= (\sqrt{a^2+1})^5+5(\sqrt{a^2+1})^4a^3+10(\sqrt{a^2+1})^3a^6\\ &\qquad +10(\sqrt{a^2+1})^2a^9+5\sqrt{a^2+1}\,a^{12}+a^{15}\\ &= (a^2+1)^2\sqrt{a^2+1}+5(a^2+1)^2a^3+10(a^2+1)\sqrt{a^2+1}\,a^6\\ &\qquad +10(a^2+1)a^9+5\sqrt{a^2+1}\,a^{12}+a^{15}\\ &= 5(a^4+2a^2+1)a^3+10(a^2+1)a^9+a^{15}+\sqrt{a^2+1}[a^4+2a^2+1\\ &\qquad +10(a^2+1)a^6+5a^{12}]\\ &= 5a^7+10a^5+5a^3+10a^{11}+10a^9+a^{15}+\sqrt{a^2+1}(a^4+2a^2+1\\ &\qquad +10a^8+10a^6+5a^{12})\\ &= a^{15}+10a^{11}+10a^9+5a^7+10a^5+5a^3+\sqrt{a^2+1}(5a^{12}+10a^8\\ &\qquad +10a^6+a^4+2a^2+1).\end{aligned}$$

$$\begin{aligned}(1+2x-x^2-x^3)^3 &= (1+2x)^3-3(1+2x)^2(x^2+x^3)\\ &\qquad +3(1+2x)(x^2+x^3)^2-(x^2+x^3)^3\\ &= (1+6x+12x^2+8x^3)-(3x^2+3x^3+12x^3+12x^4+12x^4\\ &\qquad +12x^5)+(3x^4+6x^5+3x^6+6x^5+12x^6+6x^7)\\ &\qquad -(x^6+3x^7+3x^8+x^9)\\ &= 1+6x+9x^2-7x^3-21x^4+14x^6+3x^7-3x^8-x^9.\end{aligned}$$

Expand to four terms:

151. $(1-3x)^{-\frac{1}{3}}$; $(1-4x^2)^{-\frac{2}{3}}$; $(1-\frac{3}{2}x^{\frac{1}{2}})^{\frac{2}{3}}$; $(a-2x^{-\frac{1}{3}})^{-5}$.

$$\begin{aligned}(1-3x)^{-\frac{1}{3}} &= 1-\tfrac{1}{3}(-3x)+\frac{-\frac{1}{3}\cdot-\frac{4}{3}}{2}(-3x)^2+\frac{-\frac{1}{3}\cdot-\frac{4}{3}\cdot-\frac{7}{3}}{2\cdot 3}(-3x)^3\\ &= 1+x+2x^2+\tfrac{14}{3}x^3+\cdots\end{aligned}$$

$$(1-4x^2)^{-\frac{2}{3}}=1-\tfrac{2}{3}(-4x^2)+\frac{-\frac{2}{3}\cdot-\frac{5}{3}}{2}(-4x^2)^2+\frac{-\frac{2}{3}\cdot-\frac{5}{3}\cdot-\frac{8}{3}}{2\cdot 3}(-4x^2)^3$$
$$=1+\tfrac{8}{3}x^2+\tfrac{80}{9}x^4+\tfrac{2560}{81}x^6+\cdots$$

$$(1-\tfrac{3}{2}x^{\frac{1}{2}})^{\frac{2}{3}}=1+\tfrac{2}{3}(-\tfrac{3}{2}x^{\frac{1}{2}})+\frac{\frac{2}{3}\cdot-\frac{1}{3}}{2}(-\tfrac{3}{2}x^{\frac{1}{2}})^2+\frac{\frac{2}{3}\cdot-\frac{1}{3}\cdot-\frac{4}{3}}{2\cdot 3}(-\tfrac{3}{2}x^{\frac{1}{2}})^3$$
$$=1-x^{\frac{1}{2}}-\tfrac{1}{4}x-\tfrac{1}{6}x^{\frac{3}{2}}-\cdots$$

$$(a-2x^{-\frac{1}{3}})^{-5}=a^{-5}-5a^{-6}(2x^{-\frac{1}{3}})+\frac{(-5)(-6)}{2}a^{-7}(-2x^{-\frac{1}{3}})^2$$
$$+\frac{(-5)(-6)(-7)}{6}a^{-8}(-2x^{-\frac{1}{3}})^3+\cdots$$
$$=a^{-5}-10a^{-6}x^{-\frac{1}{3}}+60a^{-7}x^{-\frac{2}{3}}+280a^{-8}x^{-1}+\cdots$$

152. Find the eighty-seventh term of $(2x-y)^{90}$.

$$\frac{90\cdot 89\cdot 88\cdots 5}{1\cdot 2\cdot 3\cdots 86}(2x)^4(-y)^{86}=\frac{90\cdot 89\cdot 88\cdot 87}{1\cdot 2\cdot 3\cdot 4}(16x^4)y^{86}$$
$$=15\cdot 16\cdot 22\cdot 89\cdot 87x^4y^{86}.$$

Resolve into partial fractions:

153. $\dfrac{3-2x}{1-3x+2x^2}$; $\dfrac{3-2x}{(1-x)(1-3x)}$; $\dfrac{1}{1-x^3}$.

$$\frac{3-2x}{1-3x+2x^2}=\frac{A}{1-2x}+\frac{B}{1-x}=\frac{A(1-x)+B(1-2x)}{1-3x+2x^2}$$
$$\therefore A+B=3,\ -A-2B=-2$$
$$A=4,\ B=-1$$
$$\therefore \frac{3-2x}{1-3x+2x^2}=\frac{4}{1-2x}-\frac{1}{1-x}.$$

$$\frac{3-2x}{(1-x)(1-3x)}=\frac{A}{1-x}+\frac{B}{1-3x}=\frac{A(1-3x)+B(1-x)}{(1-x)(1-3x)}$$
$$\therefore A+B=3,\ -3A-B=-2$$
$$A=-\tfrac{1}{2},\ B=\tfrac{7}{2}$$
$$\therefore \frac{3-2x}{(1-x)(1-3x)}=-\frac{1}{2(1-x)}+\frac{7}{2(1-3x)}$$

$$\frac{1}{1-x^3}=\frac{A}{1-x}+\frac{Bx+C}{1+x+x^2}$$
$$=\frac{A(1+x+x^2)+(1-x)(Bx+C)}{1-x^3}.$$

$$\therefore A + C = 1,\ A + B - C = 0,\ A - B = 0$$

$$A = \tfrac{1}{3},\ B = \tfrac{1}{3},\ C = \tfrac{2}{3}$$

$$\therefore \frac{1}{1 - x^3} = \frac{1}{3(1 - x)} + \frac{x + 2}{3(1 + x + x^2)}.$$

154. Expand to five terms $\dfrac{3 - 2x}{1 - 3x + 2x^2}$.

$$\frac{3 - 2x}{1 - 3x + 2x^2} = A + Bx + Cx^2 + Dx^3 + Ex^4 + \cdots$$

$$\begin{aligned}
3 - 2x = A + Bx &+ Cx^2 + Dx^3 + Ex^4 + \cdots \\
- 3Ax &- 3Bx^2 - 3Cx^3 - 3Dx^4 - \cdots \\
&+ 2Ax^2 + 2Bx^3 + 2Cx^4 + \cdots
\end{aligned}$$

$$\begin{aligned}
3 - 2x = A &+ (B - 3A)x + (C - 3B + 2A)x^2 \\
&+ (D - 3C + 2B)x^3 + (E - 3D + 2C)x^4 + \cdots
\end{aligned}$$

$$\therefore A = 3,\ B - 3A = -2,\ C - 3B + 2A = 0$$

$$D - 3C + 2B = 0,\ E - 3D + 2C = 0$$

$$A = 3,\ B = 7,\ C = 15,\ D = 31,\ E = 63, \cdots$$

$$\therefore \frac{3 - 2x}{1 - 3x + 2x^2} = 3 + 7x + 15x^2 + 31x^3 + 63x^4 + \cdots$$

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